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EVALUATION OF THE PARAMETERS OF A COMPLEX STATIONARY GAUSS-MARKOV PROCESS

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1. We shall consider a two-dimensional stationary stochastic process whose components $\xi(t)$ and $\eta(t)$ satisfy the following stochastic differential equations:

$$d\xi = -\lambda\xi\,dt - \omega\eta\,dt + d\varphi,$$
$$d\eta = \omega\xi\,dt - \lambda\eta\,dt + d\psi, \qquad (1)$$

where $\phi(t)$ and $\psi(t)$ are two independent Wiener processes with

$$M\,d\varphi = M\,d\psi = 0, \quad M(d\varphi)^2 = M(d\psi)^2 = a\,dt.$$

Putting

$$\zeta = \xi + i\eta, \quad \chi = \varphi + i\psi, \quad \gamma = \lambda - i\omega,$$

we can write the system (1) in a form of one equation

$$d\zeta = -\gamma\zeta dt + d\chi. \qquad (1\,a)$$

The complex covariance function of our process is of the form

$$C(\tau) = A(\tau) + iB(\tau) = M[\zeta(t)\,\overline{\zeta(t+\tau)}] = \sigma^2 \exp(-\lambda|\tau| - i\omega\tau), \qquad (2)$$

where $\sigma^2 = a/\lambda$.

If the process is observed on the interval $[0, T]$, then it is possible to determine its empirical covariance function

$$c(\tau) = a(\tau) + ib(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} \zeta(t)\,\overline{\zeta(t+\tau)}\,dt. \qquad (3)$$

With probability one the empirical covariance function has its right derivative at zero

$$c'(0) = -a - \frac{1}{T}s_1^2 + \frac{1}{T}s_2^2 - ir,$$

where a is a parameter, introduced above, which characterizes the intensity of the "white noises" $\phi'(t)$ and $\psi'(t)$, and

$$s_1^2 = \frac{1}{2}[|\zeta(0)|^2 + |\zeta(T)|^2], \quad s_2^2 = \frac{1}{T}\int_0^T |\zeta(t)|^2\,dt, \quad r = \frac{1}{T}\int_0^T |\zeta(t)|^2\,d\theta.$$

The integration is the expression for r is performed with respect to the angular variable θ, which is determined from the equation:

$$\zeta(t) = |\zeta(t)|\,e^{i\theta(t)}.$$

In Figure 1 is given the empirical covariance function $c(\tau)$ for Chandler's variations of the coordinates of the earth's pole.*

2. The parameter a is precisely determined from the complete realization of the process. It remains

* The instantaneous axis of the earth's rotation is displaced relative to the small axis of the earth's ellipsoid (the so-called "free nutation"). These displacements also contain a periodic component with an annual period. After these displacements are eliminated there remain the Chandler's displacements, which have the tendency to oscillate with a period of the order of 14 months. These oscillations, however, are not strictly periodic but have large, primarily smooth changes in the amplitude (the waves being of the order of 10–20 years). Figure 1 shows that this Chandler's component of polar displacement satisfies quite well the hypothesis stated at the beginning of our note.

to investigate the problem of estimation of the parameters λ and ω. Denote by P the probability meas-
ure in the space of the sample functions of our process for the interval $[0, T]$. In the same space we
ılso introduce the standard measure

$$V = L \times W,$$

vhere L is the usual Lebesgue measure on the plane $\zeta(0)$, and W is the two-dimensional Wiener
ıeasure on the space of increments $\zeta(t) - \zeta(0)$ with the same characteristics which are assumed for
he stochastic process $\chi(t)$.

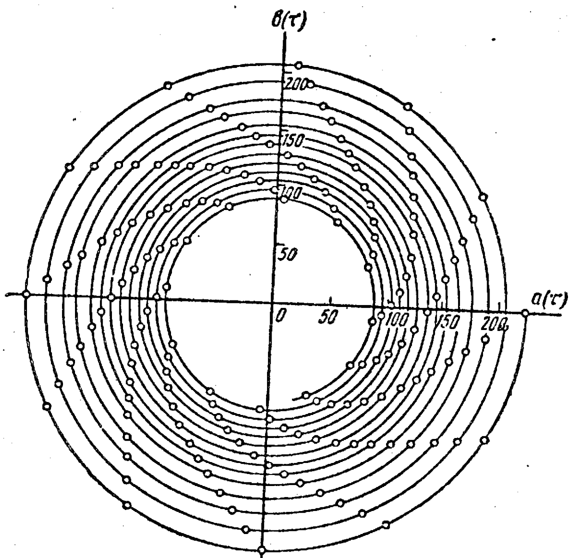

Figure 1. $\tau = 0{,}1 \cdot n;\ n = 0{,}1, \ldots, 156$

an be shown (see [2, 3]), that

$$\frac{dP}{dV} = C\lambda \exp\left[-\frac{\lambda^2+\omega^2}{2a} T s_2^2 - \frac{\lambda}{a} s_1^2 + \lambda T + \frac{\omega}{a} T r\right], \tag{4}$$

re C is a constant. The formula (4) shows that the system of three statistics is a sufficient sys-
of statistics for this problem.

Differentiating the expression

$$L = \log \frac{dP}{dV} = c' + \log\lambda - \frac{\lambda^2+\omega^2}{2a} T s_2^2 - \frac{\lambda}{a} s_1^2 + \lambda T + \frac{\omega}{a} T r$$

respect to ω and λ, we get the equations

$$\frac{\partial L}{\partial \omega} = -\frac{\omega}{a} T s_2^2 + \frac{T}{a} r = 0; \tag{5}$$

$$\frac{\partial L}{\partial \lambda} = \frac{1}{\lambda} - \frac{\lambda}{a} T s_2^2 - \frac{s_1^2}{a} + T = 0. \tag{6}$$

e equations serve for the determination of the maximum likelihood estimates $\hat{\omega}$ and $\hat{\lambda}$. From equa-
(5) we obtain

$$\hat{\omega} = \frac{r}{s_2^2}. \tag{*}$$

* Figure 1 was obtained as a result of the data processing of Table 6 from A. Ja. Orlov's paper [1]. From
oordinates $x(t)$ and $y(t)$ in this table a component with an annual period is singled out and the residual is
to be $\xi(t)$ and $\eta(t)$. The points corresponding to increments of τ in the 0.1 of a year are denoted on this
by small crosses. It can be immediately seen from the figure that the period $2\pi/\omega$ is approximately equal
months. A regular character of the obtained spiral may suggest a supposition that the parameter λ also can
timated in a very precise way. This, however, is not the case, as will be explained at the end of this note.
e detailed exposition of the Technique of the calculations and a discussion of the results obtained will be
hed elsewhere.

It can be shown that

$$\frac{\hat{\omega}-\omega}{\sigma(\hat{\omega})}, \quad \sigma^2(\hat{\omega}) = \frac{a}{Ts_2^2}$$

is normally (0.1) distributed (this is an exact result and not an asymptotic one!). The equation (6) always has a unique positive solution. Denoting $\lambda T = \kappa$, $\hat{\lambda} T = \hat{\kappa}$, we obtain the following equation for $\hat{\kappa}$:

$$h_2 \hat{\kappa}^2 + (h_1 - 1)\hat{\kappa} - 1 = 0,$$

where $h_1 = s_1^2/aT$, $h_2 = s_2^2/aT$.

The distribution of statistics h_1 and h_2 and, hence, also the distribution of $\hat{\kappa}$ depends only on the parameter κ. Since $\hat{\kappa}$ has a continuous distribution, it is possible to find a k such that for any α, $0 < \alpha < 1$ and any κ, $0 < \kappa < \infty$,

$$\mathrm{P}(\hat{\kappa} > k|\kappa) = \alpha. \tag{7}$$

Inverting the function

$$k = k_\alpha(\kappa),$$

we get the function

$$\kappa = \kappa_\alpha(k).$$

(It has been established that the function $k_\alpha(\kappa)$ varies monotonically from 0 to ∞ as κ varies from 0 to ∞; thus, the inversion is possible and is unique.)

Clearly,

$$\mathrm{P}[\kappa < \kappa_\alpha(\hat{\kappa})|\,\kappa] \equiv \alpha. \tag{8}$$

We have organized the calculations of the function $\kappa_\alpha(\hat{\kappa})$ for $\alpha = 0.1$; 0.05; 0.025; 0.01; 0.005; 0.001; 0.9; 0.95; 0.975; 0.99; 0.995; 0.999. The results will be published after these calculations have been completed.

For small values of $\hat{\kappa}$ the equation (8) is equivalent to:

$$\mathrm{P}(\hat{\varkappa} < u\varkappa) = \exp\left(-\frac{1}{u}\right), \tag{9}$$

i.e., the ratio $\hat{\kappa}/\kappa$ has χ^2 distribution with two degrees of freedom.

For large $\hat{\kappa}$ the equation (8) is equivalent to:

$$\mathrm{P}\left(\varkappa < \hat{\varkappa} + u\sqrt{\hat{\varkappa}}\right) \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2}\, dt, \tag{10}$$

i.e., the estimate of $\hat{\kappa}$ is asymptotically normal with variance

$$\sigma^2(\hat{\kappa}) \sim \hat{\kappa}. \tag{11}$$

3. In the case of the displacement of the earth's pole, mentioned at the beginning of this note, the following values were obtained on the basis of observations for $T = 60$ years*

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* The introduction of the Wiener functions ϕ and ψ in the equation (1), i.e., of the perturbations of "white noise" type is, of course, a crude idealization for the case of the earth's pole displacement. It would have been more correct to write:

$$\xi' = -\lambda\xi - \omega\eta + f, \quad \eta' = \omega\xi - \lambda\eta + g.$$

However, the data from [1] shows that the values of functions $f(t)$ and $g(t)$ at the periods of time t, which are several years apart, are practically independent. Thus, the substitution of functions f and g by the "equivalent white noise" is legitimate. The error introduced in the determination of the intensity a of this equivalent white noise is, apparently, sufficiently small that it does not significantly influence the results of the estimation of parameter λ. The value $\hat{\omega}$ is calculated from a discrete analogy of the formula (*), which was obtained using the maximum likelihood method for the "system with discrete time."

See also [6] concerning the estimation of parameters λ and ω for the case of the displacement of the earth's pole. The results obtained in [6] are close to our results: $\lambda = 1/15$, $2\pi/\omega = 1.193$. Similar values also have been given by Jeffreys [7]. However, in papers [5, 8] sharply differing values are indicated: $\lambda = 0.3$ and $\lambda = 0.01$. The reason for these deviations will be explained elsewhere.

$$\hat{\omega} = 5.274, \quad \hat{\kappa} = 3.6, \quad 2\pi : \hat{\omega} = 1.191, \quad \sigma(2\pi : \hat{\omega}) = 0.006.$$

The asymptotical formula (11) gives

$$\sigma^2(\hat{\kappa}) = 3.6.$$

Since κ is a fortiori positive, while the formula (10) gives for $\alpha < 0.03$ a negative estimate k_α, it is evident that the asymptotics of the formula (10) is not yet suitable.

The calculations performed give the following estimates:

$$\varkappa_{0.90} = 5.5, \quad \varkappa_{0.95} = 6.2, \quad \varkappa_{0.975} = 7.8,$$
$$\varkappa_{0.10} = 1.27, \quad \varkappa_{0.05} = 0.82, \quad \varkappa_{0.025} = 0.46,$$

These values correspond to the following estimates of λ:

$$\lambda_{0.90} = 0.09, \quad \lambda_{0.95} = 0.10, \quad \lambda_{0.975} = 0.13,$$
$$\lambda_{0.10} = 0.02, \quad \lambda_{0.05} = 0.01, \quad \lambda_{0.025} = 0.008.$$

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