On the number of tangencies among 1-intersecting curves

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Abstract

Let $\mathcal{C}$ be a set of curves in the plane such that no three curves in $\mathcal{C}$ intersect at a single point and every pair of curves in $\mathcal{C}$ intersect at exactly one point which is either a crossing or a touching point. According to a conjecture of János Pach the number of pairs of curves in $\mathcal{C}$ that touch each other is $O(|\mathcal{C}|)$. We prove this conjecture for x-monotone curves.

1 Introduction

We study the number of tangencies within a family of 1-intersecting x-monotone planar curves. A planar curve is a *Jordan arc*, that is, the image of an injective continuous function from a closed interval into $\mathbb{R}^2$. If no two points on a curve have the same x-coordinate, then the curve is *x-monotone*. We consider families of curves such that every pair of curves intersect at a finite number of points. Such a family is called t-intersecting if every pair of curves intersects at at most t points. An intersection point p of two curves is a *crossing point* if there is a small disk D centered at p which contains no other intersection point of these curves, each curve intersects the boundary of D at exactly two points and in the cyclic order of these four points no two consecutive

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points belong to the same curve. If two curves intersect at exactly one point which is not a crossing point, then we say that they are *touching* or *tangent* at that point.

The number of *tangencies* is the number of tangent pairs of curves. If more than two curves are allowed to intersect at a common point, then every pair of curves might be tangent, e.g., for the graphs of the functions x^{2i}, $i = 1, 2, \ldots, n$, in the interval $[-1, 1]$. Therefore, we restrict our attention to families of curves in which no three curves intersect at a common point. It is not hard to construct such a family of n (x-monotone) 1-intersecting curves with $\Omega(n^{4/3})$ tangencies based on a famous construction of Erdős (see [10]) of n lines and n points admitting that many point-line incidences. János Pach [9] conjectured that requiring every pair of curves to intersect (either at crossing or a tangency point) leads to significantly less tangencies.

Conjecture 1 ([9])**.** *Let $\mathcal{C}$ be a set of n curves such that no three curves in $\mathcal{C}$ intersect at a single point and every pair of curves in $\mathcal{C}$ intersect at exactly one point which is either a crossing or a tangency point. Then the number tangencies among the curves in $\mathcal{C}$ is $O(n)$.*

Györgyi, Hujter and Kisfaludi-Bak [7] proved Conjecture 1 for the special case where there are constantly many faces in the arrangement of $\mathcal{C}$ that together contain all the endpoints of the curves. In this paper we show that Conjecture 1 also holds for x-monotone curves.

Theorem 2. *Let $\mathcal{C}$ be a set of n x-monotone curves such that no three curves in $\mathcal{C}$ intersect at a single point and every pair of curves in $\mathcal{C}$ intersect at exactly one point which is either a crossing or a tangency point. Then the number tangencies among the curves in $\mathcal{C}$ is $O(n)$.*

We prove Theorem 2 by considering two types of tangencies according to whether a tangency point is between two curves such that their projections on the x-axis are nested or overlapping. In each case we consider the tangencies graph whose vertices represent the curves and whose edges represent tangent pairs of curves. In the latter case we show that it is possible to disregard some ratio of the edges using the pigeonhole principle and the dual of Dilworth's Theorem and then order the remaining edges such that there is no long monotone increasing path with respect to this order. In the first case, we show that after disregarding some ratio of the edges the remaining edges induce a forest.

Related Work. It follows from a result of Pach and Sharir [13] that n x-monotone 1-intersecting curves admit $O\left(n^{4/3} (\log n)^{2/3}\right)$ tangencies. Note that this bound almost matches the lower bound mentioned above. It also follows from [13] that for bi-infinite x-monotone 1-intersecting curves the maximum number of tangencies is $\Theta(n \log n)$.

Pálvölgyi et nos [2] showed that there are $O(n)$ tangencies among families of n 1-intersecting curves that can be partitioned into two sets such that all the curves within each set are pairwise disjoint. Variations of this bipartite setting were also studied in [1, 8, 14].

Pach, Rubin and Tardos [11, 12] settled a long-standing conjecture of Richter and Thomassen [15] concerning the number of crossing points determined by pairwise intersecting curves. In particular, they showed that in any set of curves admitting linearly many tangencies the number of crossing points is superlinear with respect to the number of tangencies. This implies that for any fixed t every set of n t-intersecting curves admits $o(n^2)$ tangencies. Salazar [17] already pointed that out for such families which are also pairwise intersecting. Better bounds for families of t-intersecting curves were found in [4, 8]. Specifically, it follows from [8] that n 1-intersecting curves determine $O(n^{7/4})$ tangencies.

There are several other problems in combinatorial geometry that can be phrased in terms of bounding the number of tangencies between certain curves, see, e.g., [3]. The most famous of which is the unit distance problem of Erdős [5] which asks for the maximum number of unit distances among n points in the plane. It is easy to see that this problem is equivalent to asking for the maximum number of tangencies among n unit circles.

2 Proof of Theorem 2

Let $\mathcal{C}$ be a set of n x-monotone curves such that no three curves in $\mathcal{C}$ intersect at a single point and every pair of curves in $\mathcal{C}$ intersect at exactly one point which is either a crossing or a tangency point. By slightly extending the curves if needed, we may assume that every intersection point of two curves is an interior point of both of them and that all the endpoints of the curves are distinct.

Let $p = (x_1, y_1)$ and $q = (x_2, y_2)$ be two points. We write $p <_x q$ if $x_1 < x_2$ and we write $p <_y q$ if $y_1 < y_2$. We mainly consider the order of points from left to right, so when we use terms like 'before', 'after' and 'between' they should be understood in this sense. For a curve $c \in \mathcal{C}$ we denote by $L(c)$ and $R(c)$ the left and right endpoints of c, respectively. If $p, q \in c$, then $c(p, q)$ denotes the part of c between these two points. We denote by $c(-, p)$ (resp., $c(p, +)$) the part of c between $L(c)$ (resp., $R(c)$) and p. For another curve $c' \in \mathcal{C}$ we denote by $I(c, c')$ the intersection point of c and c'. We may also write, e.g., $c(c', q)$ instead of $c(I(c, c'), q)$

Suppose that an x-monotone curve c_1 lies *above* another x-monotone curve c_2, that is, the two curves are non-crossing (but might be touching) and there is no vertical line ℓ such that $I(c_1, \ell) <_y I(c_2, \ell)$. Assuming the endpoints of c_1 and c_2 are distinct there are four possible cases: (1) $L(c_1) <_x L(c_2) <_x R(c_2) <_x R(c_1)$;

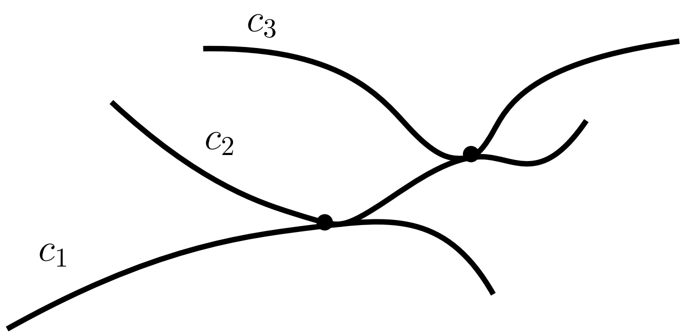


Figure 1: If $c_1 \prec_4 c_2 \prec_4 c_3$ then c_1 and c_3 do not intersect.

(2) $L(c_2) <_x L(c_1) <_x R(c_1) <_x R(c_2)$; (3) $L(c_1) <_x L(c_2) <_x R(c_1) <_x R(c_2)$; and (4) $L(c_2) <_x L(c_1) <_x R(c_2) <_x R(c_1)$. We denote by $c_2 \prec_i c_1$ the relation that corresponds to case i, for $i = 1, 2, 3, 4$. It is not hard to see that each $\prec_i$ is indeed a partial order.

Proposition 3. *For every $i = 1, 2, 3, 4$ there are no three curves $c_1, c_2, c_3 \in \mathcal{C}$ such that $c_1 \prec_i c_2 \prec_i c_3$.*

Proof. It is easy to see that if $c_1 \prec_i c_2 \prec_i c_3$ then c_1 and c_3 do not intersect. See Figure 1 for an illustration of the case $i = 4$. □

We say that the tangency point of two touching curves $c_1, c_2 \in \mathcal{C}$ is of *Type i* if $c_1 \prec_i c_2$. We will count separately tangency points of Types 1 and 2 and tangency points of Types 3 and 4.

Lemma 4. *There are $O(n)$ tangency points of Type 1 or 2.*

Proof. Since all the curves in $\mathcal{C}$ are pairwise intersecting and x-monotone there is a vertical line ℓ that intersects all of them. By slightly shifting ℓ if needed we may assume that no two curves intersect ℓ at the same point. We assume without loss of generality that at least half of all the tangency points of Types 1 and 2 are to the right of ℓ, for otherwise we may reflect all the curves about ℓ. We may further assume that at least half of the tangency points of Types 1 and 2 to the right of ℓ are of Type 2, for otherwise we may reflect all the curves about the x-axis. Henceforth, we consider only Type 2 tangency points to the right of ℓ.

By Proposition 3 a curve cannot touch one curve from above and another curve from below at Type 2 tangency points. Thus, we may partition the curves into *blue* curves and *red* curves such that at every tangency point a blue curve touches a red curve from below (we ignore curves that contain no tangency points among the ones that we consider).

Proposition 5. *Every pair of blue curves cross each other.*

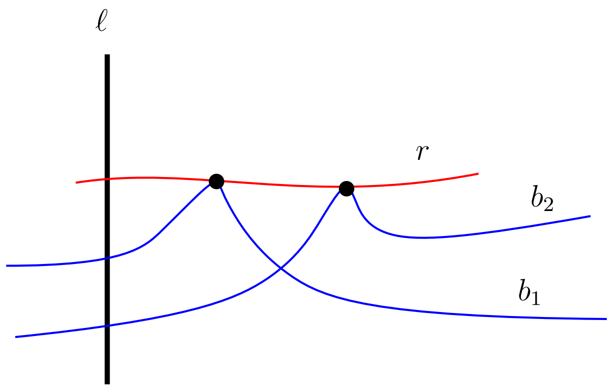


Figure 2: An illustration for the proof of Proposition 6: If two blue curves touch the same red curve, then they cross at a point between these two tangency points.

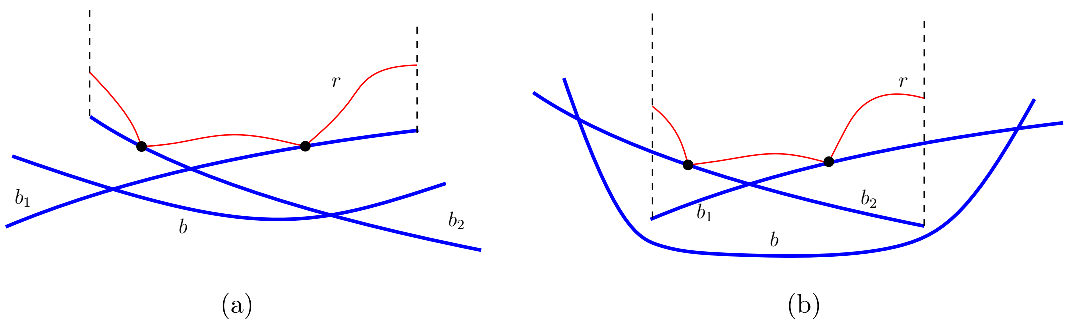


Figure 3: An illustration for the proof of Proposition 7. If $I(b_1, b_2)$ is above b, then r and b do not intersect.

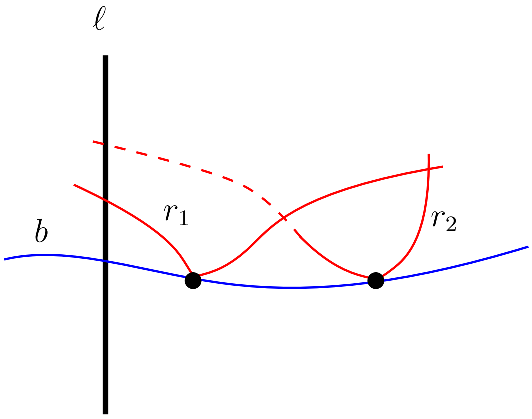
Proof. Suppose that a blue curve b_1 touches another blue curve b_2 from below (the tangency point may be of any type). Since b_2 is a blue curve there is a red curve r which it touches from below. Since the tangency point of b_2 and r is of Type 2 it follows that b_1 and r do not intersect. □

Proposition 6. *Let r be a red curve and let b_1 and b_2 be two blue curves that touch r, such that $I(r, b_1) <_x I(r, b_2)$. Then $I(r, b_1) <_x I(b_1, b_2) <_x I(r, b_2)$.*

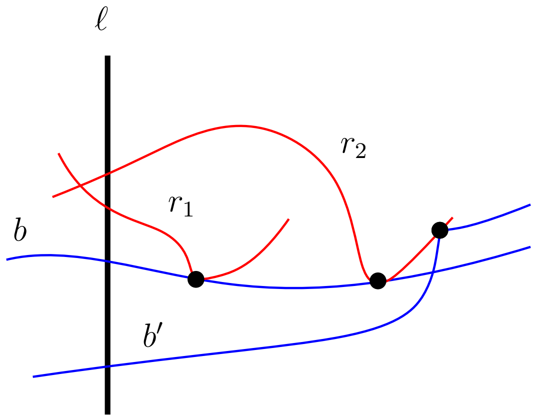
Proof. Since $L(b_i) <_x L(r) <_x R(r) <_x R(b_i)$, for $i = 1, 2$, it is easy to see that the blue curves cross at a point between their tangency points with r, see Figure 2. □

Proposition 7. *Let b_1 and b_2 be two blue curves such that that b_1 is below b_2 before $I(b_1, b_2)$. Let b be another blue curve such that $I(b_1, b_2)$ is between $I(b_1, b)$ and $I(b_2, b)$. If there is a red curve r that touches both b_1 and b_2 then $I(b_1, b_2)$ is below b.*

Proof. Observe that r lies above the upper envelope of b_1 and b_2. Furthermore, since we consider Type 2 tangencies $L(b_i) <_x L(r) <_x R(r) <_x R(b_i)$, for $i = 1, 2$. Therefore if $I(b_1, b_2)$ is above b then r and b do not intersect, see Figure 3. □



(a) If $I(r_1, r_2) >_x I(b, r_2)$ then r_2 must also intersect $r_1(b, r_2)$.



(b) If $I(r_1, r_2) <_x I(b, r_1)$ then b' and r_1 do not intersect.

Figure 4: Illustrations for the proof of Proposition 8: if r_1 and r_2 touch b then $I(r_1, r_2)$ is between these two tangency points.

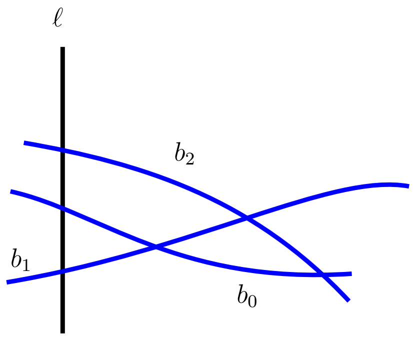
We proceed by marking the rightmost tangency point on every red curve. Clearly, at most n tangency points are marked. Henceforth, we consider only unmarked tangency points.

Proposition 8. *Let b be a blue curve and let r_1 and r_2 be two red curves that touch b, such that $I(b, r_1) <_x I(b, r_2)$ (and both of these tangency points are unmarked). Then $I(b, r_1) <_x I(r_1, r_2) <_x I(b, r_2)$.*

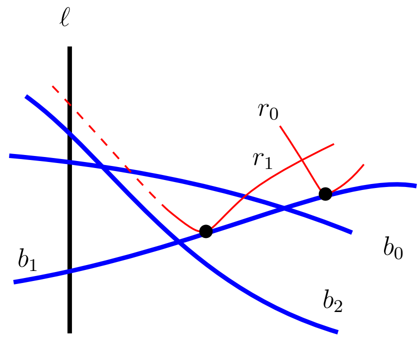
Proof. If $I(r_1, r_2) >_x I(b, r_2)$ then r_2 must intersect $r_1(b, r_2)$ since it intersects ℓ, see Figure 4a. Suppose now that $I(r_1, r_2) <_x I(b, r_1)$. This implies that $R(r_1) <_x I(b, r_2)$ for otherwise r_1 and r_2 intersect also to the right of $I(b, r_1)$. Since $I(b, r_2)$ is not the rightmost tangency point on r_2, there is a blue curve b' that touches r_2 to the right of $I(b, r_2)$ at a Type 2 tangency point. However, it follows from Proposition 6 that $I(b, r_2) <_x I(b, b') <_x I(b', r_2)$ which implies that b' lies below b to the left of $I(b, b')$. Therefore, b' and r_1 do not intersect (see Figure 4b). □

Let G be the (bipartite) *tangencies graph* of the blue and red curves. That is, the vertices of G correspond to the blue and red curves and its edges correspond to pairs of touching blue and red curves (recall that we consider only unmarked tangency points of Type 2 to the right of ℓ). We will show that G is a forest and hence has at most $n - 1$ edges.

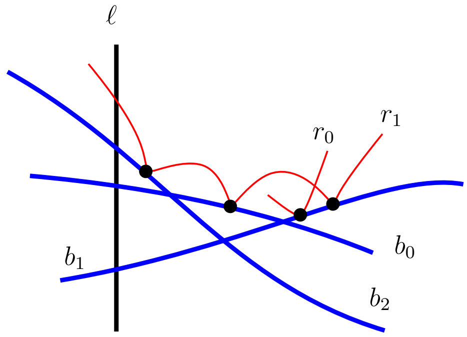
Proposition 9. *Let $b_0 - r_0 - b_1 - r_1 - b_2$ be a path in G such that: (a) b_0, b_1 and b_2 correspond to distinct blue curves and r_0 and r_1 correspond to distinct red curves; (b) $I(b_1, \ell) <_y I(b_0, \ell) <_y I(b_2, \ell)$. Then b_0 and b_2 intersect to the left of ℓ.*



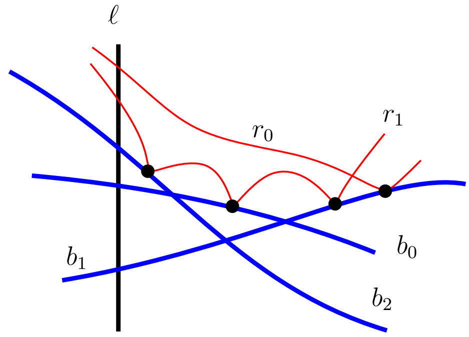
(a) b_2 intersects b_1 and then b_0.



(b) If $I(b_1, r_1) \in b_1(I(b_1, b_2), I(b_1, b_0))$ then r_1 must cross b_0 twice.



(c) If $I(b_1, r_1) \in b_1(I(b_1, b_0), +)$, then r_1 touches b_0. If $I(b_1, r_0)$ precedes $I(b_1, r_1)$, then r_0 must intersect r_1 twice or crosses b_0 to be able to intersect ℓ.



(d) If $I(b_1, r_1)$ precedes $I(b_1, r_0)$, then r_0 does not intersect b_0 to the right of ℓ.

Figure 5: Illustrations for the proof of Proposition 9. $I(b_0, b_2)$ is to the right of ℓ.

Proof. It follows from Proposition 6 that $I(b_0, b_1)$ and $I(b_1, b_2)$ are to the right of ℓ. Suppose for contradiction that $I(b_0, b_2)$ is to the right of ℓ. Recall that b_2 intersect ℓ above b_0 and b_0 intersects ℓ above b_1. If, going from left to right, b_2 intersects first b_1 (necessarily to the right of $I(b_0, b_1)$) and then it intersects b_0, then $I(b_1, b_2)$ is above b_0 which contradicts Proposition 7 (see Figure 5a).

Therefore, b_2 intersects b_0 and then b_1 which implies that b_1 intersects b_2 and then b_0. The curve r_1 lies above the upper envelope of b_1 and b_2, therefore it may touch b_1 at a point which is either in $b_1(b_2, b_0)$ or in $b_1(b_0, +)$. Consider the first case. It follows from Proposition 8 that $I(r_0, r_1)$ is between $I(b_1, r_1)$ and $I(b_1, r_0)$ and therefore r_1 must cross b_0 after it touches b_1, since r_0 is above the upper envelope of b_0 and b_1. However, in this case r_1 must cross b_0 once more to the left of $I(b_1, r_1)$, since it also touches b_2 and therefore must lie above it (see Figure 5b).

Consider now the case that r_1 touches b_1 at a point in $b_1(b_0, +)$. Then r_1 must

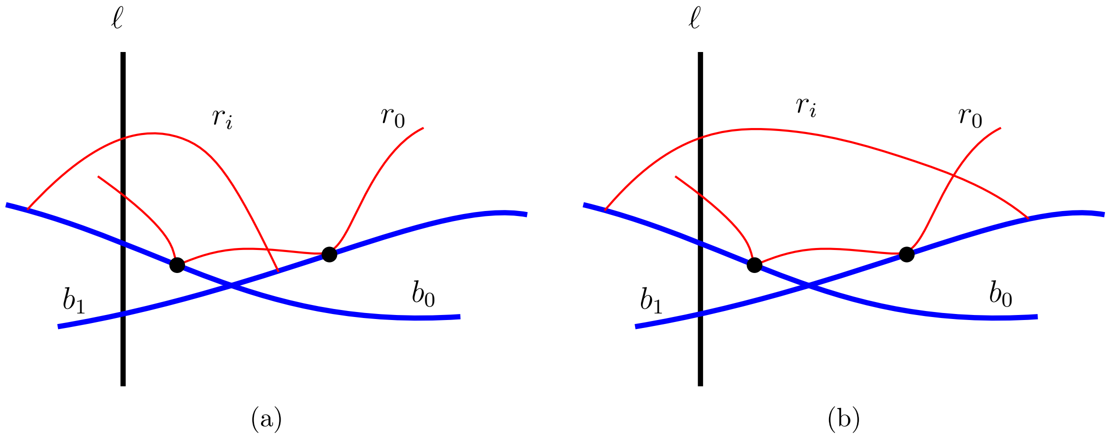


Figure 6: Illustrations for the statement of Proposition 10: r_i intersects ℓ above r_0 and intersects $b_0(-,\ell)$, $r_0(b_0,+)$ and $b_1(b_0,+)$.

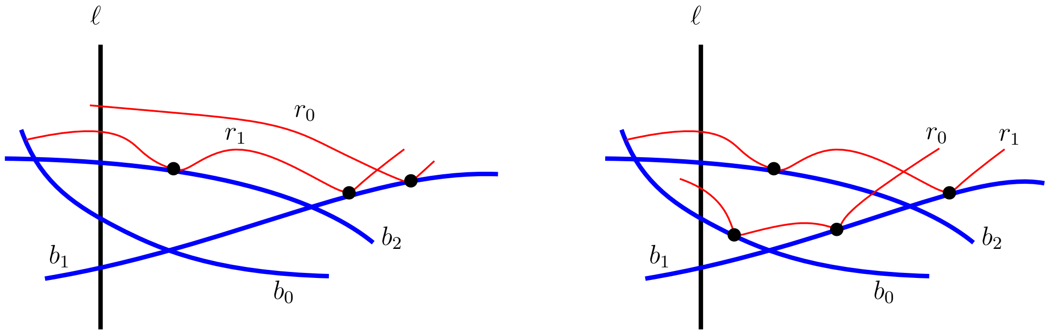
intersect $b_0(b_2, b_1)$ since this is the only part of b_0 which lies above the upper envelope of b_1 and b_2. Since r_1 may not cross b_2 or intersect b_0 twice, it follows that r_1 must touch b_0 (see Figure 5c). Note that r_0 also touches b_1 at $b_1(b_0,+)$. If $I(b_0,r_0)$ precedes $I(b_0,r_1)$ on b_0, then $r_0(b_0,+)$ must intersect r_1 by Proposition 8 (see Figure 5c). However, then r_0 must intersect r_1 once more or cross b_0 which is impossible. If, on the other hand, $I(b_0,r_1)$ precedes $I(b_0,r_0)$ on b_0, then $r_1(b_1,+)$ must intersect r_0 by Proposition 8. However, then r_0 cannot touch b_0 to the right of ℓ, see Figure 5d. □

Suppose that G contains a cycle and let $C = b_0 - r_0 - b_1 - r_1 - \ldots - b_k - r_k - b_0$ be a shortest cycle in G, such that b_i corresponds to a blue curve and r_i corresponds to a red curve, for every $i = 0, 1, \ldots, k$. We may assume without loss of generality that b_1 has lowest intersection with ℓ among the blue curves in C and that $I(b_0,\ell) <_y I(b_2,\ell)$.

Proposition 10. *For every $i \geq 1$ the curve r_i intersects ℓ above r_0 and intersects $b_0(-,\ell)$, $r_0(b_0,+)$ and $b_1(b_0,+)$. See Figure 6 for an illustration.*

Proof. We prove the claim by induction. Consider the case $i = 1$. Before showing that r_1 satisfies the claim, we first look at b_2. It intersects ℓ above b_0 and crosses b_1 to the right of ℓ by Propositions 5 and 6. It follows from Proposition 9 that b_0 and b_2 intersect to the left of ℓ (refer to Figure 7a). Clearly b_2 intersects $b_1(b_0,+)$. Since r_1 is above b_2 it must also intersect b_0 to the left of ℓ and intersect $b_1(b_0,+)$. It remains to show that r_1 intersects ℓ above r_0 and intersects $r_0(b_0,+)$.

Consider r_0 and note that it must touch $b_0(\ell, b_1)$ and (further to the right) touch $b_1(b_0,+)$. If $I(b_1,r_1) <_x I(b_1,r_0)$ then $I(r_0,r_1)$ is between these points by Proposition 8 and it follows that $r_0(\ell,r_1)$ is above $r_1(\ell,r_0)$ and hence r_0 cannot touch b_0 (see Figure 7a). Therefore, $I(b_1,r_0) <_x I(r_0,r_1) <_x I(b_1,r_1)$ which implies that $I(r_0,\ell) <_y I(r_1,\ell)$. Since r_0 touches b_0 before touching b_1, this also implies that r_1



(a) If $I(b_1, r_1) <_x I(b_1, r_0)$ then r_0 cannot touch b_0.

(b) $I(b_1, r_0) <_x I(b_1, r_1)$. r_1 satisfies the properties of Proposition 10

Figure 7: Illustrations for the proof of Proposition 10: the induction base.

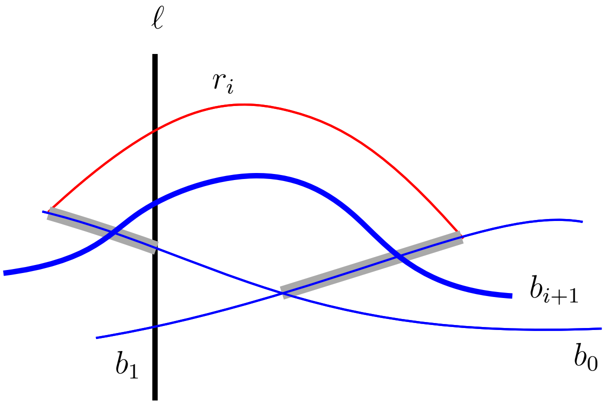


Figure 8: Proposition 10 Case 1: $I(b_0, \ell) <_y I(b_{i+1}, \ell) <_y I(r_i, \ell)$. Since $L(b_{i+1}) <_x L(r_i) <_x R(r_i) <_x R(b_{i+1})$ it follows that b_{i+1} must cross $b_0(r_i, \ell)$ and $b_1(b_0, r_i)$.

intersects $r_0(b_0, +)$. Therefore, r_1 satisfies the properties above, see Figure 7b (note that r_1 cannot intersect b_1 to the left of $I(b_1, r_0)$ as in Figure 6(a)).

Suppose now that the claim holds for r_i, $i \geq 1$. Observe that b_{i+1} intersects ℓ above b_1 (as all the blue curves in C) and recall that r_i and r_{i+1} touch b_{i+1} at Type 2 tangency points. We consider two cases based on whether b_{i+1} intersects ℓ above or below b_0.

Case 1: b_{i+1} intersects ℓ above b_0. Since $L(b_{i+1}) <_x L(r_i) <_x R(r_i) <_x R(b_{i+1})$ it follows that b_{i+1} must cross $b_0(r_i, \ell)$ and $b_1(b_0, r_i)$, see Figure 8. Since r_{i+1} lies above b_{i+1} it follows that, as b_{i+1}, it intersects b_0 to the left of ℓ and $b_1(b_0, +)$. It remains to show that $I(r_0, \ell) <_y I(r_{i+1}, \ell)$ and that r_{i+1} intersects $r_0(b_0, +)$. We proceed by considering two subcases.

Case 1a: $I(r_0, \ell) <_y I(b_{i+1}, \ell)$. Since $I(b_{i+1}, \ell) <_y I(r_{i+1}, \ell)$ it follows that $I(r_0, \ell) <_y I(r_{i+1}, \ell)$. Furthermore, b_{i+1} must intersect $r_0(b_0, +)$ since $R(r_i) <_x R(b_{i+1})$ and by the

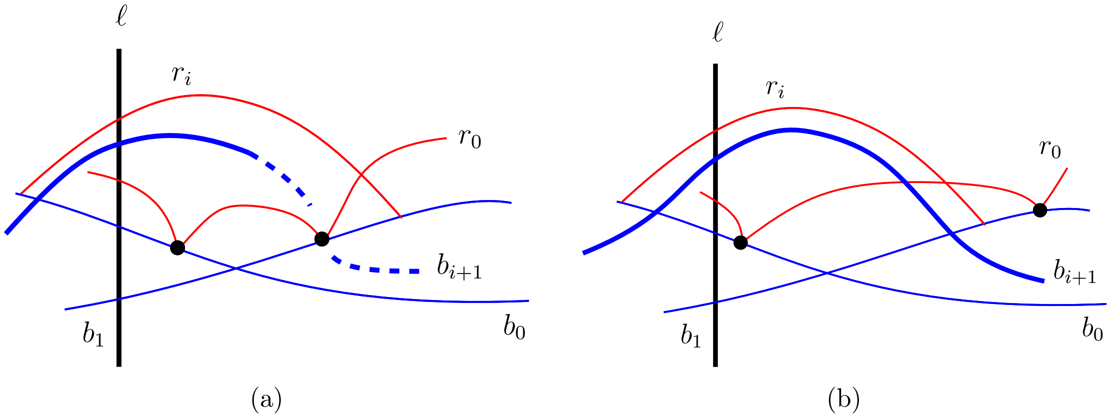


Figure 9: $I(r_0, \ell) <_y I(b_{i+1}, \ell)$. Since b_{i+1} has the desired properties so does r_{i+1}

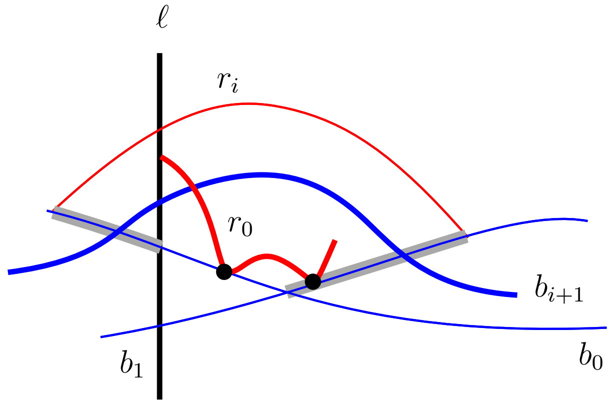


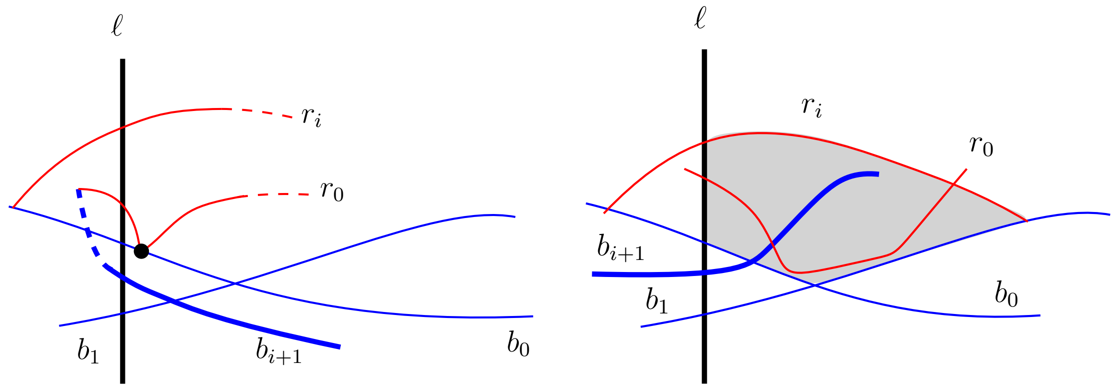
Figure 10: If $I(b_{i+1}, \ell) <_y I(r_0, \ell)$ then r_i does not intersect $r_0(b_0, +)$.

induction hypothesis $r_0(b_0, +)$ intersects r_i. Therefore r_{i+1} intersects $r_0(b_0, +)$ as well (see Figure 9).

Case 1b: $I(b_{i+1}, \ell) <_y I(r_0, \ell)$. Observe that r_0 touches b_0 at $b_0(\ell, b_1)$. Since b_{i+1} does not intersect $b_0(\ell, +)$ it must intersect $r_0(\ell, b_0)$, see Figure 10. Therefore $b_{i+1}(r_0, +)$ does not intersect $r_0(b_0, +)$. Since r_i lies above b_{i+1} it follows that r_i does intersect $r_0(b_0, +)$ which is a contradiction.

Case 2: b_{i+1} intersects ℓ below b_0. Since $I(b_0, b_1) <_x I(b_1, r_i) <_x R(b_{i+1})$ the curve b_{i+1} must cross either $b_0(\ell, b_1)$ or $b_1(\ell, b_0)$. In the latter case b_{i+1} cannot intersect $b_0(b_1, +)$ by Proposition 7, therefore it must intersect $b_0(-, \ell)$. But then $L(r_i) <_x L(b_{i+1})$ (see Figure 11a) Therefore, b_{i+1} intersects $b_0(\ell, b_1)$.

Consider the region bounded by $\ell(r_i, b_0)$, $b_0(\ell, b_1)$, $b_1(b_0, r_i)$ and $r_i(\ell, b_1)$, see Figure 11b. Then b_{i+1} enters this region at $b_0(\ell, b_1)$ and leaves it at $b_1(b_0, r_i)$. Note that b_{i+1} must intersect r_0 at this region since only within this region b_{i+1} has a part above the upper envelope of b_0 and b_1 (where r_0 lies). Furthermore, b_{i+1} must touch r_0, for otherwise it must cross r_0 twice (see Figure 11b).



(a) If b_{i+1} intersects $b_1(\ell, b_0)$ then $L(r_i) <_x L(b_{i+1})$.

(b) b_{i+1} intersects $b_0(\ell, b_1)$. If b_{i+1} crosses r_0, then it must cross it twice.

Figure 11: $I(b_1, \ell) <_y I(b_{i+1}, \ell) <_y I(b_0, \ell)$.

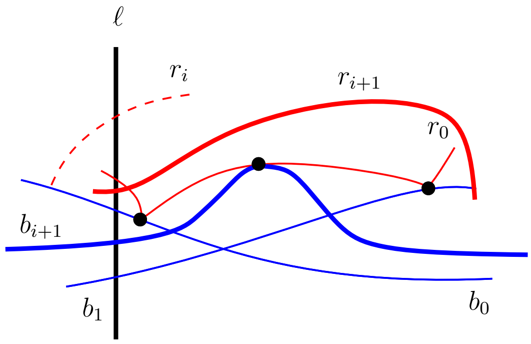
It follows that b_{i+1} crosses $b_0(r_0, b_1)$, then touches r_0 and then crosses $b_1(b_0, r_i)$. One property that we wish to show is that $I(r_0, \ell) <_y I(r_{i+1}, \ell)$. Suppose that $I(r_{i+1}, \ell) <_y I(r_0, \ell)$. Since r_{i+1} lies above b_{i+1} it may intersect b_1 only at $b_1(b_{i+1}, +)$. It follows that r_{i+1} crosses $r_0(\ell, b_{i+1})$ and intersects $b_1(r_0, +)$, see Figure 12a. However, this implies that $R(r_0) <_x R(r_{i+1}) <_x R(b_{i+1})$. Furthermore, $L(b_{i+1}) <_x L(r_i) <_x L(r_0)$, thus $I(b_{i+1}, r_0)$ is a Type 2 tangency point. Since $I(b_{i+1}, r_0) <_x I(b_1, r_0)$, it also holds that $I(b_{i+1}, r_0)$ is not the rightmost tangency point on r_0 and therefore (b_{i+1}, r_0) is an edge in G. But then $r_0 - b_1 - r_1 - \ldots - b_{i+1} - r_0$ is a shorter cycle than C.

Thus $I(r_0, \ell) <_y I(r_{i+1}, \ell)$. If r_{i+1} intersects $r_0(-, b_{i+1})$ then it must touch it for otherwise r_{i+1} cannot intersect $b_1(b_{i+1}, +)$ (the only part of b_1 that lies above b_{i+1} and may intersect r_{i+1}), see Figure 12b. However, then $R(r_0) <_x R(r_{i+1}) <_x R(b_{i+1})$ which implies at before that $I(b_{i+1}, r_0)$ is an unmarked Type 2 tangency point and there is a shorter cycle than C. Therefore r_{i+1} intersects $r_0(b_{i+1}, +)$ (and hence, $r_0(b_0, +)$), $b_0(-, \ell)$ and $b_1(b_0, +)$, as desired (see Figure 12c). □

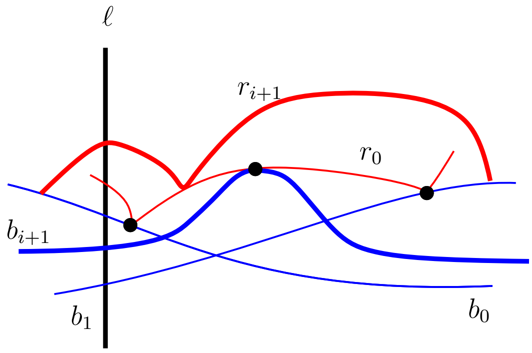
We now return to the proof of Lemma 4 and consider the cycle C. It follows from Proposition 10 that r_k intersects b_0 to the left of ℓ and therefore (b_0, r_k) cannot be an edge in G. Thus G is a forest and has at most $n - 1$ edges. This implies that there are at most $2n - 1$ Type 2 tangency points to the right of ℓ and at most $8n - 4$ tangency points of Types 1 and 2. □

Lemma 11. *There are $O(n)$ tangency points of Type 3 or 4.*

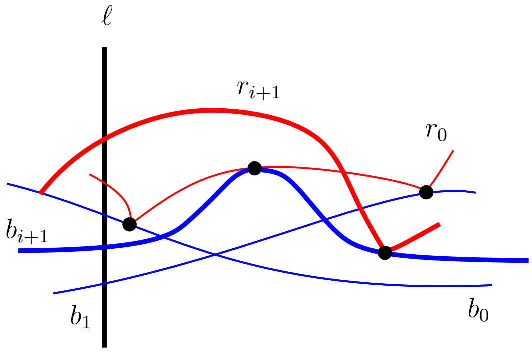
Proof. Since all the curves in $\mathcal{C}$ are pairwise intersecting and x-monotone there is a vertical line ℓ that intersects all of them. By slightly shifting ℓ if needed we may assume that no two curves intersect ℓ at the same point. We assume without loss of generality that at least half of all the tangency points of Type 3 or 4 are to the right of ℓ, for otherwise we may reflect all the curves about ℓ. We may further assume that at least



(a) If $I(r_{i+1}, \ell) <_y I(r_0, \ell)$ then (b_{i+1}, r_0) is an edge in G that creates a shorter cycle than C.



(b) If $I(r_0, \ell) <_y I(r_{i+1}, \ell)$ and r_{i+1} intersects $r_0(-, b_{i+1})$ then (b_{i+1}, r_0) is an edge in G that creates a shorter cycle than C.



(c) If $I(r_0, \ell) <_y I(r_{i+1}, \ell)$ then r_i has the desired properties.

Figure 12: Concluding cases in the proof of Proposition 10.

half of the tangency points of Type 3 or 4 to the right of ℓ are of Type 4, for otherwise we may reflect all the curves about the x-axis. Henceforth, we consider only Type 4 tangency points to the right of ℓ.

By Proposition 3 a curve cannot touch one curve from above and another curve from below at Type 4 tangency points. Thus, we may partition the curves into *blue* curves and *red* curves such that at every tangency point a blue curve touches a red curve from below (we ignore curves that contain no tangency points among the ones that we consider).

Clearly, there are no Type 4 tangencies among the blue curves, however, there might be tangencies of other types among them. Next we wish to obtain a subset of the blue curves such that every pair of them are crossing and they together contain a percentage of the tangency points that we consider. It follows from Proposition 3 that the largest chain in the partially ordered set of the blue curves with respect to $\prec_1$ is of length two. Therefore, by Mirsky's Theorem (the dual of Dilworth's Theorem) the blue curves can be partitioned into two antichains with respect to $\prec_1$. The blue curves of one of these antichains contain at least half of the tangency points that we

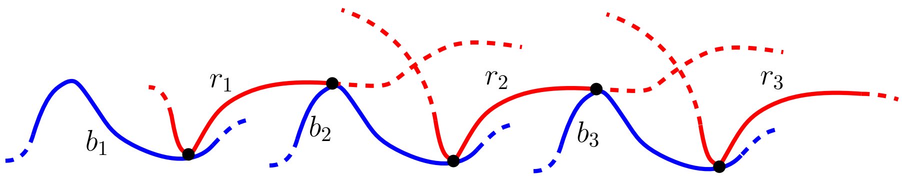


Figure 13: Since the curves intersect ℓ and $R(b_i) <_x R(r_{i-1})$, it follows that $I(b_2, r_1) <_x I(r_1, r_2) < I(b_2, r_2)$ and $I(b_3, r_2) <_x I(r_2, r_3) < I(b_3, r_3)$.

consider. By continuing with this set of blue curves and applying the same argument twice more with respect to $\prec_2$ and $\prec_3$ we obtain a set of pairwise crossing blue curves that together contain at least $1/8$ of the tangency points of Type 4 to the right of ℓ. Henceforth we consider these blue curves and the red curves that touch at least one of them at a Type 4 tangency point to the right of ℓ.

Let $G = (B \cup R, E)$ be the (bipartite) *tangencies graph* of these blue and red curves. That is, B corresponds to the blue curves, R corresponds to the red curves and E corresponds to pairs of touching blue and red curves (at Type 4 tangency points to the right of ℓ). We order the edges of G according to the order of their corresponding tangency points from left to right. We will show that G has linearly many edges using the following fact, attributed to Rödl [16] in [6].

Proposition 12. *Let $G = (V, E)$ be a graph and let $<$ be a total order of its edges. Let k be an integer and suppose that G does not contain a monotone increasing path of k edges, that is, a path $e_1 - e_2 - \ldots - e_k$ such that $e_1 < e_2 < \ldots < e_k$. Then $|E| < \binom{k}{2}|V|$.*

Recall that we order the edges of G according to the order of their corresponding tangency points from left to right. The lemma follows from Proposition 12 and the next claim.

Proposition 13. *G does not contain a monotone increasing path of 7 edges starting at B.*

Proof. Suppose that G contains a monotone increasing path $b_1 - r_1 - \ldots - b_4 - r_4$, such that $b_i \in B$ and $r_i \in R$, for $i = 1, 2, 3, 4$. Since all the curves intersect ℓ and $R(b_i) <_x R(r_{i-1})$, we have:

Observation 14. *For $i = 1, 2, 3$ we have that $I(b_{i+1}, r_i) <_x I(r_i, r_{i+1}) < I(b_{i+1}, r_{i+1})$ and $r_{i+1}(-, r_i)$ lies above $r_i(-, r_{i+1})$ (see Figure 13).*

Considering consecutive blue curves in the path, observe that $I(b_i, b_{i+1})$ cannot be to the right of $I(b_{i+1}, r_i)$, since in such a case b_{i+1} must intersect b_i or r_i twice to be able to intersect ℓ, see Figure 14.

Observation 15. *For $i = 1, 2, 3$ we have that $I(b_i, b_{i+1}) <_x I(b_{i+1}, r_i)$ and $b_i(-, b_{i+1})$ lies above $b_{i+1}(-, b_i)$.*

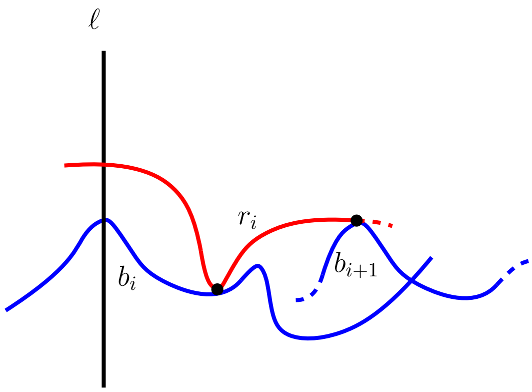
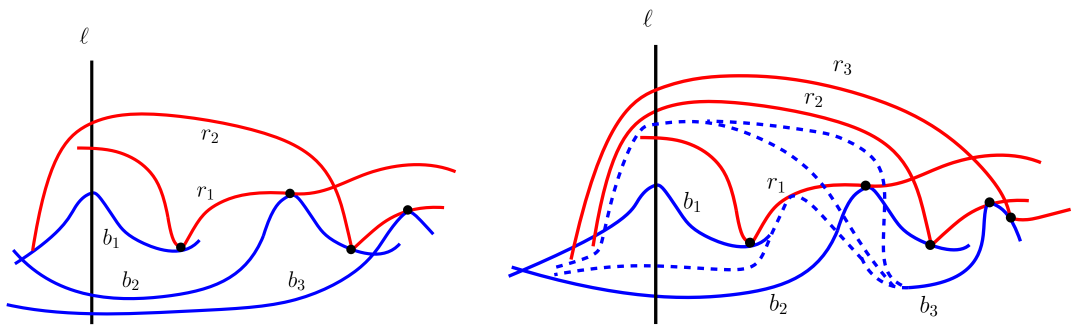


Figure 14: $I(b_i, b_{i+1})$ cannot be to the right of $I(b_{i+1}, r_i)$.



(a) If b_3 intersects $b_2(r_2, +)$ then it cannot intersect r_1.

(b) If b_3 intersects $b_2(-, r_2)$ then b_2 and r_3 do not intersect.

Figure 15: Case 1: If $I(b_1, b_2) <_x I(b_1, r_1)$ then $I(b_1, b_2) <_x I(b_1, r_2) <_x L(r_1)$.

Thus, $I(b_1, b_2)$ is to the left of $I(b_2, r_1)$. We consider two cases based on its location with respect to $I(b_1, r_1)$.

Case 1: $I(b_1, b_2) <_x I(b_1, r_1)$. This implies that $R(b_1) <_x I(b_2, r_1)$. Since $I(b_2, r_1) <_x I(r_1, r_2)$ and $r_2(-, r_1)$ lies above $r_1(-, r_2)$ which lies above b_1, it follows that r_2 and b_1 may intersect only to the left of $L(r_1)$. We must also have $I(b_1, b_2) <_x I(b_1, r_2)$ since $L(b_2) < L(r_2)$ and r_2 lies above b_2 which is above b_1 to the left of $I(b_1, b_2)$ by Observation 15 (see Figure 15a).

Considering b_3, we observe that it cannot intersect $b_2(r_2, +)$ since then it does not intersect r_1. Indeed, suppose that b_3 intersects $b_2(r_2, +)$ and refer to Figure 15a. b_3 lies below r_2 and $R(b_3) <_x R(r_2)$, therefore b_3 may not intersect $r_1(r_2, +)$ which lies below r_2. Since $I(r_1, r_2) <_x I(b_2, r_2) <_x I(b_2, b_3)$ it follows that $b_3(b_2, +)$ cannot intersect r_1. The other part of b_3, $b_3(-, b_2)$, lies below b_2 which lies below r_1 and has its left endpoint to the left of $L(r_1)$. Therefore $b_3(-, b_2)$ cannot intersect r_1 as well.

Therefore, b_3 crosses $b_2(-, r_2)$. This implies that $R(b_2) <_x I(b_3, r_2)$, see Figure 15b. We claim that b_3 and r_2 'block' r_3 from intersecting b_2. Indeed, since $R(b_2) <_x I(b_3, r_2) <_x I(r_2, r_3)$ and $r_3(-, r_2)$ lies above $r_2(-, r_3)$ which lies above b_2 it follows

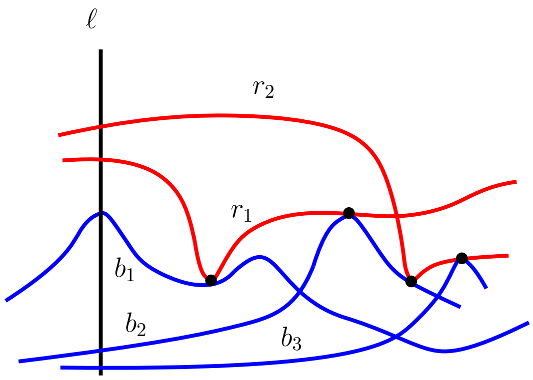


Figure 16: Case 2: $I(b_1, r_1) <_x I(b_1, b_2) <_x I(b_2, r_1)$. If $I(b_2, r_2) <_x I(b_2, b_3)$ then b_3 and r_1 do not intersect.

that r_3 may intersect b_2 only to the left of $L(r_2)$. However, $r_2(r_1, +)$ 'blocks' b_3 from intersecting r_1 to the right of $I(r_1, r_2)$, therefore b_3 must intersect $r_1(-, r_2)$, which implies that b_3 must cross b_2 to the right of $L(r_1)$ which is to the right of $L(r_2)$, see Figure 15b. But then b_3 is above b_2 to the left of $L(r_2)$, and since $L(b_3) <_x L(r_2)$ and $L(b_3) <_x L(r_3)$ it follows that b_3 'blocks' b_2 from intersecting r_3 to the left of $L(r_2)$. Therefore, b_2 and r_3 do not intersect.

This concludes Case 1. Note that we have not used the existence of b_4 and r_4, that is, we only considered the path $b_1 - r_1 - b_2 - r_2 - b_3 - r_3$ in G.

Case 2: $I(b_1, r_1) <_x I(b_1, b_2) <_x I(b_2, r_1)$. We claim that $I(b_2, b_3) <_x I(b_2, r_2)$. Indeed, suppose that $I(b_2, r_2) <_x I(b_2, b_3)$ and refer to Figure 16. By Observation 15 $b_3(-, b_2)$ lies below $b_2(-, b_3)$. Since b_2 lies below r_1 and $L(b_2) <_x L(r_1)$ it follows that $b_3(-, b_2)$ cannot intersect r_1. Considering the other part of b_3, $b_3(b_2, +)$, it lies below r_2 and its right endpoint is to the left of $R(r_2)$. Since r_2 is below r_1 to the right of $I(r_1, r_2)$ and $I(r_1, r_2) <_x I(b_2, r_2) <_x I(b_2, b_3)$, it follows that $b_3(b_2, +)$ cannot intersect r_1.

Therefore $I(b_2, b_3) <_x I(b_2, r_2)$. However, then we are in Case 1 for the path $b_2 - r_2 - b_3 - r_3 - b_4 - r_4$, which is impossible. $\square$

Returning to the proof of Lemma 11 we conclude from Propositions 12 and 13 that G has at most $28n$ edges. This in turn implies that there are at most $8 \cdot 2 \cdot 2 \cdot 28n = 896n$ tangency points of Types 3 and 4. $\square$

By Lemmata 4 and 11 there are at most $904n - 4$ tangency points among the curves in $\mathcal{C}$. This concludes the proof of Theorem 2.

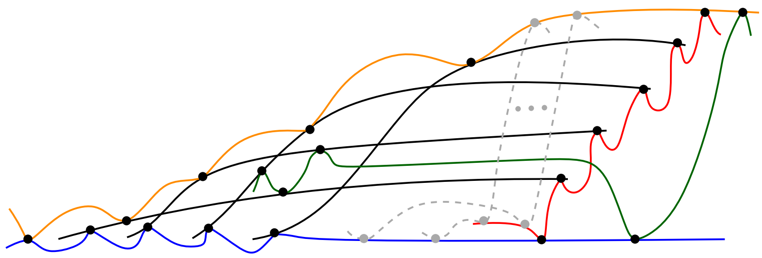


Figure 17: n x-monotone pairwise intersecting 1-intersecting curves might determine $3n - 4$ tangencies.

3 Discussion

We have shown that n x-monotone pairwise intersecting 1-intersecting curves determine $O(n)$ tangencies. The constant hiding in the big-O notation is rather large, since, for simplicity, we did not make much of an effort to get a smaller constant. In particular, our upper bound can be improved by considering more cases. For example, in the proof of Lemma 11 we may consider tangencies among blue curves and avoid using the dual of Dilworth's Theorem. It is also enough to forbid a monotone increasing path of 5 edges in Proposition 13, again by considering more cases. It would be interesting to determine the exact maximum number of tangencies among a set of n x-monotone curves each two of which intersect at exactly one point. The best lower bound we came up with is $3n - 4$, see Figure 17.

Suppose that we allow more than two curves to intersect at a single point but count the number of *tangency points* rather than the number of tangent pairs of curves. Is it still true that there are linearly many tangency points? For n 1-intersecting curves which are not necessarily x-monotone one can get as many as $\Omega(n^{4/3})$ tangency points via the construction of that many point-line incidences, see Figure 18 for an illustration.

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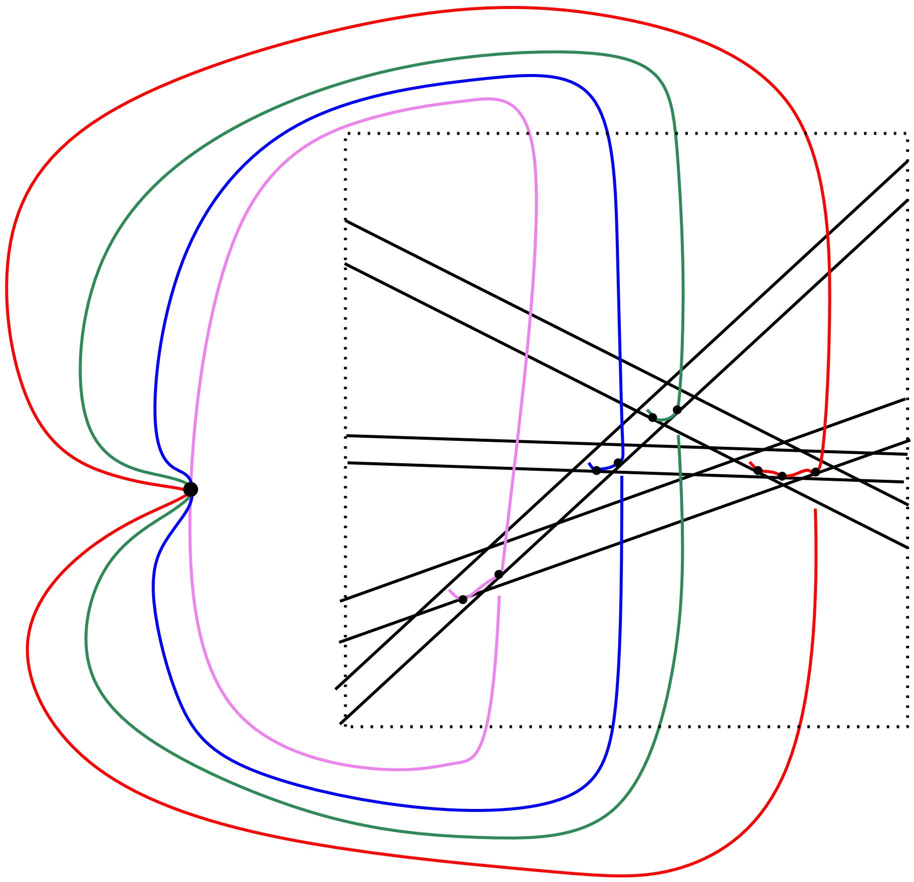

Figure 18: n pairwise intersecting 1-intersecting curves might determine $\Omega(n^{4/3})$ tangency points.

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