

Quartet excitations and cluster spectra in light nuclei

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Abstract

The relation of quarteting and clustering in atomic nuclei is discussed based on symmetry-considerations. This connection enables us to predict a complete high-energy cluster spectrum from the description of the low-energy quartet part. As an example the ^{28}Si nucleus is considered, including its well-established ground-state region, the recently proposed superdeformed band, and the high-lying molecular resonances.

Keywords: quarteting and clustering, excitation spectrum, multichannel dynamical symmetry

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Most of the atomic nuclei are typical mesoscopic systems, which allow neither *ab initio*, nor statistical description. Therefore, models play the crucial role in the understanding the nuclear structure. The fundamental structure models are based on different physical pictures, e.g. shell, cluster or liquid drop, therefore, their interrelation is not trivial. Symmetry-considerations are very helpful in finding their connection, as well as in describing complex spectra. In this letter we show how the nucleon-quarteting, which is a shell model phenomenon, is related to the clusterization, i.e. to the appearance of a molecule-like configuration. We do so by applying a semimicroscopic algebraic description for both phenomena, which reveals a special symmetry, called multichannel dynamical symmetry. This symmetry allows us to obtain a high-lying cluster spectrum from the quartet model fitted to the low-energy part. We do not know any other method of this ability.

The investigation of quarteting and clustering has a long history, and a large variety of models have been invented for their description. When the cluster is an alpha-particle, which is the most typical and best studied case, the two structures are obviously related to each other: in both cases the basic building block is composed of two protons and two neutrons. In the phenomenological approaches, which do not respect the Pauli-exclusion principle, the wavefunction of the shell-like and molecule-like configurations (or those of two different cluster configurations) are orthogonal to each other. In fact, however, the anti-symmetrization modifies the simple geometric picture, and as a result, the overlap can be finite, up to 100 percent. One needs microscopically constructed model spaces for the study of this connection. (Whether the interactions are also microscopic or not, i.e. if the description is fully microscopic, or semimicroscopic is less relevant in this respect.)

In what follows we apply semimicroscopic algebraic models for the description of both quarteting and clus-

tering. This approach takes into account the exclusion principle, furthermore, due to its fully algebraic nature it has rather transparent symmetry properties. (We call a model fully algebraic when not only the basis states, but the physical operators as well are characterized by group representations.)

The semimicroscopic algebraic quartet model (SAQM) [1] is a symmetry-governed truncation of the no-core shell model [2], that describes the quartet excitations in a nucleus. A quartet is formed by two protons and two neutrons, which interact with each other very strongly, as a consequence of the short-range attractive forces between the nucleons inside a nucleus [3]. The interaction between the different quartets is weaker. In this approach the L-S coupling is applied, the model space has a spin-isospin sector, characterized by Wigner's $U^{ST}(4)$ group [4], and a space part described by Elliott's $U(3)$ [5]. Four nucleons form a quartet [6] when their spin-isospin symmetry is $\{1,1,1,1\}$, and their permutational symmetry is $\{4\}$. This definition allows two protons and two neutrons to form a quartet even if they sit in different shells. As a consequence the quartet model space incorporates 0, 1, 2, 3, 4, ... major shell excitations (in the language of the shell model), contrary to the original interpretation of [3], when the four nucleons had to occupy the same single-particle orbital, therefore, only 0, 4, 8, ... major shell excitations could be described.

The model is fully algebraic, therefore, group theoretical methods can be applied in calculating the matrix elements. The operators contain parameters to fit to the experimental data, that is why the model is called semimicroscopic: phenomenologic operators are combined with microscopic model space. Due to the quartet symmetry only a single $\{1,1,1,1\} U^{ST}(4)$ sector plays a role in the calculation of the physical quantities, thus the $U(3)$ space-group and its

subgroups are sufficient for characterizing the situation:

$$U(3) \supset SU(3) \supset SO(3) \supset SO(2) \\ \llbracket [n_1, n_2, n_3], (\lambda, \mu), K, L, M \rrbracket. \quad (1)$$

In Eq. (1) we have indicated also the representation labels of the groups which serve as quantum numbers of the basis states. Here $n = n_1 + n_2 + n_3$ is the number of the oscillator quanta, and $\lambda = n_1 - n_2$, $\mu = n_2 - n_3$. The angular momentum content of a (λ, μ) representation is as follows [5]: $L = K, K + 1, \dots, K + \max(\lambda, \mu)$, $K = \min(\lambda, \mu), \min(\lambda, \mu) - 2, \dots, 1$ or 0 , with the exception of $K_L = 0$, for which $L = \max(\lambda, \mu), \max(\lambda, \mu) - 2, \dots, 1$ or 0 . In the limiting case of the dynamical symmetry, when the Hamiltonian is expressed in terms of the invariant operators of this group-chain, an analytical solution is available for the energy-eigenvalue problem (an example is shown below).

The SAQM can be considered as an effective model in the sense of [7]: the bands of different quadrupole shapes are described by their lowest-grade $U(3)$ irreducible representations (irreps) without taking into account the giant-resonance excitations, built upon them, and the model parameters are renormalised for the subspace of the lowest $U(3)$ irreps.

The semimicroscopic algebraic cluster model (SACM) [8], just like the other cluster models, classifies the relevant degrees of freedom of the nucleus into two categories: they belong either to the internal structure of the clusters, or to their relative motion. In other words: the description is based on a molecule-like picture. The internal structure of the clusters is handled in terms of Elliott's shell model [5] with $U^{ST}(4) \otimes U(3)$ group structure (as discussed beforehand). The relative motion is taken care of by the vibron model [9], which is an algebraic model of the dipole motion, and it has a $U(3)$ basis, too. For a two-cluster-configuration this model has a group-structure of $U_{C_1}^{ST}(4) \otimes U_{C_1}(3) \otimes U_{C_2}^{ST}(4) \otimes U_{C_2}(3) \otimes U_R(4)$.

The model space is constructed also in this case in a microscopic way, i.e. the Pauli-forbidden states are excluded. It requires the truncation of the basis of the vibron model, as given by the Wildermuth-condition (see below for some specific examples). This condition determines the lowest-allowed quantum number of the relative motion, i.e. the allowed major shells of the (united) nucleus. Furthermore, one needs to distinguish between the Pauli-allowed and forbidden states within a major shell, too. Different methods can be applied to this purpose; e.g. by making an intersection with the $U(3)$ shell model basis of the nucleus, which is constructed to be free from the forbidden states. The SACM is fully algebraic, and semimicroscopic in the sense discussed above.

When we are interested only in spin-isospin zero states of the nucleus (a typical problem in cluster studies, and being our case here, too), then only the space symmetries are relevant (apart from the construction of the model space).

Considering, for the sake of simplicity, a binary cluster configuration the corresponding group-chain is:

$$U_{C_1}(3) \otimes U_{C_2}(3) \otimes U_R(4) \supset U_C(3) \otimes U_R(3) \supset \\ U(3) \supset SU(3) \supset SO(3) \supset SO(2). \quad (2)$$

The basis defined by this chain is especially useful for treating the exclusion principle, since the $U(3)$ generators commute with those of the permutation group, therefore, all the basis states of an irrep are either Pauli-allowed, or forbidden [10]. In particular, this $U(3)$ basis allows us to pick up the allowed cluster states from the $U(3)$ shell model basis (1).

A Hamiltonian corresponding to the dynamical symmetry of group-chain (2) reads as:

$$\hat{H} = \hat{H}_{C_1} + \hat{H}_{C_2} + \hat{H}_{U_R(4)} + \hat{H}_{U_C(3)} + \hat{H}_{U_R(3)} + \\ \hat{H}_{U(3)} + \hat{H}_{SU(3)} + \hat{H}_{SO(3)}. \quad (3)$$

We note here, that the first part

$$\hat{H}_{CM} = \hat{H}_{C_1} + \hat{H}_{C_2} + \hat{H}_{U_R(4)} + \hat{H}_{U_C(3)} + \hat{H}_{U_R(3)} \quad (4)$$

is an operator that corresponds to the pure cluster picture, while the second part

$$\hat{H}_{SM} = \hat{H}_{U(3)} + \hat{H}_{SU(3)} + \hat{H}_{SO(3)} \quad (5)$$

is a shell model Hamiltonian (of the united nucleus).

The multichannel dynamical symmetry (MUSY) [11, 12] connects different cluster configurations (including the shell model limit) in a nucleus. Here the word channel refers to the reaction channel, that defines the cluster configuration.

The simplest case is a two-channel symmetry connecting two different clusterizations. It holds, when both cluster configurations can be described by an $U(3)$ dynamical symmetry and in addition a further symmetry connects them to each other. This latter symmetry is that of the Talmi-Moshinsky transformation. It acts in the pseudo space of the particle indices, or geometrically it corresponds to the transformations between the different sets of Jacobi-coordinates associated to the cluster configurations [13, 12]. The \hat{H}_{SM} Hamiltonian of Eq. (5) is symmetric with respect to these transformations, therefore, it is invariant under the changes from one clusterization to the other. The cluster part of the Hamiltonian \hat{H}_{CM} is affected by the transformation from one configuration to the other, of course. Nevertheless, it may remain invariant, which is the case for simple operators, like the harmonic oscillator Hamiltonian, or the quadrupole operator [12]. Due to this symmetry of the quadrupole operator, the $E2$ transitions of different clusterizations also coincide, when the MUSY holds, just like the energy eigenvalues of the symmetric Hamiltonians [12].

The MUSY is a composite symmetry of a composite system. Its logical structure is somewhat similar to that

of the dynamical supersymmetry (SUSY) of nuclear spectroscopy. In the SUSY case the system has two components, a bosonic and a fermionic one, each of them showing a dynamical symmetry, and a further symmetry connects them to each other. The connecting symmetry is that of the supertransformations which change bosons into fermions or vice versa. In the MUSY case the system has two (or more) different clusterizations, each of them having dynamical symmetries which are connected to each other by the symmetry of the (Talmi-Moshinsky) transformations that change from one configuration to the other.

When the multichannel dynamical symmetry holds then the spectra of different clusterizations are related to each other by very strong constraints. The MUSY provides us with a unified multiplet structure of different cluster configurations, furthermore the corresponding energies and E2 transitions coincide exactly. Of course, it can not be decided a priori whether the MUSY holds or not, rather one can suppose the symmetry and compare its consequences with the experimental data. In what follows we derive the spectra of two clusterizations from the quartet spectrum of the ^{28}Si nucleus.

The ^{28}Si nucleus provides us with many reasons to be chosen as an illustrative example. i) It has a well-established band-structure in the low-energy region, and to several bands SU(3) quantum numbers could be associated as a joint conclusion of experimental and theoretical investigations [14]. ii) More recently a new candidate was proposed for the superdeformed (SD) band [15]. Theoretical studies predicted the SD band [16, 17] in line with the experimental observation. iii) There are two cluster configurations: $^{24}\text{Mg}+^4\text{He}$, and $^{16}\text{O}+^{12}\text{C}$, belonging to reaction channels in which fine-resolution measurements revealed a rich spectrum of resonances.

In [11] the connection of these two cluster configurations has been discussed in terms of the multichannel dynamical symmetry. In the present work we go beyond the former description in several aspects. We calculate the quartet spectrum of the ^{28}Si nucleus, and obtain the spectra of both clusterizations from the quartet excitations by projection, without fitting anything to the cluster states, i.e. the cluster spectra appear as pure predictions. In doing so we apply a simple Hamiltonian with less number of parameters than in [11]. In addition to the energy spectra we give the E2 transition ratios as well. The new superdeformed candidate band is also taken into account.

Quartet excitations.

The lower most part of Figure 1 shows the experimental bands of the ^{28}Si nucleus, as established in [14] together with the recently found superdeformed (SD) band [15]. An especially favourable circumstance is that SU(3) quantum numbers are associated to several experimental bands, without any reference to the quartet or cluster studies. (In the experimental spectrum β means β -instabil, while O and P stand for oblate and prolate, respectively.)

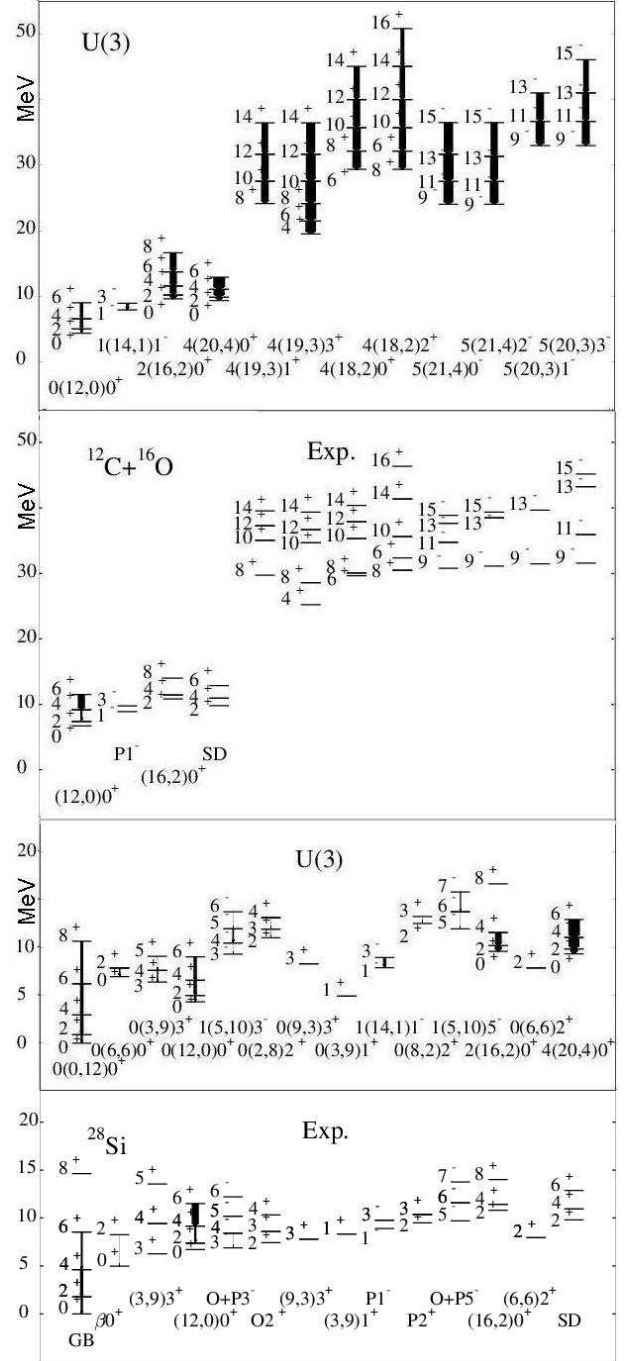


Figure 1: The spectrum of the semimicroscopic algebraic quartet model in comparison with the experimental data of the ^{28}Si nucleus (lower part). The experimental bands are labeled by the available quantum numbers, and the model states by the $n(\lambda, \mu)K^\pi$ labels. The width of the arrow between the states is proportional to the strength of the E2 transition. The upper part shows the $^{12}\text{C}+^{16}\text{O}$ cluster spectrum, which is obtained as a projection from the quartet spectrum, without any further fitting.

The U(3) spectrum (the second one from below in Figure 1) is calculated within the SAQM approach [1]. The experimental states are described by the lowest-lying model bands with the appropriate spin-parity content. We have applied a U(3) dynamically symmetric Hamiltonian, i.e. an operator expressed in terms of the invariant operators of the group-chain: $U(3) \supset SU(3) \supset SO(3)$:

$$\hat{H} = (\hbar\omega)\hat{n} + a\hat{C}_{SU3}^{(2)} + b\hat{C}_{SU3}^{(3)} + d\frac{1}{2\theta}\hat{L}^2. \quad (6)$$

The first term is the harmonic oscillator Hamiltonian (linear invariant of the U(3)), with a strength obtained from the systematics [18] $\hbar\omega = 45A^{-\frac{1}{3}} - 25A^{-\frac{2}{3}}$ MeV = 12.11 MeV. The second order invariant of the SU(3) ($\hat{C}_{SU3}^{(2)}$) represents the quadrupole-quadrupole interaction, while the third order Casimir-operator ($\hat{C}_{SU3}^{(3)}$) distinguishes between the prolate and oblate shapes. θ is the moment of inertia calculated classically for the rigid shape determined by the U(3) quantum numbers (for a rotor with axial symmetry) [19], and the a , b and d parameters were fitted to the experimental data: $a = -0.133$ MeV, $b = 0.000444$ MeV, $d = 1.003$ MeV. The $B(E2)$ value is given as [1]:

$$B(E2, I_i \rightarrow I_f) = \frac{2I_f + 1}{2I_i + 1} \alpha^2 | \langle (\lambda, \mu) K I_i, (11) 2 | \langle (\lambda, \mu) K I_f \rangle |^2 C(\lambda, \mu), \quad (7)$$

where $\langle (\lambda, \mu) K I_i, (11) 2 | \langle (\lambda, \mu) K I_f \rangle$ is the SU(3) \supset SO(3) Wigner coefficient, and α^2 (= 0.366 W.u.) is a parameter fitted to the experimental value of the $2_1^+ \rightarrow 0_1^+$ transition of 13.2 W.u.

Cluster spectra.

The MUSY can connect the quartet (shell) model state to other clusterizations, too. Here we show, how the $^{24}\text{Mg} + ^4\text{He}$, and $^{16}\text{O} + ^{12}\text{C}$ cluster spectra can be obtained from the quartet spectrum by simple projections.

In this description the clusters are considered to be in their intrinsic ground states. Each of the four clusters of the present study have spin-isospin zero quantum numbers, i.e. they belong to scalar representations of Wigner's $U^{ST}(4)$ group. Their space symmetry is given by Elliott's U(3) group, which is known to be approximately valid for these light nuclei, therefore, simple leading representation characterize their ground states as follows: ^4He : $\{0,0,0\}$, ^{12}C : $\{4,4,0\}$, ^{16}O : $\{4,4,4\}$, ^{24}Mg : $\{16,8,4\}$.

A state of an $\{n_1, n_2, n_3\}$ symmetry is present in a binary cluster configuration $C_1 + C_2$, if the triple product matches with it:

$$\{n_1^{c_1}, n_2^{c_1}, n_3^{c_1}\} \otimes \{n_1^{c_2}, n_2^{c_2}, n_3^{c_2}\} \otimes \{n_R, 0, 0\} = \{n_1, n_2, n_3\} \oplus \dots \quad (8)$$

where $\{n_R, 0, 0\}$ stands for the relative motion, and n_R is limited from below, due to the Pauli-principle (known as the Wildermuth-condition [20]).

In case of the $^{24}\text{Mg} + ^4\text{He}$ clusterization the lowest allowed value of n_R is 8, showing that in the unification of

Table 1: SU(3) quantum numbers of the 0 $\hbar\omega$ states in the ^{28}Si nucleus. The superscripts indicate multiplicity.

$\hbar\omega$	Quartet	$^{24}\text{Mg} + \alpha$	$^{12}\text{C} + ^{16}\text{O}$
0	$(12, 0)^1, (0, 12)^1,$ $(3, 9)^1, (9, 3)^1,$ $(6, 6)^1, (2, 8)^2,$ $(8, 2)^2, (5, 5)^2,$ $(3, 6)^2, (6, 3)^2,$ $(1, 7)^1, (7, 1)^1,$ $(4, 4)^4, (2, 5)^1,$ $(5, 2)^1, (0, 6)^3,$ $(6, 0)^3, (3, 3)^3,$ $(1, 4)^1, (4, 1)^1,$ $(2, 2)^3, (0, 0)^2$	$(12, 0)^1, (0, 12)^1,$ $(3, 9)^1, (9, 3)^1,$ $(6, 6)^1, (2, 8)^1,$ $(8, 2)^1, (5, 5)^1,$ $(3, 6)^1, (6, 3)^1,$ $(4, 4)^1$	$(12, 0)^1$

the two nuclei the 4 nucleons of the ^4He has to be lifted to the 2 $\hbar\omega$ major shell in order not to violate the exclusion principle. For the $^{16}\text{O} + ^{12}\text{C}$ clusterization the values below 16 are excluded.

We note here that in our case the results of the triple product have always single multiplicity. This is because one of the clusters has a closed-shell structure, i.e. it is an U(3) scalar. As a consequence a single U(3) irrep is multiplied by the single-row irrep of the relative motion $\{n_R, 0, 0\}$.

For illustration we show in Table I. the SU(3) quantum numbers of the quartet model as well as the two cluster model spaces for 0 $\hbar\omega$.

Until the basis states are determined by the U(3) (and its subgroups) symmetry, and the interactions are dynamically symmetric, i.e. the MUSY holds, the corresponding energies and E2 transition rates in the quartet and cluster descriptions coincide. Therefore, by applying the selection rule (8), not only the cluster model basis states, but also their energy eigenvalues, as well as the E2 transition probabilities between them can be selected. In other words the cluster spectrum is obtained from the quartet one by a simple projection.

The $^{12}\text{C} + ^{16}\text{O}$ spectrum of Figure 1 shows those bands of the low-energy part, which are present in this cluster configuration, as well as the resonance spectrum from the heavy ion experiments, according to the compilation of [21]. The latter one is organised into bands according to their energy-differences. The corresponding U(3) spectrum is calculated with Eq. (6), without fitting anything to the high-lying resonances. In other words the $^{12}\text{C} + ^{16}\text{O}$ resonance energies are predicted from the quartet excitations of the ^{28}Si . In particular, the projection was done by taking the intersection of the quartet and cluster spectra in the superdeformed valley (in the second minimum of the energy-versus-deformation function, where the SD state corresponds to the "ground"-band). In order to characterize the breaking (or the goodness) of the MUSY quan-

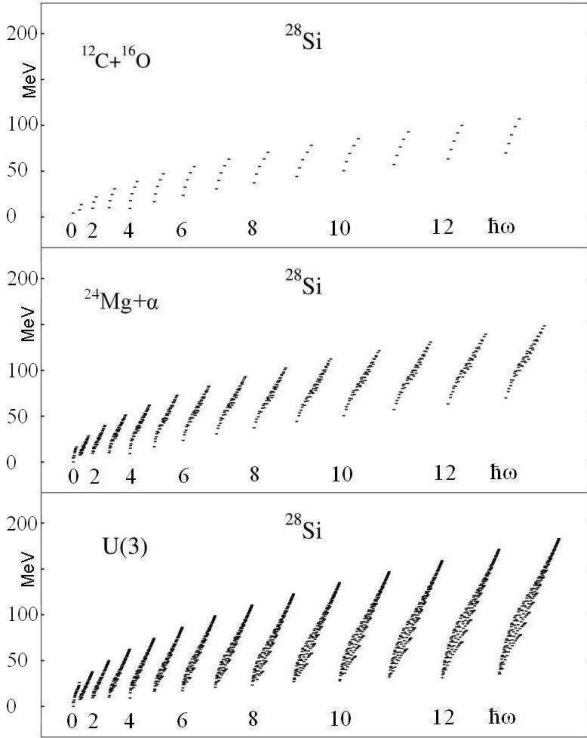


Figure 2: The landscape of the quartet and cluster band-heads in the ^{28}Si nucleus.

tatively, we have calculated the

$$sb = \frac{\sum_i |E_i^{exp} - E_i^{th}|}{\sum_i E_i^{exp}} \quad (9)$$

ratio. It turned out to be 13 % for the spectrum of Figure 1 (including both the low- and the high-energy parts). When the resonances are also taken into account in the fitting procedure (e.g. with a weight of 0.1 compared to the weight of 1.0 of the states in the well-established bands), a slightly better agreement of $sb = 12\%$ can be obtained. The low-energy bands $0(12,0)0^+$, $1(14,1)1^-$, $2(16,2)0^+$, $4(20,4)0^+$ have single multiplicity in the shell-model space, therefore, the overlap of the wavefunctions of their states in the quartet and cluster descriptions is 100%. (In the shell-model expansion of the cluster states there is only a single term.)

The $^{24}\text{Mg}+^4\text{He}$ cluster spectrum contains all the states shown in Figure 1. In the low-energy spectrum (lower part of Figure 1) all the bands, except the $0(2,8)2^+$, and $0(8,2)2^+$ have single multiplicity in the shell-model. As a consequence their wavefunctions are identical with those of the $^{24}\text{Mg}+^4\text{He}$ cluster configuration, as well as with those of the $^{12}\text{C}+^{16}\text{O}$ clusterization, when it is allowed.

For further illustration we show in Figure 2 the landscape of the bandhead-states in the 0-13 $\hbar\omega$ major shells for the quartet and cluster spectra.

As for the other possible binary clusterizations (e.g. $^{20}\text{Ne}+^8\text{Be}$) of the ^{28}Si nucleus the following can be said. From the theoretical point of view they are available for

this kind of analysis, too, though technically some parts might be more involved, due to the non-closed structure (non SU(3) scalar nature) of the clusters. At the same time, they are much less known from the experimental side.

Further extension to non-alpha-like nuclei is also possible. From the quartet side extra nucleons can be included when the semimicroscopic model is applied, like here (as opposed to the phenomenologic quartet model), since this approach is based on the nucleon degrees of freedom [1]. The semimicroscopic algebraic cluster model allows the treatment of the cluster with odd mass number, as well [22], due to the same reason.

In conclusion we can say that the semimicroscopic algebraic models are able to describe the quartet and cluster spectra in light nuclei in a unified framework. In particular: the multichannel dynamical symmetry gives the cluster spectra from that of the quartet model by simple projections, therefore, it has a very strong predictive power. In case of ^{28}Si e.g. the high-lying spectrum of the $^{12}\text{C}+^{16}\text{O}$ clusterization is predicted from the low-lying quartet spectrum in remarkable agreement with the experimental observation.

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