ON THE EXOTIC ISOMETRY FLOW OF THE QUADRATIC WASSERSTEIN SPACE OVER THE REAL LINE

GYÖRGY PÁL GEHÉR, TAMÁS TITKOS, AND DÁNIEL VIROSZTEK

Abstract. Kloeckner discovered that the quadratic Wasserstein space over the real line (denoted by $\mathcal{W}_2(\mathbb{R})$) is quite peculiar, as its isometry group contains an exotic isometry flow. His result implies that it can happen that an isometry Φ fixes all Dirac measures, but still, Φ is not the identity of $\mathcal{W}_2(\mathbb{R})$. This is the only known example of this surprising and counterintuitive phenomenon. Kloeckner also proved that the image of each finitely supported measure under these isometries (and thus under all isometry) is a finitely supported measure. Recently we showed that the exotic isometry flow can be represented as a unitary group on $L^2\big((0,1)\big)$. In this paper, We calculate the generator of this group, and we show that every exotic isometry (and thus every isometry) maps the set of all absolutely continuous measures belonging to $\mathcal{W}_2(\mathbb{R})$ onto itself.

1. Introduction

The aim of this paper is to take a closer look at Kloeckner's result on isometries of the quadratic Wasserstein space over the real line, denoted by $\mathcal{W}_2(\mathbb{R})$. In [25, Theorem 1.1] Kloeckner showed that $\mathcal{W}_2(\mathbb{R})$ admits a flow $(\Phi^q)_{q\in\mathbb{R}}$ of exotic isometries. But even these wildly behaving isometries preserve finitely supported measures, in particular, they map the set of all Dirac masses onto itself. These results were originally defined by means of an extension argument using geometric tools. In [16] we gave an operator theoretic description of $(\Phi^q)_{q\in\mathbb{R}}$ and we showed that it can be represented as an operator semigroup (in fact a unitary group) $(U_q)_{q\in\mathbb{R}}$ on $L^2\big((0,1)\big)$. In **Theorem A** we extend this result by calculating the skew-selfadjoint generator of the group. As an application, in **Theorem B** we complement Kloeckner's result on preserver properties: we will show that every element Φ^q of the flow (and thus all isometry of $\mathcal{W}_2(\mathbb{R})$) maps the set of all absolutely continuous measures belonging to $\mathcal{W}_2(\mathbb{R})$ onto itself.

1991 *Mathematics Subject Classification.* Primary: 54E40; 46E27 Secondary: 60A10; 60B05.

Key words and phrases. Wasserstein space, isometric embeddings, isometric rigidity, exotic isometry flow.

Gy. P. Gehér was supported by the Leverhulme Trust Early Career Fellowship (ECF-2018-125), and also by the Hungarian National Research, Development and Innovation Office (Grant no. K134944).

T. Titkos was supported by the Hungarian National Research, Development and Innovation Office - NKFIH (grant no. PD128374 and grant no. K134944) and by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

D. Virosztek was supported by the Momentum program of the Hungarian Academy of Sciences under grant agreement no. LP2021-15/2021, and partially supported by the ERC Synergy Grant No. 810115.

2. Definitions and statement of the results

Let (X, ϱ) be a complete and separable metric space, and denote the set of Borel probability measures on X and $X \times X$ by $\mathcal{P}(X)$ and $\mathcal{P}(X \times X)$, respectively. For $\mu, \nu \in \mathcal{P}(X)$ we say that $\pi \in \mathcal{P}(X \times X)$ is a coupling ($\pi \in C(\mu, \nu)$, in notation) if

$$\pi(A \times X) = \mu(A) \qquad \text{and} \qquad \pi(X \times A) = \nu(A)$$

for all Borel sets A. For a given real number $p \geq 1$ the p-Wasserstein space $\mathcal{W}_p(X)$ is the set

$$\left\{ \mu \in \mathcal{P}(X) \,\middle|\, \exists \hat{x} \in X : \int_X \varrho^p(x, \hat{x}) \,\mathrm{d}\mu(x) < +\infty \right\}$$

endowed with the metric

$$d_p(\mu, \nu) = \inf_{\pi \in C(\mu,\nu)} \left\{ \iint_{X \times X} \varrho^p(x, y) \,\mathrm{d}\pi(x, y) \right\}^{1/p}.$$

We will denote the set of Dirac measures by Δ_1, and in general, the symbol Δ_n stands for the set of those elements of $\mathcal{W}_p(X)$ that can be written as $\sum_{i=1}^n \lambda_i \delta_{x_i}$. For more details on $\mathcal{W}_p(X)$ we refer the reader to the textbooks [1,9,33,36,37]. In this paper we will focus on the case when $X \in \{\mathbb{R}, [0,1]\}$ and $p \in \{1, 2\}$.

Given a metric space (Y, r) a map $\Phi : Y \to Y$ is called an isometry ($\Phi \in \mathrm{Isom}(Y)$ in notations) if it is surjective and $r(\Phi(y_1), \Phi(y_2)) = r(y_1, y_2)$ for all $y_1, y_2 \in Y$. Isometries of p-Wasserstein spaces and various other metric spaces of measures have been intensively studied in recent years, see e.g. [2, 6, 7, 13–16, 18, 19, 24–28, 34, 38], not to mention isometries of other important metric structures, see e.g. [3–5, 10–12, 20–22, 29–32]. The most important contribution concerning the current topic has been done by Kloeckner in [25]. He managed to describe $\mathrm{Isom}(\mathcal{W}_2(\mathbb{R}))$. In order to explain his results, we need to recall some technical details.

A special feature of Wasserstein spaces over the unit interval and over the real line is that that the p-Wasserstein distance of measures can be calculated by means of their cumulative distribution functions and quantile functions. Recall that the *cumulative distribution function* of a Borel probability measure μ is defined as

$$F_\mu \colon \mathbb{R} \to [0,1], \quad x \mapsto F_\mu(x) := \mu\left((-\infty, x]\right).$$

We will use the notation $F_\mu(x-) := \lim_{t \uparrow x} F_\mu(t)$ for the limit from the left. The *quantile function* of μ (or the right-continuous generalized inverse of F_μ) is

$$F_\mu^{-1} \colon (0,1) \to \mathbb{R}, \quad y \mapsto F_\mu^{-1}(y) := \sup\left\{x \in \mathbb{R} \,|\, F_\mu(x) \leq y\right\}.$$

The distance $d_2(\mu, \nu)$ in $\mathcal{W}_2(\mathbb{R})$ can be calculated for all $\mu, \nu \in \mathcal{W}_2(\mathbb{R})$ as

$$d_2(\mu, \nu) = \left(\int_0^1 \left| F_\mu^{-1}(t) - F_\nu^{-1}(t) \right|^2 \mathrm{d}t \right)^{\frac{1}{2}} = \|F_\mu^{-1} - F_\nu^{-1}\|_{L^2\left((0,1)\right)}$$

(see for instance [37, Remarks 2.19]), and according to Vallender [35], the distance $d_1(\mu,\nu)$ in $\mathcal{W}_1([0,1])$ can be calculated for all $\mu,\nu \in \mathcal{W}_1([0,1])$ as

$$d_1(\mu,\nu) = \int_0^1 \left|F_\mu^{-1}(t) - F_\nu^{-1}(t)\right| \mathrm{d}t = \int_0^1 \left|F_\mu(t) - F_\nu(t)\right| \mathrm{d}t.$$

A very important property of p-Wasserstein spaces ($p \geq 1$) is that X embeds into $\mathcal{W}_p(X)$ isometrically by the map $x \mapsto \delta_x$, and that $\mathcal{F}(X) = \bigcup_{n\in\mathbb{N}} \Delta_n(X)$ is a dense subset of $\mathcal{W}_p(X)$ (see e.g. Example 6.3 and Theorem 6.16 in [36]). We have a similarly natural embedding for isometry groups, namely $\mathrm{Isom}(X)$ embeds into $\mathrm{Isom}(\mathcal{W}_p(X))$ by a group homomorphism. Indeed, if $\psi : X \to X$ is an isometry of X, then its push-forward $\psi_\#$ is an isometry of $\mathcal{W}_p(X)$, where

$$(\psi_\#\mu)(A) = \mu(\psi^{-1}(A))$$

for all Borel sets $A \subseteq X$. Finally, let us introduce some notations. Let $m(\mu)$ denote the center of mass of a $\mu \in \mathcal{W}_2(\mathbb{R})$:

$$m(\mu) = \int_0^1 F_\mu^{-1}(x)\ \mathrm{d}x.$$

The symbols $\mathbf{1}$, $\mathbf{x}$, and $\mathbf{1}\otimes\mathbf{1}$ will stand for $\mathbf{1}(t) = 1$ ($t \in (0,1)$), $\mathbf{x}(t) = t$ ($t \in (0,1)$), and $(\mathbf{1}\otimes\mathbf{1})f = \int_0^1 f(s)\mathrm{d}s \cdot \mathbf{1}$ ($f \in L^2\big((0,1)\big)$. Finally, let us denote by r_c the map $x \mapsto 2c - x$, which is called the reflection through $c \in \mathbb{R}$. Kloeckner showed in [25, Theorem 1.1] that every isometry of $\mathcal{W}_2(\mathbb{R})$ is a composition of some of the following maps:

- a trivial isometry, that is, $\psi_\#$ for some $\psi \in \mathrm{Isom}(\mathbb{R})$;
- the map $\mu \mapsto \left(r_{m(\mu)}\right)_\#(\mu)$, that is, the isometry that reflects every measure through its center of mass;
- an exotic isometry Φ^q for some $q \in \mathbb{R}$.

In order to define Φ^q, Kloeckner parametrized $\Delta_2(\mathbb{R})$ by $x, p \in \mathbb{R}$, $\sigma \geq 0$ as follows

$$\mu(x,\sigma,p) := \frac{e^{-p}}{e^p + e^{-p}} \cdot \delta_{x-\sigma e^p} + \frac{e^p}{e^p + e^{-p}} \cdot \delta_{x+\sigma e^{-p}}. \tag{2.1}$$

Now let $q \in \mathbb{R}$ be fixed, and define Φ^q on $\Delta_2(\mathbb{R})$ by

$$\Phi^q\left(\mu(x,\sigma,p)\right) := \mu(x,\sigma,p+q) \quad (x,\sigma,p \in \mathbb{R}, \sigma \geq 0). \tag{2.2}$$

He proved that this indeed defines an isometry on $\Delta_2(\mathbb{R})$ and that it extends uniquely to an isometry of $\mathcal{W}_2(\mathbb{R})$. He also pointed out that even though the above definition is constructive, it is not very explicit outside $\Delta_2(\mathbb{R})$.

Let us identify $\mathcal{W}_2(\mathbb{R})$ with the corresponding set of quantile functions. In that way, Φ^q can be considered as a map defined on a subset of $L^2\big((0,1)\big)$. We proved in [16] that this map extends to a real unitary operator $U_q\colon L^2\big((0,1)\big) \to L^2\big((0,1)\big)$ which can be written in terms of a composition operator, the Volterra operator, a multiplication operator and a rank-one projection. Our first result complements this description.

Theorem A. *Let q be a real number. Then the action of the exotic isometry Φ^q is given by the following formula:*

$$F^{-1}_{\Phi^q(\mu)}(x) = (1-e^q)\cdot m(\mu) + \left\{e^q + (e^{-q}-e^q)h_q(x)\right\}\cdot F^{-1}_\mu(h_q(x)) + (e^q - e^{-q})\cdot \int_0^{h_q(x)} F^{-1}_\mu(s)\ \mathrm{d}s \qquad (\mu\in\mathcal{W}_2(\mathbb{R}), 0<x<1), \tag{2.3}$$

where

$$h_q(x) = \frac{xe^{2q}}{1+(e^{2q}-1)x} \qquad (x\in(0,1)). \tag{2.4}$$

Moreover, the exotic isometry flow $\{\Phi^q\colon q\in\mathbb{R}\}$ extends into a strongly continuous one-parameter (real) unitary group $\{U_q\colon q\in\mathbb{R}\} = \{\exp(qA)\colon q\in\mathbb{R}\}$ on $L^2\big((0,1)\big)$. The skew-symmetric generator $(A, D(A))$ of this operator semigroup is

$$\left(Af\right)(x) = (1-2x)\cdot f(x) + 2x(1-x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}f(x) + \int_0^x f(s)\mathrm{d}s - \int_x^1 f(s)\mathrm{d}s \tag{2.5}$$

on the domain

$$D(A) = \left\{ f\in L^2\big((0,1)\big) \,\middle|\, f \text{ is absolutely continuous, } \mathbf{x(1-x)}\tfrac{\mathrm{d}f}{\mathrm{d}x}\in L^2\big((0,1)\big) \right\}. \tag{2.6}$$

Before stating Theorem B, we remark that it is not true in general that an isometry of a Wasserstein space preserves Δ_n or preserves absolute continuity of measures. To see such an example, consider the Wasserstein space $\mathcal{W}_1([0,1])$, and the isometry j which is defined as $j(\mu) := \nu$ if $F_\nu = F^{-1}_\mu$ (for more details see [16, Section 2.1]).

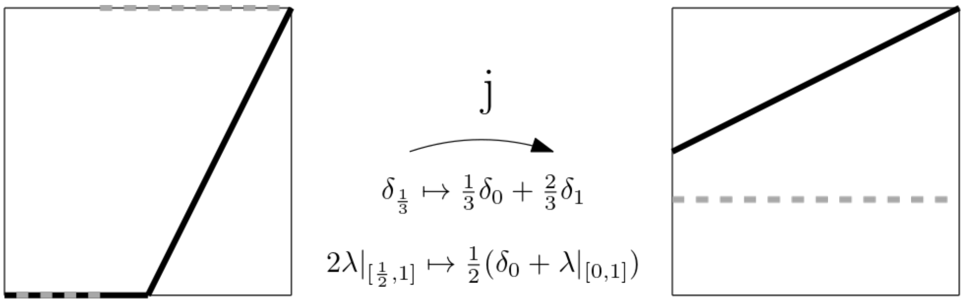

FIGURE 1. The cumulative distribution functions (restricted to $[0,1]$) of $\delta_{\frac13}$ (gray dashed line), $\frac12(\delta_0+\lambda|_{[0,1]})$ (black line), and their images.

For this isometry

- it is not true that the image of Δ_n is Δ_n for all $n\in\mathbb{N}$. In fact, Dirac measures (i.e., elements of Δ_1) are typically mapped into Δ_2, as $j(\delta_t) = t\delta_0 + (1-t)\delta_1$,
- it is not true that the image of each absolutely continuous measure is absolutely continuous. One such example is $\mu = 2\lambda|_{[\frac12,1]}$, where λ is the Lebesgue measure. The image of μ is $\frac12(\delta_0+\lambda|_{[0,1]})$.

Theorem B. *Let us denote by $\mathcal{W}_2^{ac}(\mathbb{R})$ the set of all absolutely continuous measures belonging to $\mathcal{W}_2(\mathbb{R})$. Then for all $q\in\mathbb{R}$, Φ^q maps $\mathcal{W}_2^{ac}(\mathbb{R})$ onto itself.*

3. Proofs

3.1. **Proof of Theorem A.** The proof of (2.3) appeared originally in [16, Theorem 3.18.]. In the following proof, we briefly recall some details (mainly notations) for the reader's convenience.

Proof. We look at the Wasserstein space $\mathcal{W}_2(\mathbb{R})$ as a convex and closed subset of $L^2\big((0,1)\big)$ whose linear span is dense in $L^2\big((0,1)\big)$, via the identification $\mu \mapsto F_\mu^{-1}$. Therefore by [39, Theorem 11.4] and (2.2) the exotic isometry Φ^q can be extended to a unique linear isometric embedding which we denote by U_q. Let us point out that the linear span of $\{F_\mu^{-1} \,|\, \mu \in \Delta_2(\mathbb{R})\}$ is dense in $L^2\big((0,1)\big)$, therefore U_q is the unique bounded operator on $L^2\big((0,1)\big)$ such that

$$U_q\left(F^{-1}_{\mu(x,\sigma,p)}\right) = F^{-1}_{\mu(x,\sigma,p+q)} \qquad (x,\sigma,p \in \mathbb{R}, \sigma \geq 0). \tag{3.1}$$

Therefore, it is enough to find a bounded linear operator that satisfies (3.1), that operator will then be equal to U_q. Observe that (3.1) is equivalent to

$$U_q\mathbf{1} = \mathbf{1} \quad \text{and} \quad U_q\left(F^{-1}_{\mu(0,1,p)}\right) = F^{-1}_{\mu(0,1,p+q)} \qquad (p \in \mathbb{R}). \tag{3.2}$$

Let us introduce some notations: let $M_{\mathbf{1}-\mathbf{x}}$ stand for the multiplication operator by the function $\mathbf{1}-\mathbf{x}$, and V for the Volterra operator: $(Vf)(t) = \int_0^t f(s)\,\mathrm{d}s$ $(t \in (0,1))$. The composition operator with symbol h_q (see (2.4)) will be denoted by C_q

$$C_q \colon L^2\big((0,1)\big) \to L^2\big((0,1)\big), \quad (C_q f)(x) = f(h_q(x)) \quad (x \in (0,1)).$$

Notice that C_q is a bounded operator, as h_q maps $[0,1]$ bijectively onto itself; it is a smooth function on a neighbourhood of $[0,1]$, and its derivative is bounded from below by $e^{-2|q|}$ on $[0,1]$. In [16, Theorem 3.18] we showed that U_q can be written as

$$U_q = C_q \cdot \left[(1-e^q)\cdot(\mathbf{1}\otimes\mathbf{1}) + e^q \cdot I + (e^{-q}-e^q)\cdot M_{\mathbf{x}} + (e^q - e^{-q})\cdot V\right], \tag{3.3}$$

which implies (2.3) for almost every $x \in (0,1)$. Since two right-continuous functions are equal almost everywhere on $(0,1)$ if and only if they coincide on $(0,1)$, we conclude (2.3), and thus (using (3.2) and (3.3)) $(U_q)_{q\in\mathbb{R}}$ is strongly continuous. The next step is to find its generator. That is, the operator $A : D(A) \to L^2\big((0,1)\big)$ such that $\{Uq \,|\, q \in \mathbb{R}\} = \{\exp(qA) \,|\, q \in \mathbb{R}\}$. The domain $D(A)$ is the collection of all $f \in L^2\big((0,1)\big)$ such that the limit $\lim_{q\downarrow 0} \frac{1}{q}(U_q f - f)$ exists. Note that for all $q \in \mathbb{R}, f \in L^2\big((0,1)\big)$ we have

$$\begin{aligned}\frac{1}{q}\left(U_q f - f\right)(x) = &\frac{1-e^q}{q}\int_0^1 f(s)\,\mathrm{d}s + \frac{e^q f(h_q(x)) - f(x)}{q} \\ &+ \frac{e^{-q}-e^q}{q} h_q(x) f(h_q(x)) + \frac{e^q - e^{-q}}{q}\int_0^{h_q(x)} f(s)\,\mathrm{d}s\end{aligned}$$

for (Lebesgue) almost every $x \in (0,1)$. Let us calculate the limit $\lim_{q\downarrow 0}\frac{1}{q}\left(U_q f - f\right)(x)$: using that $\frac{\mathrm{d}}{\mathrm{d}q}\left(e^q f(h_q(x))\right)\Big|_{q=0} = f(x) + f'(x)2x(1-x)$, we have

$$\begin{aligned}\lim_{q\downarrow 0}\frac{1}{q}\left(U_q f - f\right)(x) &= -\int_0^1 f(s)\,\mathrm{d}s + \frac{\mathrm{d}}{\mathrm{d}q}\left(e^q f(h_q(x))\right)\Big|_{q=0} - 2xf(x) + 2\int_0^x f(s)\,\mathrm{d}s \\ &= (1-2x)\cdot f(x) + 2x(1-x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}f(x) + \int_0^x f(s)\mathrm{d}s - \int_x^1 f(s)\mathrm{d}s\end{aligned}$$

for all $f \in L^2\big((0,1)\big)$ such that the map $x \mapsto x(1-x)f'(x)$ belongs to L^2. According to Stone's theorem [8, Theorem 3.24], $(A, D(A))$ is a skew-selfadjoint operator. $\square$

3.2. **Proof of Theorem B.** We close the paper with an application of Theorem A. Using (2.3), one can show that the slopes of secant lines of $F_\mu^{-1} \circ h_q$ and $F_{\Phi^q(\mu)}^{-1}$ are comparable (see (3.4) below), which is the key observation of the proof below.

Proof. As we pointed out earlier, the function h_q from (2.4) is a monotone increasing bijection of $[0,1]$ which is smooth in a neighbourhood of $[0,1]$, and whose derivative satisfies

$$e^{-2|q|} \leq \tfrac{d}{dx} h_q(x) \leq e^{2|q|} \qquad (x \in [0,1]).$$

The inverse will be denoted by $h_q^{-1}\colon [0,1] \to [0,1]$. Note that for all $\mu \in \mathcal{W}_2(\mathbb{R})$, $q \in \mathbb{R}$ and $0 < x_1 < x_2 < 1$ we have

$$\begin{aligned}
&F_{\Phi^q(\mu)}^{-1}\left(h_q^{-1}(x_2)\right) - F_{\Phi^q(\mu)}^{-1}\left(h_q^{-1}(x_1)\right) \\
&= e^q \cdot \left(F_\mu^{-1}(x_2) - F_\mu^{-1}(x_1)\right) + \left(e^{-q} - e^q\right) \cdot \left(x_2 F_\mu^{-1}(x_2) - x_1 F_\mu^{-1}(x_1) - \int_{x_1}^{x_2} F_\mu^{-1}(s)\, \mathrm{d}s\right) \\
&= e^q \cdot \left(F_\mu^{-1}(x_2) - F_\mu^{-1}(x_1)\right) + \left(e^{-q} - e^q\right) \cdot \int_{x_1}^{x_2} s\, \mathrm{d}F_\mu^{-1}(s),
\end{aligned}$$

where we used integration by parts. Since

$$0 \leq \int_{x_1}^{x_2} s\, \mathrm{d}F_\mu^{-1}(s) \leq \int_{x_1}^{x_2} 1\, \mathrm{d}F_\mu^{-1}(s) = F_\mu^{-1}(x_2) - F_\mu^{-1}(x_1),$$

we conclude the following for all $\mu \in \mathcal{W}_2(\mathbb{R})$, $q \in \mathbb{R}$ and $0 < x_1 < x_2 < 1$:

$$\begin{aligned}
&e^{-|q|} \cdot \left(F_\mu^{-1}(x_2) - F_\mu^{-1}(x_1)\right) \\
&\leq F_{\Phi^q(\mu)}^{-1}\left(h_q^{-1}(x_2)\right) - F_{\Phi^q(\mu)}^{-1}\left(h_q^{-1}(x_1)\right) \leq e^{|q|} \cdot \left(F_\mu^{-1}(x_2) - F_\mu^{-1}(x_1)\right).
\end{aligned} \tag{3.4}$$

Since $(\Phi^q)^{-1} = \Phi^{-q}$, what is left to prove is that if $\mu \in \mathcal{W}_2^{ac}(\mathbb{R})$, then $\Phi^q(\mu)$ is also absolutely continuous. We shall do this indirectly. Assume that μ is absolutely continuous but $\Phi^q(\mu)$ is not. Then F_μ^{-1} cannot be constant on any non-degenerate interval. Consequently by (3.4), the same holds for $F_{\Phi^q(\mu)}^{-1}$, thus $F_{\Phi^q(\mu)}$ is continuous on $\mathbb{R}$. By definition, there exists an $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there exists a finite set of pair-wise disjoint intervals $\big\{[a_{n,j}, b_{n,j}) \,\big|\, j = 1, \dots, N_n\big\}$ with $N_n \in \mathbb{N}$ and satisfying

$$\sum_{j=1}^{N_n} b_{n,j} - a_{n,j} < \tfrac{\varepsilon}{n} \quad \text{and} \quad \sum_{j=1}^{N_n} F_{\Phi^q(\mu)}\left(b_{n,j}\right) - F_{\Phi^q(\mu)}\left(a_{n,j}\right) > \varepsilon. \tag{3.5}$$

By the continuity of $F_{\Phi^q(\mu)}$, without loss of generality we may assume from now on that

$$0 < F_{\Phi^q(\mu)}\left(a_{n,j}\right) < F_{\Phi^q(\mu)}(c) < F_{\Phi^q(\mu)}\left(b_{n,j}\right) < 1 \quad (a_{n,j} < c < b_{n,j}) \tag{3.6}$$

always holds. Set

$$x_{n,j} := F_{\Phi^q(\mu)}\left(a_{n,j}\right) \qquad \text{and} \qquad y_{n,j} := F_{\Phi^q(\mu)}\left(b_{n,j}\right) \in (0,1)$$

for all j, n. Then by (3.6) we have $F_{\Phi^q(\mu)}^{-1}(x_{n,j}) = a_{n,j}$ and $F_{\Phi^q(\mu)}^{-1}(y_{n,j}-) = b_{n,j}$. Therefore (3.5) can be written in the form

$$\sum_{j=1}^{N_n} F_{\Phi^q(\mu)}^{-1}(y_{n,j}-) - F_{\Phi^q(\mu)}^{-1}(x_{n,j}) < \tfrac{\varepsilon}{n} \quad \text{and} \quad \sum_{j=1}^{N_n} y_{n,j} - x_{n,j} > \varepsilon. \tag{3.7}$$

By replacing $y_{n,j}$ by $\tilde{y}_{n,j} \in (x_{n,j}, y_{n,j}]$ close enough to $y_{n,j}$, we may assume without loss of generality that we have $F_{\Phi^q(\mu)}^{-1}(y_{n,j})$ everywhere in (3.7) instead of $F_{\Phi^q(\mu)}^{-1}(y_{n,j}-)$. Set $u_{n,j} := h_q(x_{n,j})$ and $v_{n,j} := h_q(y_{n,j})$. Observe that (3.4), (3.7) and the properties of h_q imply

$$\sum_{j=1}^{N_n} F_{\mu}^{-1}(v_{n,j}) - F_{\mu}^{-1}(u_{n,j}) < \tfrac{\varepsilon}{n} \cdot e^{|q|} \quad \text{and} \quad \sum_{j=1}^{N_n} v_{n,j} - u_{n,j} > \varepsilon \cdot e^{-2|q|}. \tag{3.8}$$

Recall that F_{μ}^{-1} is not constant on any non-degenerate interval. Therefore setting $c_{n,j} := F_{\mu}^{-1}(u_{n,j})$ and $d_{n,j} := F_{\mu}^{-1}(v_{n,j})$ gives

$$\sum_{j=1}^{N_n} d_{n,j} - c_{n,j} < \tfrac{\varepsilon}{n} \cdot e^{|q|} \quad \text{and} \quad \sum_{j=1}^{N_n} F_{\mu}(d_{n,j}) - F_{\mu}(c_{n,j}) > \varepsilon \cdot e^{-2|q|},$$

which contradicts μ being absolutely continuous. □

Acknowledgement. We are grateful to the anonymous referee for his/her valuable comments and suggestions.

References

[1] L. Ambrosio, N. Gigli, *A user's guide to optimal transport. Modelling and optimisation of flows on networks*, Lecture Notes in Math., 2062, Fond. CIME/CIME Found. Subser., Springer, Heidelberg, 2013.

[2] J. Bertrand, B. Kloeckner, *A geometric study of Wasserstein spaces: isometric rigidity in negative curvature*, Int. Math. Res. Notices **5** (2016), 1368–1386.

[3] F. Botelho, J. Jamison, L. Molnár, *Surjective isometries on Grassmann spaces*, Journal of Functional Analysis, **265** 2013, 2226–2238.

[4] F. Botelho, D. Ilisevic, *On isometries with finite spectrum* Journal of Operator Theory **86** 2021, 255–273.

[5] F. Botelho, J. Jamison, *Surjective isometries on absolutely continuous vector-valued function spaces*, Contemp. Math., **687** (2017), 55–65.

[6] G. Dolinar, B. Kuzma, D. Mitrovic, *Isometries of probability measures with respect to the total variation distance*, J. Math. Anal. Appl. **507** (2021), Paper No. 125829.

[7] G. Dolinar, L. Molnár, *Isometries of the space of distribution functions with respect to the Kolmogorov–Smirnov metric*, J. Math. Anal. Appl. **348** (2008), 494–498.

[8] K-J. Engel, R. Nagel, *A short course on operator semigroups*, Universitext. Springer, New York, 2006.

[9] A. Figalli and F. Glaudo, *An Invitation to Optimal Transport, Wasserstein Distances, and Gradient Flows*, EMS Textbooks in Mathematics, EMS Press, Berlin, 2021.

[10] R.J. Fleming,J.E. Jamison, *Isometries on Banach Spaces: Function Spaces*, Chapman and Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 129, Boca Raton, FL, 2003.

[11] R.J. Fleming, J.E. Jamison, *Isometries on Banach Spaces: Function Spaces Vol. 2.: Vector-valued Function Spaces*, Chapman and Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 138, Boca Raton, FL, 2008.

[12] K. Jarosz, V.D. Pathak, *Isometries between function spaces*, Trans. Amer. Math. Soc., **305** (1988), 193–206.

[13] Gy.P. Gehér, *Surjective Kuiper isometries*, Houston J. Math. **44** (2018), 263–281.

[14] Gy.P. Gehér, T. Titkos, *A characterisation of isometries with respect to the Lévy-Prokhorov metric*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, Vol. **XIX** (2019), 655–677.

[15] Gy.P. Gehér, T. Titkos, D. Virosztek, *On isometric embeddings of Wasserstein spaces – the discrete case*, J. Math. Anal. Appl. **480** (2019), 123435.

[16] Gy.P. Gehér, T. Titkos, D. Virosztek, *Isometric study of Wasserstein spaces – the real line*, Trans. Amer. Math. Soc. **373** (2020), 5855-–5883.

[17] Gy.P. Gehér, T. Titkos, D. Virosztek, *Dirac masses and isometric rigidity*, RIMS Kokyuroku, **2125** (2020), 34–41.

[18] Gy. P. Gehér, T. Titkos, D. Virosztek, *The isometry group of Wasserstein spaces: the Hilbertian case*, J. London Math. Soc., **106** (2022), 3836–3894.

[19] Gy. P. Gehér, T. Titkos, D. Virosztek, *Isometric rigidity of Wasserstein tori and spheres*, Mathematika, **69** (2023), 20–32.

[20] O. Hatori, T. Miura, H. Takagi, *Characterizations of Isometric Isomorphisms between Uniform Algebras via Nonlinear Range-Preserving Properties*, Proc. Amer. Math. Soc. **134** (2006), 2923–2930.

[21] O. Hatori, L. Molnár, *Generalized isometries of the special unitary group*, Arch. Math. **106** (2016), 155–163.

[22] O. Hatori, L. Molnár, *Isometries of the unitary group*, Proc. Amer. Math. Soc. **140** (2012), 2141–2154.

[23] R.V. Kadison, *Isometries of Operator Algebras*, Annals of Mathematics Second Series **54** (1951), 325–338.

[24] G. Kiss, T. Titkos, *Isometric rigidity of Wasserstein spaces: the graph metric case*, Proc. Amer. Math. Soc. **150** (2022), 4083–4097.

[25] B. Kloeckner, *A geometric study of Wasserstein spaces: Euclidean spaces*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze **IX** (2010), 297–323.

[26] L.Molnár, *Kolmogorov-Smirnov isometries and affine automorphisms of spaces of distribution functions*, Cent. Eur. J. Math. **9** (2011), 789–796.

[27] L. Molnár, P. Szokol, *Kolmogorov-Smirnov isometries of the space of generalized distribution functions*, Math. Slovaca **64** (2014), 433–444.

[28] L. Molnár, *Lévy isometries of the space of probability distribution functions*, J. Math. Anal. Appl. **380** (2011), 847–852.

[29] L. Molnár, *Bures isometries between density spaces of C^*-algebras*, Linear Algebra Appl. **557** (2018), 22–33.

[30] L. Molnár, *Busch-Gudder metric on the cone of positive semidefinite operators and its isometries*, Integral Equations Operator Theory **90** (2018), Article number: 20.

[31] F.J. Fernández-Polo, A.M. Peralta, *On the extension of isometries between the unit spheres of von Neumann algebras*, Journal of Mathematical Analysis and Applications **466** (2018), 127–143.

[32] F.J. Fernández-Polo, A.M. Peralta, *On the extension of isometries between the unit spheres of a C^*-algebra and $B(H)$*, Trans. Amer. Math. Soc. Ser. B **5** (2018), 63–80.

[33] F. Santambrogio, *Optimal Transport for Applied Mathematicians*, Progress in Nonlinear Differential Equations and Their Applications 87, Birkhauser Basel (2015).

[34] J. Santos-Rodríguez, *On isometries of compact L^p–Wasserstein spaces*, Adv. Math. **409** (2022), Article No. 108632.

[35] S.S. Vallender, *Calculation of the Wasserstein distance between probability distributions on the line,* Theory Probab. Appl. **18** (1973), 784–786.

[36] C. Villani, *Optimal Transport: Old and New*, Grundlehren der mathematischen Wissenschaften, Springer, 2009.

[37] C. Villani, *Topics in optimal transportation,* Graduate studies in Mathematics vol. 58, American Mathemtical Society, Providence, RI, 2003.

[38] D. Virosztek, *Maps on probability measures preserving certain distances — a survey and some new results,*, Acta Sci. Math. (Szeged) **84** (2018), 65–80.

[39] J.H. Wells, L.R. Williams, *Embeddings and extensions in analysis*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 84. Springer-Verlag, New York-Heidelberg, 1975.

György Pál Gehér, Department of Mathematics and Statistics, University of Reading, Whiteknights, P.O. Box 220, Reading RG6 6AX, United Kingdom

E-mail address: `gehergyuri@gmail.com`

Tamás Titkos, Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda u. 13-15., Budapest H-1053, Hungary, and BBS University of Applied Sciences, Alkotmány u. 9., Budapest H-1054, Hungary

E-mail address: `titkos.tamas@renyi.hu`

Dániel Virosztek, Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda u. 13-15., Budapest H-1053, Hungary

E-mail address: `virosztek.daniel@renyi.hu`