Selected Transl. in Math.
Statist. and Probability
Volume 14, 1978

ON THE STATISTICAL EXAMINATION OF CONTINUOUS STATE MARKOV PROCESSES. III*

MÁTYÁS ARATÓ

Introduction

The first two parts [1], [2] of this paper dealt with statistical examination of continuous-time, discrete, one-dimensional stationary Gaussian Markov processes, whereas in this part two-dimensional (complex) processes of similar type will be examined. The importance of examining such processes by statistical methods became apparent to workers at the Department of Probability at Moscow State University in connection with the investigation of a geophysical problem. The latter will be presented in a separate section of this paper, where further possibilities in connection with complete investigation of the physical phenomena are pointed out. These investigations were regularly discussed at the department; the results obtained, in the elaboration of which the author of this paper participated, are presented here. On these results a joint paper [4] by A. N. Kolmogorov, Ja. G. Sinaĭ and the author has been published, and the calculations and mathematical procedures were given in a joint paper by Rykova, Sinaĭ and the author [5]. Detailed proofs were not included in [4]; but, except for §3, which contains some results of Sinaĭ and of Arató [3], they were worked out by the author.

The Continuous-Time, Stationary Normal Complex Case

§1. Description of continuous-time processes

Let us take a two-dimensional stochastic process whose components $\xi(t)$ and $\eta(t)$ satisfy the differential equations

$$d\xi = -\lambda\xi dt + \omega\eta dt + d\varepsilon_1,$$
$$d\eta = \omega\xi dt - \lambda\eta dt + d\varepsilon_2, \tag{1.1}$$

where $\epsilon_1(t)$ and $\epsilon_2(t)$ are two independent Wiener processes with the parameters

$$\mathsf{E}\, d\varepsilon_1 = \mathsf{E}\, d\varepsilon_2 = 0, \quad \mathsf{E}\,(d\varepsilon_1)^2 = \mathsf{E}\,(d\varepsilon_2)^2 = a \cdot dt.$$

Assuming that

$$\zeta(t) = \xi(t) + i\eta(t), \quad \chi(t) = \varepsilon_1(t) + i\varepsilon_2(t), \quad \gamma = \lambda - i\omega,$$

we can write the system (1.1) as a single differential equation:

$$(1.1') \qquad d\zeta = -\gamma\zeta dt + d\chi.$$

The complex correlation function of the process $\zeta(t)$ is (see [1], Theorem 1.2)

$$(1.2) \qquad C(\tau) = A(\tau) + iB(\tau) = \mathsf{E}\,[\zeta(t)\overline{\zeta(t+\tau)}] = \sigma^2 \exp(-\lambda|\tau| - i\omega\tau)$$

where $\sigma^2 = a/\lambda$ (from (1.1')). Using a realization of $\zeta(t)$ on the interval $(0, T)$, the empirical correlation function is

$$(1.3) \qquad c(\tau) = a(\tau) + ib(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} \zeta(t)\overline{\zeta(t+\tau)}\, dt,$$

which is differentiable from the right at zero with probability 1 and

$$(1.4) \qquad c'(0) = -a - \frac{1}{T}s_1^2 + \frac{1}{T}s_2^2 - ir,$$

where a is a parameter characterizing the intensity of the "white noise" processes $\epsilon_1'(t)$ and $\epsilon_2'(t)$, while

$$s_1^2 = \frac{1}{2}[|\zeta(0)|^2 + |\zeta(T)|^2], \quad s_2^2 = \frac{1}{T}\int_0^T |\zeta(t)|^2\, dt, \quad r = \frac{1}{T}\int_0^T |\zeta(t)|^2\, d\theta.$$

The variable θ is determined in the last integral by

$$\zeta(t) = |\zeta(t)|\, e^{i\theta(t)}.$$

The proof of (1.4) is given as follows. It can easily be shown that

$$(1.5) \qquad \frac{c(\tau+h) - c(\tau)}{h} = \frac{1}{T-\tau}\int_0^{T-\tau} \zeta(t)\, d\overline{\zeta(t)} + \frac{1}{(T-\tau)^2}\int_0^{T-\tau} \zeta(t)\overline{\zeta(t+h)}\, dt - \frac{1}{T-\tau}\zeta(T-\tau)\overline{\zeta(T)} + o(1).$$

On the other hand, using the relation

$$\sum_2^n |\zeta(t_i) - \zeta(t_{i-1})|^2 \to 2aT,$$

valid for a complex process (see [1], (1.10)), it is easy to see that the value of the integral $\int_0^T \zeta(t) d\zeta(t)$, understood as the limit of the sum

$$\sum_2^n \zeta(t_{i-1})[\zeta(t_i) - \zeta(t_{i-1})]$$

is equal to

$$(1.6) \qquad -aT + \frac{|\zeta(T)|^2 - |\zeta(0)|^2}{2} - i\int_0^T |\zeta(t)|^2\, d\theta.$$

In fact, with simple calculations it can be seen that

$$\sum_2^n |\zeta(t_i) - \zeta(t_{i-1})|^2 = \sum_2^n [\zeta(t_i)\overline{\zeta(t_i)} - \zeta(t_{i-1})\overline{\zeta(t_i)} - \zeta(t_i)\overline{\zeta(t_{i-1})} - \zeta(t_{i-1})\overline{\zeta(t_{i-1})}]$$

$$= -2\sum_2^n \zeta(t_{i-1})[\overline{\zeta(t_i) - \zeta(t_{i-1})}] + |\zeta(T)|^2 - |\zeta(0)|^2 + \sum_2^n [\zeta(t_{i-1})\overline{\zeta(t_i)} - \zeta(t_i)\overline{\zeta(t_{i-1})}]$$

$$= -2\sum_2^n \zeta(t_{i-1})[\overline{\zeta(t_i) - \zeta(t_{i-1})}] + |\zeta(T)|^2 - |\zeta(0)|^2$$

$$+ \sum_2^n |\zeta(t_i)|\,|\zeta(t_{i-1})|\,[e^{i(\theta(t_{i-1}) - \theta(t_i))} - e^{i(\theta(t_i) - \theta(t_{i-1}))}].$$

From (1.5) and (1.6) we now get (1.4).

§2. Estimation of parameters and their distribution

In §1 we mentioned that, as in the one-dimensional case, the diffusion parameter a can be precisely estimated from a single realization, since

$$\sum_1^n |\xi(t_i) - \xi(t_{i-1})|^2 \to aT, \quad \text{for } n \to \infty, \ \max|t_i - t_{i-1}| \to 0$$

and

$$\sum_1^n |\eta(t_i) - \eta(t_{i-1})|^2 \to aT, \quad \text{for } n \to \infty, \ \max|t_i - t_{i-1}| \to 0.$$

Therefore the unknown parameters are λ and ω. Let P be the measure on the space of realizations on $(0, T)$ associated with the process $\zeta(t)$. Let us introduce on the same space the standard measure $V = L \times W$, where L is the usual Lebesgue measure in the $\zeta(0)$ plane, and let W be the two-dimensional Wiener measure on the space of increments $\zeta(t) - \zeta(0)$ with the parameters of the process $\chi(t)$ (see (1.1')). It is known (see [14]; cf. [1], (2.1)) that

$$\text{(2.1)} \qquad \frac{dP}{dV} = \frac{\lambda}{\pi \cdot a^2} \exp\left[-\frac{\lambda^2+\omega^2}{2a} Ts_2^2 - \frac{\lambda}{a} s_1^2 + \lambda T + \frac{\omega}{a} Tr\right]^{(1)}$$

According to (2.1), the system s_1^2, s_2^2, r is a sufficient system for the problem. Taking derivatives with respect to ω and λ in the formula

$$L = \log \frac{dP}{dV} = -\log \pi a^2 + \log \lambda - \frac{\lambda^2+\omega^2}{2a} Ts_2^2 - \frac{\lambda}{a} s_1^2 + \lambda T + \frac{\omega}{a} Tr$$

we get

$$\text{(2.2)} \qquad \frac{\partial L}{\omega\omega} = -\frac{\omega}{a} Ts_2^2 + \frac{T}{a} r = 0;$$

$$\text{(2.3)} \qquad \frac{\partial L}{\partial \lambda} = \frac{1}{\lambda} - \frac{\lambda}{a} Ts_2^2 - \frac{s_1^2}{a} + T = 0,$$

from which the maximum likelihood estimators $\hat{\omega}$ and $\hat{\lambda}$ can be obtained. From (2.2) we have

$$\text{(2.2')} \qquad \breve{\omega} = \frac{r}{s_2^2}$$

and, denoting $\lambda T = \kappa$ and $\hat{\lambda} T = \hat{\kappa}$, $\hat{\kappa}$ can be determined from (2.3) as a solution of the equation

(1) The Radon-Nikodým derivative can also be derived heuristically in a manner similar to that used in part I, by means of functionals of $\epsilon_1(t)$ and $\epsilon_2(t)$. Since in the complex case the Radon-Nikodým derivative has the form

$$C(\lambda) \exp\left\{-\frac{|\zeta(0)|^2}{a}\lambda - \frac{1}{2a}\int_0^T \frac{(d\xi + \lambda\xi\, dt + \omega\eta\, dt)^2}{dt} - \frac{1}{2a}\int_0^T \frac{(d\eta - \omega\xi\, dt + \lambda\eta\, dt)^2}{dt}\right\}$$

$$= C(\lambda) \exp\left\{-\frac{|\zeta(0)|^2}{a}\lambda - \frac{\lambda^2+\omega^2}{2a}\int_0^T |\zeta(t)|\, dt - \frac{\lambda}{a}\int_0^T (\xi d\xi + \eta d\eta)\right.$$

$$\left. - \frac{1}{2a}\int_0^T \left[\frac{(d\xi)^2}{dt} + \frac{(d\eta)^2}{dt}\right] - \frac{\omega}{a}\int_0^T (\eta d\xi - \xi d\eta)\right\},$$

expression (2.1) can formally be obtained, as will be seen from expressions related to the integrals mentioned above.

$$\text{(2.3')} \qquad h_2\hat{\kappa}^2 + (h_1 - 1)\hat{\kappa} - 1 = 0$$

where $h_1 = s_1^2/aT$ and $h_2 = s_2^2/aT$. The distribution of the single positive solution of (2.3') will be treated further in §3.

Let $\sigma^2(\hat{\omega}) = 2a/Ts_2^2$. The following theorem will be proved.

THEOREM 2.1. *For the maximum likelihood estimate* $\hat{\omega}$ *the random variable* $(\hat{\omega} - \omega)/\sigma(\hat{\omega})$ *has a normal distribution with parameters* (0, 1).

REMARK. An interesting feature of this theorem is that for the estimate $\hat{\omega}$ it gives an exact, and not asymptotic, distribution.

PROOF. It can be seen that

$$\frac{\hat{\omega} - \omega}{\sigma(\hat{\omega})} = \sqrt{\frac{1}{2aT}} \frac{\int_0^T |\zeta(t)|^2 (d\Theta - \omega\, dt)}{\left(\int_0^T |\zeta(t)|^2\, dt\right)^{1/2}}$$

and it remains only to show (when integrating over an interval of length dt instead of T) that the random variable

$$|\zeta(t)|\,(d\theta - \omega dt)\sqrt{2adt}$$

has a normal distribution. From (1.1') we get

$$|\zeta|\,(d\theta - \omega dt) = d\varepsilon_2 \cdot \cos\theta - d\varepsilon_1 \cdot \sin\theta,$$

which gives the desired result, because $d\epsilon_2$ and $d\epsilon_1$ are independent normally distributed variables.

The above theorem points out the character of the motion described by the process $\zeta(t)$: a point moving with mean angular velocity ω at a distance $|\zeta|$ from the origin.(2)

(2) Very useful in changing to the discrete-time case is the relation

$$(*) \qquad \int_0^T |\zeta(t)|^2 d\theta = \int_0^T (\xi d\eta - \eta d\xi)$$

which can easily be proved starting from the identity

$$\sum [\zeta(t_i)\overline{\zeta(t_{i-1})} - \zeta(t_{i-1})\overline{\zeta(t_i)}] = 2i \sum [\eta(t_i)(\xi(t_i) - \xi(t_{i-1})) - \xi(t_i)(\eta(t_i) - \eta(t_{i-1}))]$$

in the same way as in proving (1.6).

§3. Construction of confidence intervals for the parameter λ.

In this section we shall assume that, except for λ, all parameters of a complex stationary Gaussian Markov process $\zeta(t)$ are known. For the sake of simplicity let $\mathsf{E}\xi(t) = \mathsf{E}\eta(t) = 0$ and $\mathsf{E}\xi^2(t) = \mathsf{E}\eta^2(t) = 1/2\lambda$. Under these assumptions we shall determine the characteristic function of a sufficient system of statistics and show a method for obtaining the corresponding confidence intervals. As mentioned above, these results are given along with necessary calculations by Ja. G. Sinaĭ, L. V. Rykova and the author [5].

It can be seen from (2.1) that the statistics

$$\chi_1(T) = \frac{1}{2}\{\xi^2(0)+\eta^2(0)+\xi^2(T)+\eta^2(T)\}, \tag{3.1}$$

$$\chi_2(T) = \int_0^T (\xi^2(t)+\eta^2(t))\,dt$$

form a sufficient system. To determine the characteristic function of these variables, the following partial differential equation may be considered (the problem of existence and uniqueness in differential equations connected with various functionals are dealt with in Dynkin's paper [7]). Let

$$u(T, x, y) = \mathsf{E}\,\{e^{i\alpha_1\chi_1(T)+i\alpha_2\chi_2(T)}|_{\substack{\xi(0)=x\\ \eta(0)=y}}\}, \tag{3.2}$$

i.e. $u(T, x, y)$ is the conditional characteristic function of χ_1 and χ_2 under the condition that $\xi(0) = x$ and $\eta(0) = y$. Evidently

$$u(T+\Delta T, x, y) = \frac{1}{2\pi\Delta T}\iint_{-\infty}^{\infty} e^{-\frac{(x_1-x+\lambda x\Delta T+\omega y\Delta T)^2}{2\Delta T}-\frac{(y_1-y-\omega x\Delta T+\lambda y\Delta T)^2}{2\Delta T}}$$

$$\cdot u(T, x, y\Big\{[1+i\alpha_2\Delta T(x^2+y^2)]\Big[1-i\alpha_1\Big(\frac{(x_1-x)^2}{2}+\frac{(y_1-y)^2}{2} \tag{3.3}$$

$$+x(x_1-x)+y(y_1-y)\Big)+\frac{\alpha_1^2}{2}(x^2(x_1-x)^2+y^2(y_1-y)^2$$

$$+2y(x_1-x)(y_1-y)+\dots)\Big]\Big\}dx_1\,dy_1$$

and as $\Delta T \to 0$ the following equation for $u(T, x, y)$ is obtained:

$$\frac{\partial u}{\partial T} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial x}(-i\alpha_1 x-\lambda x-\omega y)+\frac{\partial u}{\partial v}(-i\alpha_1 y+\omega x-\lambda y) \tag{3.4}$$

$$+u\left[-i\alpha_1+i\alpha_1\lambda(x^2+y^2)-\frac{\alpha_1^2}{2}(x^2+y^2)+i\alpha_2(x^2+y^2)\right].$$

Let

$$u(T, x, y) = u_1(T, x, y)e^{i\alpha_1\frac{x^2+y^2}{2}};$$

hence we have from (3.4)

$$\frac{\partial u_1}{\partial T} = \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}\right)+\frac{\partial u_1}{\partial x}(\lambda x+\omega y)+\frac{\partial u_1}{\partial y}(-\omega x+\lambda y)+u_1 i\alpha_2(x^2+y^2), \tag{3.5}$$

or in polar coordinates

$$\frac{\partial u_1}{\partial T} = \frac{1}{2}\frac{\partial^2 u_1}{\partial r^2}+\frac{\partial u_1}{\partial r}\left(\frac{1}{2r}+\lambda r\right)+u_1 i\alpha_2 r^2. \tag{3.5'}$$

After the transformation $u_1(T, r) = v(T, r^2) = v(T, \rho)$, the last function satisfies

$$\frac{\partial v}{\partial T} = 2\varrho\frac{\partial^2 v}{\partial \varrho^2}+2\frac{\partial v}{\partial \varrho}(1-\lambda\varrho)+i\alpha_2\varrho v \tag{3.6}$$

with the initial condition

$$v(0, \varrho) = e^{\frac{i\alpha_1}{2}\varrho}. \tag{3.7}$$

Let w be the Laplace transform of v, i.e.

$$w(T, \gamma) = \int_0^\infty e^{-\gamma\varrho}v(T, \varrho)\,d\varrho.$$

From (3.6) and (3.7) we have

$$\frac{\partial w}{\partial T} = \frac{\partial w}{\partial \gamma}(-2\gamma^2+2\lambda\gamma-id_2)+w(2\lambda-2\gamma), \tag{3.8}$$

$$w(0, \gamma) = \frac{1}{\gamma-\frac{i\alpha_1}{2}}. \tag{3.9}$$

The solution of (3.8) can easily be obtained by commonly known methods (see, for example, [13], Chapter VIII, §2). Let the solutions of the equation

$$\gamma^2-\lambda\gamma+\frac{i\alpha_1}{2}=0,\quad \gamma_{1,2}=\frac{\lambda\pm\sqrt{\lambda^2-2i\alpha_2}}{2} \tag{3.10}$$

be denoted by γ_1 and γ_2. The first integrals of (3.8) are

$$\begin{aligned} c_1 &= T-\frac{1}{2(\gamma_1-\gamma_2)}\log\frac{\gamma-\gamma_1}{\gamma-\gamma_2} \\ c_2 &= \log w+\frac{1}{2}\log(\gamma-\gamma_1)(\gamma-\gamma_2)-\frac{\lambda}{2(\gamma_1-\gamma_2)}\log\frac{\gamma-\gamma_1}{\gamma-\gamma_2}. \end{aligned} \tag{3.11}$$

For $T=0$ we have

$$\begin{aligned} \frac{\gamma-\gamma_1}{\gamma-\gamma_2} &= e^{-2(\gamma_1-\gamma_2)c_1}, \\ \gamma &= \frac{\gamma_1-\gamma_2 e^{-2(\gamma_1-\gamma_2)c_1}}{1-e^{-2(\gamma_1-\gamma_2)c_1}}, \end{aligned} \tag{3.12}$$

and therefore

$$\begin{aligned} \log w+\tfrac{1}{2}\log\frac{e^{-2(\gamma_1-\gamma_2)c_1}(\gamma_1-\gamma_2)^2}{(1-e^{-2(\gamma_1-\gamma_2)c_1})^2}+\lambda c_1 &= c_2, \\ w=\frac{(1-e^{-2(\gamma_1-\gamma_2)c_1})}{e^{-c_1(\gamma_1-\gamma_2)}(\gamma_1-\gamma_2)}\,e^{c_2-\lambda c_1}. \end{aligned} \tag{3.13}$$

From (3.9) and (3.13) we get

$$\frac{1-e^{-2c_1(\gamma_1-\gamma_2)}}{e^{-c_1(\gamma_1-\gamma_2)}(\gamma_1-\gamma_2)}e^{c_2-\lambda c_1}-\frac{1}{\dfrac{\gamma_1-\gamma_2 e^{-2(\gamma_1-\gamma_1)c_1}}{1-e^{-2(\gamma_1-\gamma_2)c_1}}-\dfrac{i\alpha_1}{2}}=0.$$

Substituting (3.11), we obtain

$$\frac{1-\dfrac{\gamma-\gamma_2}{\gamma-\gamma_2}e^{-2T(\gamma_1-\gamma_2)}}{e^{-T(\gamma_1-\gamma_2)}\sqrt{\dfrac{\gamma-\gamma_1}{\gamma-\gamma_2}}(\gamma_1-\gamma_1)}\,e^{\log w+\frac{1}{2}\log(\gamma-\gamma_1)(\gamma-\gamma_2)-\frac{\lambda}{2(\gamma_1-\gamma_2)}\log\frac{\gamma-\gamma_1}{\gamma-\gamma_2}-\lambda T}$$

$$\cdot e^{\frac{\lambda}{2(\gamma_1-\gamma_2)}\log\frac{\gamma-\gamma_1}{\gamma-\gamma_2}}=\frac{1}{\dfrac{\gamma_1-\gamma_2\dfrac{\gamma-\gamma_1}{\gamma-\gamma_2}e^{-2T(\gamma_1-\gamma_2)}}{1-\dfrac{\gamma-\gamma_1}{\gamma-\gamma_2}e^{-2T(\gamma_1-\gamma_2)}}-\dfrac{i\alpha_1}{2}},$$

whence

$$w=\frac{(\gamma_1-\gamma_2)e^{\lambda T-T(\gamma_1-\gamma_2)}}{(\gamma-\gamma_2)\left(\gamma_1-\dfrac{i\alpha_1}{2}\right)+(\gamma-\gamma_1)\left(\dfrac{i\alpha_1}{2}-\gamma_2\right)e^{-2T(\gamma_1-\gamma_2)}} \tag{3.14}$$

By means of the substitution $\gamma=\lambda-i\alpha_1/2$ and (3.10) the (unconditioned) characteristic function of the random variables $\chi_1(T)$ and $\chi_2(T)$ can be obtained from (3.14):

$$\begin{aligned} u_{\chi_1,\chi_2}(T) &= M(e^{i\alpha_1\chi_1(T)+i\alpha_2\chi_2(T)}) = w\left(T,\lambda-\frac{i\alpha_1}{2}\right) \\ &= \frac{4\lambda(\lambda^2-2i\alpha_2)^{1/2}e^{\lambda T-T\sqrt{\lambda^2-2i\alpha_2}}}{(\lambda-i\alpha_1+\sqrt{\lambda^2-2i\alpha_2})^2-(\lambda-i\alpha_1-\sqrt{\lambda^2-2i\alpha_2})^2e^{-2T\sqrt{\lambda^2-2i\alpha_2}}}. \end{aligned} \tag{3.15}$$

Let $\kappa=\lambda T$; the characteristic function of the random variables $\lambda\chi_1$ and $\lambda^2\chi_2$ will be

$$\frac{4(1-2i\alpha_2)^{1/2}e^{\kappa}}{(1-i\alpha_1+\sqrt{1-2i\alpha_2})^2e^{\kappa\sqrt{1-2i\alpha_2}}-(1-\alpha_1\sqrt{1-2i\alpha_2})^2e^{-\kappa\sqrt{1-2i\alpha_2}}}. \tag{3.16}$$

As follows from (2.3), the maximum likelihood equation for the unknown parameter λ will take the following form:

$$\frac{1}{\lambda}-(\chi_1-T)-\lambda\chi_2=0.$$

This has the unique positive solution

$$\hat\lambda=\frac{-(\chi_1-T)+\sqrt{(\chi_1-T)^2+4\chi_2}}{2\chi_2}. \tag{3.17}$$

In order to determine the distribution of the estimator $\hat\lambda$ let us consider the relation

$$P_\lambda\{\hat\lambda<x\}=P_\lambda\left\{\chi_2-\frac{1}{x^2}-\frac{1}{x}(T-\chi_1)>0\right\}=P_\lambda\{x^2\chi_2+x\chi_1>Tx+1\} \tag{3.18}$$

from which, substituting $x=\lambda y$, $\zeta_1=\lambda y\chi_1+\lambda^2y^2\chi_2$ and $\lambda T=\kappa$, we obtain

$$P_\lambda\{\hat\lambda<\lambda y\}=P_\kappa\{\zeta_y>\kappa y+1\}, \tag{3.19}$$

where the characteristic function of ζ_y becomes

$$(3.20)\quad \frac{4(1-2iy^2\alpha)^{1/2}e^{\varkappa}}{(1-iy\alpha+\sqrt{1-2iy^2\alpha})^2 e^{\varkappa\sqrt{1-2iy^2\alpha}}-(1-iy\alpha-\sqrt{1-2iy^2\alpha})^2 e^{-\varkappa\sqrt{1-2iy^2\alpha}}},$$

as follows from (3.16). If κ and y are given, the relevant probabilities can be calculated from (3.20).

Let $p = 1 - 2iy^2\alpha$; then from (3.20) the Laplace transform of the distribution function of the random variable ζ_y can be obtained:

$$\frac{8y^4e^{\varkappa}}{(y-1)^2}\frac{\sqrt{p}\,e^{-\varkappa\sqrt{p}}}{(p-1)}\left[\frac{1}{(\sqrt{p}+1)^2}+\frac{1}{(\sqrt{p}+2y-1)^2}-\frac{2}{(\sqrt{p}+1)(\sqrt{p}+2y-1)}\right].$$

$$\cdot\sum_{k=0}^{\infty}\left(\frac{1-\dfrac{p-1}{2y}-\sqrt{p}}{1-\dfrac{p-1}{2y}+\sqrt{p}}\right)^{2k}\cdot e^{-2k\varkappa\sqrt{p}},$$

or, substituting $s^2 = p$ and $a = 2y - 1$,

$$(3.21)\quad \frac{8y^4e^{\varkappa}}{(y-1)^2}\sum_{k=0}^{\infty}e^{-\varkappa(2k+1)s}\frac{s^2}{(s-1)(s+1)(s+a)}\frac{(s-1)^{2k}}{(s+1)^{2k+1}}\frac{(s-a)^{2k}}{(s+a)^{2k+1}}.$$

The first term in the sum (3.21) has the inverse Laplace transform (using e.g. the tables of Ditkin and Kuznecov [6])

$$\begin{aligned}
&\frac{1}{2}\left[1-\Phi\left(\frac{\varkappa y}{\sqrt{2(\varkappa y+1)}}-\frac{\sqrt{\varkappa y+1}}{y\sqrt{2}}\right)\right]\\
&+\left[1-\Phi\left(\frac{ky}{\sqrt{\varkappa y+1}}+\frac{\sqrt{\varkappa y+1}}{y\sqrt{2}}\right)\right]e^{2\varkappa}\left[-\frac{y^2(y^2+4y-2}{2(y-1)^4}\right.\\
&\left.+\frac{(6y-2)(\varkappa y^2+\varkappa y+1)}{2(y-1)^3}-\frac{\varkappa y+1}{(y-1)^2}-\frac{(\varkappa y^2+\varkappa y+1)^2}{y^2(y-1)^2}\right]\\
&+\left[1-\Phi\left(\frac{\varkappa y}{\sqrt{2(\varkappa y+1)}}+\frac{(2y-1)\sqrt{\varkappa y+1}}{y\sqrt{2}}\right)\right]e^{2\varkappa y+\frac{\varkappa y+1}{2y^2}[2y-1)^2-1]}\\
&\times\left[\frac{(2y-1)(4y^2-2y+1)}{2(y-1)^4}-\frac{(2y-1)^2}{y(y-1)^3}(\varkappa y^2+(2y-1)(\varkappa y+1))\right]\\
&+\sqrt{\frac{2}{\pi}}\frac{\sqrt{\varkappa y+1}}{y(y-1)^2}e^{\varkappa-\frac{\varkappa y+1}{2y^2}\cdot\frac{\varkappa^2y^2}{2(\varkappa y+1)}}\left[\frac{7y^2-5y+1}{y(y-1)}y^2+\varkappa y^2+\varkappa y+1\right],
\end{aligned}\tag{3.22}$$

where

$$\Phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du.$$

This approximation of the distribution of ζ_y is satisfactory with large values of κ. A better approximation can be obtained by recalling that the inverse Laplace transform of

$$\frac{(p-a)^{2k}}{(p+a)^{2k+1}}$$

equals

$$e^{-ax}\left[\frac{e^{2ax}}{k!}\frac{d^k(2ax)^k e^{-2ax}}{dx^k}\right];$$

consequently the corresponding Laguerre polynomials will appear. Using the theorem on the inverse Laplace transform, for small values of κ the density function of ζ_y becomes

$$f_\zeta(x)=\sum_k e^{s_k\varkappa x}b(s_k)$$

where s_k stand for the poles of (3.20) and $b(s_k)$ stand for the corresponding residues. The equation which determines the poles is

$$\frac{(1+ys+\sqrt{1+2y^2s})^2}{(1+ys-\sqrt{1+2y^2s})^2}=e^{-2\varkappa\sqrt{1+2y^2s}}.$$

The above relations enable us to construct confidence intervals for λ, but computers had to be used for numerical calculations. In the example discussed in the next section, where $T = 60$ years and $\hat{\kappa} = 3.6$, the following upper and lower limits are obtained (the subscripts denote the significance levels)

$$(3.23)\quad \begin{aligned}&\varkappa_{0.90}=5.5,\quad \varkappa_{0.95}=6.2,\quad \varkappa_{0.975}=7.8,\\ &\varkappa_{0.10}=1.27,\quad \varkappa_{0.05}=0.82,\quad \varkappa_{0.025}=0.46.\end{aligned}$$

It can be seen from (3.19) and (3.20) that for small values of κ

$$(3.24)\quad P_\varkappa\{\hat{\varkappa}<y\varkappa\}=\exp\left\{-\frac{1}{y}\right\}.$$

That means that the ratio $\hat{\kappa}/\kappa$ has a χ^2 distribution with two degrees of freedom, whereas for large values of κ

(3.25) $$P\{\varkappa < \hat{\varkappa} + y\sqrt{\hat{\varkappa}}\} \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\, dt,$$

i.e. $\hat{\kappa}$ is of normal distribution with variance Var $(\hat{\kappa}) \sim \hat{\kappa}$. Because $\hat{\kappa}$ has a continuous distribution, there exists for any $0 < \alpha < 1$ and $0 < \kappa < \infty$ a k such that $P_\kappa\{\hat{\kappa} > k\} = \alpha$. From $k = k_\alpha(\kappa)$ we get $\kappa = \kappa_\alpha(k)$, so evidently

$$P_\varkappa\{\varkappa < \varkappa_\alpha(\hat{\varkappa})\} \equiv \alpha;$$

and $\kappa_\alpha(\hat{\kappa})$ becomes the limit of the confidence interval for κ. The calculations indicate that $k_\alpha(\kappa)$ monotonically increases, and so its inverse function does exist, which means that the corresponding confidence limits can be constructed.

§4. A geophysical problem

The instantaneous axis of the earth's rotation constantly changes its position with respect to the minor axis of the ellipsoidal earth (called "free nutation"). This motion has a one-year period; if it is removed, there remains the so-called Chandler wobble, which has a period of about 14 months. This oscillation is not an exact periodic motion; moreover, its amplitude varies over a ten-to-twenty-year period. Figure 1, which represents the empirical correlation function of the so-called Chandler components, shows that the conditions discussed in §1 are well satisfied (the empirical correlation function forms a regular spiral, which has to be compared with (1.2)).

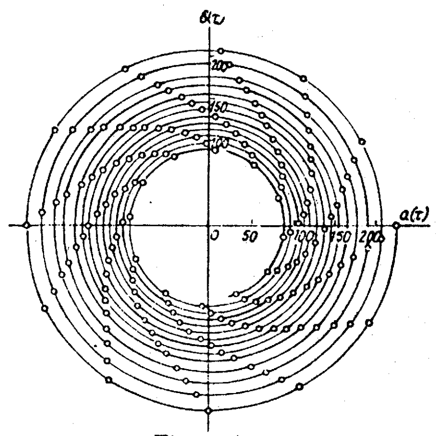

Figure 1,

$\tau = 0, 1 \cdot n; n = 0, 1, 2, \ldots, 156$

Figure 1 has been derived from Table 6 in [11] at the Department of Probability of Moscow State University. Removing the components of period one year from the $x(t)$ and $y(t)$ coordinates in Table 6, a stationary Gaussian-Markov process with coordinates $\xi(t)$ and $\eta(t)$ is obtained. In Figure 1 values belonging to increments τ of 0.1 year magnitude, corresponding to 10 data each year, are indicated; these measurements were carried out over a period of more than 70 years, not interrupted even during World War II. From Figure 1 it can be immediately seen that the $2\pi/\omega$ period is approximately 14 months long. From Theorem 2.1 it results that ω can be

exactly estimated; therefore our estimate is reliable. From Figure 1 it might be expected that the parameter λ can also be estimated; but, as we saw in §3, this is not true.

The intersection of the earth's axis of rotation with the earth's surface is defined as the pole of the earth's rotation; let its coordinates be (x, y, z), measured in the direction of the axes of the earth's ellipsoid of inertia. Assuming that the earth is a solid body, we might obtain from the Euler equations

$$x = x_0 \cos(a^2 r_0 t + \tau),$$
$$y = y_0 \sin(a^2 r_0 t + \tau),$$
$$z = z_0$$

the following motion: the pole of rotation moves along a circle with center at the pole of inertia (i.e. the intersection of the earth's surface and its axis) $I(0, 0, z_0)$. Taking the sidereal day as the time unit, we have $r_0 = 1$ and $T = 2\pi/a \sim 304$ days.

So far we have assumed that the pole of inertia I does not change its position with time; however, various shifts of mass (e.g. airflows) do effect the movement of I. Neglecting the z-components of these motions and denoting the coordinates of I in a plane parallel to the equatorial plane as φ and ψ, these become time functions which depend on periodic components (with one-year and half-year periods) and also on random components. This means that

$$\varphi(t) = \sum_k a_k \cos(\lambda_k t + \alpha_k) + \Delta_1(t),$$
$$\psi(t) = \sum_k b_k \cos(\lambda_k t + \beta_k) + \Delta_2(t),$$

where in the first approximation it can be supposed that $\Delta_1(t)$ and $\Delta_2(t)$ are of white noise type.

In the new system of coordinates the coordinates ξ and η of the pole of rotation satisfy the system

(4.1) $$\frac{d\xi}{dt} = \frac{2\pi}{T}(\eta - \psi),$$
$$\frac{d\eta}{dt} = -\frac{2\pi}{T}(\xi - \varphi);$$

solutions of this system have to involve large oscillations, which in fact were not observed. Therefore also the earth's elasticity has to be taken into account; although this causes extension of the period T, large oscillations become impossible. If the elasticity of the earth is taken into account, system (4.1) has to be replaced by

$$\frac{d\xi(t)}{dt} = c\eta + b\xi + \sum_k a_{1k}\cos(\lambda_k t + \theta_{1k}) + \Delta_1(t),$$

$$\frac{d\eta(t)}{dt} = -a\xi + b\eta + \sum_k a_{2k}\cos(\lambda_k t + \theta_{2k}) + \Delta_2(t), \tag{4.2}$$

which has been discussed in the previous sections.

From all results previously discussed in connection with the system (4.2) it follows that the assumption that the processes $\epsilon_1(t)$ and $\epsilon_2(t)$ in equation (1.1) are Wiener processes is, in examination of the motion of the earth's poles, merely a first approximation and rough idealization. Nevertheless, the data given by Orlov [13] show that if

$$\xi' = -\lambda\xi - \omega\eta + f, \quad \eta' = \omega\xi - \lambda\eta + g,$$

then the values of f and g at sufficiently far-apart points are independent, and therefore their replacement by white noise is justifiable. It appears that the error in determination of the white noise intensity is sufficiently small so that it does not interfere with the estimation of λ. However, a theory which would mathematically express this statement, although very useful, has not yet been worked out.

Working with data from a period of 60 years, the following results were obtained:(3)

$$\hat{\omega} = 5.274; \quad \hat{\varkappa} = 3.6, \quad 2\pi/\hat{\omega} = 1.191, \quad \sigma(2\pi/\hat{\omega}) = 0.006.$$

From the asymptotic formula (3.25) we get $\sigma^2(\hat{\kappa}) = 3.6$, and for confidence levels $\alpha < 0.03$ a negative lower estimate for κ could be obtained, which is impossible. Therefore the approximation with normal distribution is far from satisfactory.

BIBLIOGRAPHY

1. Mátyás Arató, *On the statistical examination of continuous state Markov processes.* I, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **14** (1964), 13–34; English transl., Selected Transl. Math. Statist. and Probability, vol. 14, Amer. Math. Soc., Providence, R.I., 1978, pp. 203–225. MR 37 #3645.

2. ———, *On the statistical examination of continuous state Markov processes.* II, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **14** (1964), 137–159; English transl., Selected Transl. Math. Statist. and Probability, vol. 14, Amer. Math. Soc., Providence, R.I., 1978, pp. 227–251. MR 37 #5998.

(3) Estimates of λ and ω can be found in [10], where values are given for λ = 1/15 and 2π/ω = 1.193. Close to these values are the results of Jeffreys [9] obtained in 1942; but papers [12] and [15] give the essentially different values λ = 0.3 and λ = 0.01 respectively. The reasons for these differences are evident from §3 of the present paper (cf. (3.23) with the lower and upper estimate of κ).

3. ———, *Some statistical questions on stationary Gaussian Markov processes,* Dissertation, Moscow State Univ., Moscow, 1962. (Russian)

4. M. Arató, A. N. Kolmogorov and Ja. G. Sinaĭ, *Estimation of the parameters of a complex stationary Gaussian Markov process,* Dokl. Akad. Nauk SSSR **146** (1962), 747–750 = Soviet Math. Dokl. **3** (1962), 1368–1371.

5. M. Arató, L. V. Rykova and Ja. G. Sinaĭ, *Confidence limits for the damping coefficient of a complex stationary Gaussian Markov process,* Teor. Verojatnost. i Primenen. (in press).*

6. V. A. Ditkin and P. I. Kuznecov, *Handbook of operational calculus,* GITTL, Moscow, 1951. (Russian) MR 13, 840.

7. E. B. Dynkin, *Functionals of trajectories of Markov random processes,* Dokl. Akad. Nauk SSSR **104** (1955), 691–694. (Russian) MR 17,501.

8. E. P. Fedorov, Trudy Poltavsk. Gravimetrič. Observatorija **2** (1948), 3. (Russian)

9. Harold Jeffries, *The variation of latitude,* Monthly Notices Roy. Astronom. Soc. **102** (1942), 139–155.

10. W. H. Munk and G. J. F. MacDonald, *The rotation of the earth,* Cambridge Univ. Press, New York, 1960. MR 22 #9277.

11. A. Ja. Orlov. *Service des latitudes,* Izdat. Akad. Nauk SSSR, Moscow, 1958.

12. N. I. Pančenko, *On the question of the decay of free nutation,* Proc. Fourteenth Astronom. Conf. USSR, Izdat. Akad. Nauk SSSR, Moscow, 1960, pp. 232–243. (Russian)

13. V. V. Stepanov, *Course in differential equations,* Fizmatgiz, Moscow, 1958; German transl., VEB Deutscher Verlag, Berlin, 1963. MR 31 #1416.

14. Charlotte T. Striebel, *Densities for stochastic processes,* Ann. Math. Statist. **30** (1959), 559–567. MR 21 #3085.

15. A. M. Walker and Andrew Young, a) *The analysis of the observations of the variation of latitude,* Monthly Notices Roy. Astronom. Soc. **115** (1955), 443–459.

b) *Further results on the analysis of the variation of latitude,* ibid. **117** (1957), 119–141.

Translated by J. GREGOR

Editor's note.* Apparently this paper has never appeared in the form cited here, but from the context its content was published in M. Arató, *Computation of confidence limits for the "damping" parameter of a complex stationary Gaussian Markov process,* Teor. Verojatnost. i Primenen. **13 (1968), 326–333 = Theor. Probability Appl. **13** (1968), 314–320. MR 40 #6711.