ON RANDOM DISC-POLYGONS IN A DISC-POLYGON

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This paper is dedicated to the memory of our dear friend and colleague Gabriella Ágnes Vas.

ABSTRACT. We prove asymptotic formulas for the expectation of the vertex number and missed area of uniform random disc-polygons in convex disc-polygons. Our statements are the r-convex analogues of the classical results of Rényi and Sulanke [10] about random polygons in convex polygons.

1. INTRODUCTION AND RESULTS

Let K be a convex body (compact convex set with non-empty interior) in d-dimensional Euclidean space $\mathbf{E}^d$, and let $X_n = \{x_1, \ldots, x_n\}$ be independent random points from K chosen according to the uniform probability distribution (the Lebesgue measure in K normalised by the volume of K). The convex hull $K_n^* = [X_n]$ of X_n is a (uniform) random polytope in K. The behaviour of the geometric properties of K_n^* have been investigated extensively. In particular, the study of the asymptotic properties of K_n^* started when, in the plane, Rényi and Sulanke [9, 10] determined the behaviour of the expectations of the vertex number of K_n^* and the $\mathrm{Area}(K \setminus K_n^*)$ missed by K_n^*, as $n \to \infty$ in the case when K is convex polygon or a sufficiently smooth disc. For a detailed overview of known results about this classical model we refer to the surveys by Bárány [2], Reitzner [8], Schneider [13], and the references therein.

In this paper we work in the Euclidean plane $\mathbf{E}^2$ and consider a modification of the classical probability model of random polygons in which we use intersections of congruent circles to generate an analogue of the classical convex hull.

Let B denote the origin centred closed unit ball of $\mathbf{E}^2$, and let $S^1 = \partial B$ be its boundary. For a fixed $r > 0$, an r-disc-polygon is a compact convex set in $\mathbf{E}^2$ that is bounded by a finite number of radius r circular arcs. Let $X \subset \mathbf{E}^2$ be a finite point set that is contained in a closed circle of radius r. The intersection of all radius r closed circular discs that contains X, denoted by $[X]_r$, is an r-disc-polygon. The vertices and edges of a disc-polygon are defined in the natural way. It is known, see, for example, [4] that if P is an r-disc-polygon and $X \subset P$, then $[X]_r \subset P$. Furthermore, for each boundary point $x \in \partial P$, there exists a point $v \in \mathbf{E}^2$ such that $x \in rS^1 + v$ and $P \subset rB + v$. We call such $rB + v$ a supporting disc of P. Note that if x is a vertex of P, then there are infinitely many vectors v with this property, therefore, in this case the supporting disc is not unique.

Let P be an r-disc-polygon in $\mathbf{E}^2$, and let $X_n = \{x_1, x_2, \ldots, x_n\}$ be a sample of n independent random points in P chosen according to the uniform probability

distribution. The closed r-hull $P_n^r = [X_n]_r$ is a *uniform random r-disc-polygon* in P.

Let $f_0(\cdot)$ be the number of vertices of a convex (disc-)polygon, and let Area$(\cdot)$ denote the area. In [10] Rényi and Sulanke proved that if P is a (classical) convex polygon, then

$$(1) \qquad \lim_{n \to \infty} \frac{\mathbb{E} f_0(P_n^*)}{\ln n} = \frac{2}{3} f_0(P).$$

In fact, their formula is more precise than (1) but we state it here in this simpler form as it fits the following discussion better. It is a natural question: what is the asymptotics of $\mathbb{E} f_0(P_n^r)$ if P is a r-disc-polygon? Our main result is the following theorem that answers this question:

Theorem 1. *If P is a convex r-disc-polygon different from rB, then*

$$(2) \qquad \lim_{n \to \infty} \frac{\mathbb{E} f_0(P_n^r)}{\ln n} = \frac{2}{3} f_0(P),$$

and

$$(3) \qquad \lim_{n \to \infty} \frac{n \mathbb{E} \operatorname{Area}(P \setminus P_n^r)}{\ln n} = \frac{2}{3} f_0(P) \operatorname{Area}(P).$$

The case $P = rB$ is rather different, and it was treated in Theorem 3.1 in [6], where we showed that $\mathbb{E} f_0(P_n^r)$ converges without any normalisation.

The quantity $\operatorname{Area}(P \setminus P_n^r)$ is often called the missed area of P, and the limit formula (3) follows from (2) by the r-convex analogue of Efron's identity, cf. [6]. Subsequently, we will prove (2) in detail.

We would like to point out that our argument is very different from the one used by Rényi and Sulanke in [10], where affine invariance played a key role in the proof of (1). This is not an option in our case as the model is not invariant under affine transformations. Therefore, in order to evaluate (13), one needs to use techniques that are more essentially based on the geometric properties of the model. This extra geometric information is described in Section 2 and it mainly concerns the behaviour of small disc-caps which determines how to divide the domain of integration in (13).

It is a natural question to ask how Theorem 1 is related to the corresponding classical result (1) of Rényi and Sulanke [9]. Our method can also be used, with some modifications, to prove (1). However, whether (2) implies (1) in the limit as $r \to \infty$ is unclear.

We call a compact convex set $K \subset \mathbf{E}^2$ R-convex (the terms R-spindle convex and R-hyperconvex are also used in the literature), if it is the intersection of all radius R closed circular discs that contain K. This condition is known to be equivalent to the property that K slides freely in a circle of radius R, that is, for any $x \in RS^1$ there exists a vector $p \in \mathbb{E}^2$ with $x \in K + p \subset RB$. The concept of R-convexity goes back, at least, to Mayer [7], and it has been investigated recently quite intensively. The importance of R-convexity comes, in part, from its connection to various old problems in which intersections of congruent balls appear, like the Kneser-Poulsen conjecture. For more information on the properties of R-convex sets we refer to [4], [6] and the references therein.

Our probability model has a natural modification for R-convex discs. If K is an R-convex disc for some $R \le r$, and $X_n = \{x_1, \ldots, x_n\}$ are independent random points chosen from K according to the uniform probability distribution, then it is

known that the random r-disc-polygon $K_n^r = [X_n]_r$ is contained in K, see [6]. The asymptotic behaviour of the expectations $\mathbb{E} f_0(K_n^r)$ and $\mathbb{E} \operatorname{Area}(K \setminus K_n^r)$ have been determined by Fodor, Kevei and Vígh in [6] in the case when K is a convex disc such that it boundary ∂K is C_+^2 and $r > 1/\kappa_m$, where $\kappa_m = \min_{x \in \partial K} \kappa(x) > 0$ and $\kappa(x)$ denotes the curvature of ∂K at x. It is known that under these conditions K is R-convex for $R \geq 1/\kappa_m$, see [12, Theorem 3.2.12 on p. 164]. The following statements were proved in [6]:

$$
(4) \qquad \lim_{n \to \infty} \mathbb{E} f_0(K_n^r) \cdot n^{-1/3} = \sqrt[3]{\frac{2}{3 \operatorname{Area}(K)}} \Gamma\left(\frac{5}{3}\right) c(K, r),
$$

$$
(5) \qquad \lim_{n \to \infty} \mathbb{E} \operatorname{Area}(K \setminus K_n^r) \cdot n^{2/3} = \sqrt[3]{\frac{2 \operatorname{Area}(K)^2}{3}} \Gamma\left(\frac{5}{3}\right) c(K, r),
$$

where

$$
c(K, r) = \int_{\partial K} \left(\kappa(x) - \frac{1}{r}\right)^{1/3} \, \mathrm{d}x.
$$

The symbol $\Gamma(\cdot)$ denotes Euler's gamma function, and integration on ∂K is with respect to arc-length.

The formulas (4) and (5) are generalisations of the corresponding classical results of Rényi and Sulanke from [9] in the sense that the asymptotic formulas of Rényi and Sulanke follow from (4) and (5) in the limit as $r \to \infty$, see Section 3 of [6] for details.

Finally, we conjecture that for any r-convex disc $K \subset \mathbf{E}^2$ different from rB the following inequalities hold for any n

$$
(6) \qquad c_1(K) \log n < \mathbb{E} f_0(K_n^r) < c_2(K) n^{1/3},
$$

for suitable constants $c_1(K)$ and $c_2(K)$, and that the orders in (6) are optimal: the left-hand inequality of is realised by r-disc-polygons and the right-hand inequality by smooth r-convex discs. We note that, due to the different behaviour of rB, it has to be excluded from the inequality (6), cf. Theorem 1.3 in [6].

The corresponding inequalities in the classical convex case for the number $f_k(\cdot)$ of k-dimensional faces were established using floating bodies and the Economic Cap Covering Theorem by Bárány and Larman [3] and by Bárány [1]: for any convex body $K \subset \mathbf{E}^d$ it holds that

$$
(7) \qquad C_1(d)(\log n)^{d-1} < \mathbb{E} f_k(K_n^*) < C_2(d) n^{\frac{d-1}{d+1}}
$$

for suitable constants $C_1(d)$ and $C_2(d)$ and any n. Here the left-hand inequality is of right order for polytopes and the right-hand one for smooth convex bodies.

Unfortunately, the analogue of the Economic Cap Covering Theorem is not known for the r-convex case, even in the plane. We conjecture that it is true, however, the methods used in its proof do not seem to translate to the r-convex setting.

2. Caps of disc-polygons

As both (2) and (3) are invariant under simultaneous scaling of K and the generating circles of P_n^r, we may and do assume from now on that $r = 1$ and omit r from the notation. Accordingly, we use the $[X]_S$ symbol for the 1-hull of the set X. In particular, the 1-hull of two points $x, y \in \mathbf{E}^2$, with $|x - y| \leq 2$ is denoted

by $[x, y]_S$ and is called the *spindle* of x and y. Subsequently, a disc-polygon always means a convex 1-disc-polygon.

Let P be a disc-polygon and let B° denote the origin centred unit radius open circular disc. A subset D of P is a *disc-cap of* P if $D = P \setminus (B^\circ + p)$ for some point $p \in \mathbf{E}^2$. Note that D is either an edge of P or $\partial B + p$ intersects ∂P in at most two points, and D contains at least one vertex of P. The boundary of a nonempty disc-cap D consists of at most two connected arcs: one arc is a subset of ∂P, and the other arc is a subset of $\partial B + p$.

For $x \in \partial P$, let $\mathcal{N}(x) \subset S^1$ denote the set of all outer unit normal vectors of P at x. If $x \in \partial P$ is not a vertex of P, then $\mathcal{N}(x) = \{u_x\}$ contains a single element. If x is a vertex, then $\mathcal{N}(x)$ determines a closed and connected arc of S^1.

Lemma 2. *Let P be a disc-polygon. Let $D = P \setminus (B^\circ + p)$ be a non-empty disc-cap of P with non-empty interior. Then there exists a unique unit vector u and a number $t > 0$ such that $B + p = B + x_0 - (1 + t)u$, where x_0 is the unique point on ∂P with $u \in \mathcal{N}(x_0)$.*

We call u the outer unit *normal*, x_0 the *vertex*, and t the *height of* D. Lemma 2 was proved in [6] for the C^2_+ case, and in higher dimension in [5] also for the C^1 case. Essentially the same argument works here too but for the sake of completeness we provide a short proof.

Proof. Let x_0 be a point of P whose distance from p is maximal. First we show that x_0 is unique. Assume on the contrary that $x_1 \neq x_0$ are both at maximal distance from p. Then the spindle $[x_0, x_1]_S$ is also in P, and one of the midpoints of the unit circular arcs connecting x_0 and x_1 is farther from p than x_0, a contradiction.

Let $u = (x_0 - p)/|x_0 - p| \in S^1$. The line through x_0 that is perpendicular to u clearly supports P at x_0 hence $u \in \mathcal{N}(x_0)$. Thus, $B + p = B + x_0 - (1 + t)u$ for some $t > 0$.

On the other hand, if $B + p = B + x - (1 + t)u$ for some $x \in \partial P$, $u \in \mathcal{N}(x)$ and $t > 0$, then $B + x - u$ supports P at x, and $(1 + t)B + p$ also supports P at x. This yields that x is the farthest point of P from p, and the uniqueness of x_0 and u follows. $\qquad\square$

Let $D(u, t)$ denote the disc-cap with normal u and height t. (Due to the strict convexity of P, u determines x_0 uniquely.) Note that for each $u \in S^1$, there exists a maximal positive constant $t^*(u)$ such that $(B + x_u - (1 + t)u) \cap P \neq \emptyset$ for all $t \in [0, t^*(u)]$. Here x_u is the unique point in ∂P with $u \in \mathcal{N}(x_u)$. Let $A(u, t) = \mathrm{Area}(D(u, t))$ and let $\ell(u, t)$ denote the arc-length of $\partial D(u, t) \cap (\partial B + x_u - (1 + t)u)$.

We recall the following notations from [6]. Let x and y be two points from P. The two unit circles passing through x and y determine two disc-caps of P, which we denote by $D_-(x, y)$ and $D_+(x, y)$, respectively, such that $\mathrm{Area}(D_-(x, y)) \leq \mathrm{Area}(D_+(x, y))$. For brevity of notation, we write $A_-(x, y) = \mathrm{Area}(D_-(x, y))$ and $A_+(x, y) = \mathrm{Area}(D_+(x, y))$ and simply $A = \mathrm{Area}(P)$.

Lemma 3. *Let P be a disc-polygon with at least three vertices. Then there exists a constant $\delta_0 > 0$, depending only on P, such that $A_+(x_1, x_2) > \delta_0$ for any two distinct points $x_1, x_2 \in P$.*

Proof. We note that $[x_1, x_2]_S$ cannot cover P because P is not a spindle. Thus, by compactness, there exists a constant $\delta_0 > 0$, depending only on P, such that
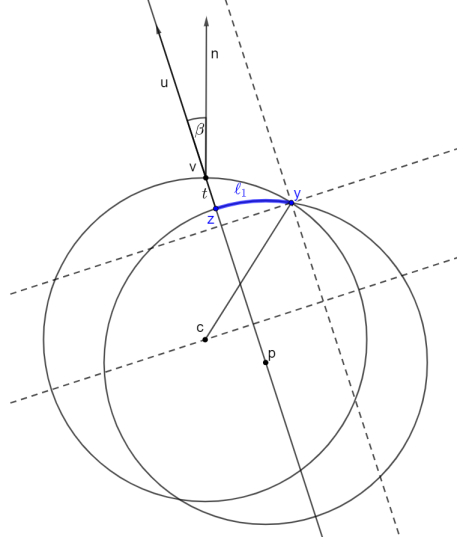
FIGURE 1. Computing ℓ_1

Area$(P \setminus [x_1, x_2]_S) > 2\delta_0$ for any two distinct points $x_1, x_2 \in P$. Now, the statement of the lemma follows from the fact that $P = D_-(x_1, x_2) \cup D_+(x_1, x_2) \cup [x_1, x_2]_S$. $\square$

Note that the statement of Lemma 3 does not hold if P has only two vertices, that is, if it is a spindle $P = [v_1, v_2]_S$.

Lemma 4. *Let $P = [v_1, v_2]_S$ be a disc-polygon with two vertices. Then there exists constants $c = c(P)$ and $\delta = \delta(P)$ such that if $x_1, x_2 \in P$ with $A_-(x_1, x_2) \leq A_+(x_1, x_2) < \delta$ then*

$$|\widehat{x_1 x_2}| > c,$$

where $|\widehat{x_1 x_2}|$ denotes the arc-length of the shorter unit circular arc joining x_1 and x_2.

Proof. Similarly as before,

$$\text{Area}\,[x_1, x_2]_S \geq \text{Area}\,[v_1, v_2]_S - A_-(x_1, x_2) - A_+(x_1, x_2) > \text{Area}\,[v_1, v_2]_S - 2\delta,$$

and the assertion follows. $\square$

Assume that for a sufficiently small t the cap $D(u, t) = P \setminus (B^\circ + p)$ contains a single vertex v of P, and denote by e and e^* the two edges of P that meet at v. Let c be the centre of the unit circle that determines e, and $n = v - c$. The circle $S^1 + p$ intersects $S^1 + c$ in y, and the segment pv in z, cf. Figure 1. Let ℓ_1 denote the shorter circular arc connecting y and z, and let β be the angle of u and n.

Lemma 5. *With the notation above*

$$(8) \qquad \lim_{(t,\beta) \to (0+, 0+)} \left(\frac{\sin \ell_1 \cdot \sin \beta}{t} - \cos \ell_1 \right) = 0.$$

Proof. We use the notations of Figure 1.

By the Pythagorean theorem

$$|y - c|^2 = 1 = (\sin \ell_1 + \sin \beta)^2 + (\cos \ell_1 - (1 + t - \cos \beta))^2.$$

After simplifying and rearranging the terms we get

$$\sin \ell_1 \sin \beta + (\cos \ell_1 - 1)(\cos \beta - 1) = (\cos \beta + \cos \ell_1 - 1)t - \frac{t^2}{2}.$$

Dividing by $t > 0$ and using the $\sin^2 x + \cos^2 x = 1$ identity lead to

$$(9) \qquad \frac{\sin \ell_1 \cdot \sin \beta}{t}\left(1 + \frac{\sin \ell_1 \cdot \sin \beta}{(1 + \cos \ell_1)(1 + \cos \beta)}\right) = \cos \beta + \cos \ell_1 - 1 - \frac{t}{2}.$$

As $\beta \to 0+$ and $t \to 0+$ the claim follows.   □

Keeping $\beta > 0$ fixed, from (9) we obtain

$$(10) \qquad\qquad\qquad \ell_1(\beta, t) \sim t \cot \beta, \quad \text{as } t \to 0+.$$

Let $A_1(\beta, t)$ denote the area of the set bounded by the arcs vy and yz, and the segment vz, see Figure 1.

Lemma 6. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $t \le \delta\beta$ and $\beta < \delta$, then*

$$\frac{1 - \varepsilon}{2} t\ell_1(\beta, t) \le A_1(\beta, t) \le \frac{1 + \varepsilon}{2} t\ell_1(\beta, t).$$

Proof. First note that the assumptions and Lemma 5 imply that ℓ_1 is small. Let i denote the length of the arc $\widehat{vy}$, and put $f(x) = x - \sin x$. Then $f(i)/2$ is the area of the set between the arc $\widehat{vy}$ and the segment vy. Therefore,

$$(11) \qquad\qquad A_1(\beta, t) = \text{Area}(yvz) + \frac{1}{2}(f(i) - f(\ell)).$$

We claim that

$$f(i) - f(\ell_1) \le \varepsilon t\ell_1.$$

By the triangle inequality in yvz we obtain

$$2\sin\frac{i}{2} - 2\sin\frac{\ell_1}{2} \le t.$$

We have

$$t \ge 2\sin\frac{i}{2} - 2\sin\frac{\ell_1}{2} = (i - \ell_1)\cos\xi \ge \frac{i - \ell_1}{2},$$

where $\xi \in (\ell_1/2, i/2)$. Thus, $i - \ell_1 \le 2t$. Furthermore, with $\xi' \in (\ell_1, i)$

$$(12) \qquad f(i) - f(\ell_1) = (i - \ell_1)f'(\xi') \le (i - \ell_1)\frac{i^2}{2} \le i^2 t \le 4\ell_1^2 t,$$

where, in the last inequality we used that $i \le 2\ell_1$. Since ℓ_1 is small, for small enough $\delta > 0$

$$(1 - \varepsilon/2)\frac{t\ell_1(\beta, t)}{2} \le \text{Area}(yvz) \le (1 + \varepsilon/2)\frac{t\ell_1(\beta, t)}{2},$$

thus the result follows from (11) and (12).   □

3. Proof of Theorem 1

We only prove (2) concerning the vertex (or edge) number. The asymptotic formula (3) for the missed area follows directly from the Efron-type identity (5.10) in [6].

First we assume that P has at least 3 vertices. Observe that the pair of random points x_1, x_2 determines an edge of P_n if and only if at least one of the disc-caps $D_-(x_1, x_2)$ and $D_+(x_1, x_2)$ does not contain any other points from X_n. Thus, using the notation from [9],

$$\mathbb{E}(f_0(P_n)) = \binom{n}{2} W_n,$$

where

$$(13) \quad W_n = \frac{1}{A^2} \int_P \int_P \left[\left(1 - \frac{A_-(x_1, x_2)}{A}\right)^{n-2} + \left(1 - \frac{A_+(x_1, x_2)}{A}\right)^{n-2} \right] \mathrm{d}x_1 \mathrm{d}x_2.$$

Note that if all points of X_n fall into the closed spindle spanned by x_1 and x_2, then x_1 and x_2 contribute two edges to P_n (since in this case $[X_n]_S = [x_1, x_2]_S$), and accordingly, this event is counted in both terms in the integrand of (13).

As $f_0(P) \geq 3$ is assumed, Lemma 3 yields that

$$\lim_{n\to\infty} \binom{n}{2} \frac{1}{A^2} \int_P \int_P \left(1 - \frac{A_+(x_1, x_2)}{A}\right)^{n-2} \mathrm{d}x_1 \mathrm{d}x_2$$

$$\leq \lim_{n\to\infty} \binom{n}{2} \frac{1}{A^2} \int_P \int_P e^{-\frac{\delta_0}{A}(n-2)} \mathrm{d}x_1 \mathrm{d}x_2$$

$$= \lim_{n\to\infty} \binom{n}{2} e^{-\frac{\delta_0}{A}(n-2)} = 0.$$

Thus, the contribution of the second term of (13) is negligible, hence, in what follows, we will consider only the first term. Note that a similar argument yields that in the first term of (13) it is enough to integrate over pairs of random points x_1, x_2 such that $A_-(x_1, x_2) < \delta_0$. Furthermore, the same conclusion holds for any fixed $\delta \leq \delta_0$. Let $\mathbf{1}(\cdot)$ denote the indicator function of an event. Then

$$(14) \quad \lim_{n\to\infty} \mathbb{E}(f_0(P_n)) \frac{1}{\ln n}$$

$$= \lim_{n\to\infty} \frac{1}{\ln n} \binom{n}{2} \frac{1}{A^2} \int_P \int_P \left(1 - \frac{A_-(x_1, x_2)}{A}\right)^{n-2} \mathbf{1}(A_-(x_1, x_2) < \delta_0) \mathrm{d}x_1 \mathrm{d}x_2.$$

Now, we re-parametrise the pair (x_1, x_2) as follows, see [6] and [11]. Let

$$(15) \qquad\qquad (x_1, x_2) = \Phi(u, t, u_1, u_2),$$

where $u, u_1, u_2 \in S^1$ and $0 \leq t \leq t_0(u)$. Here, for $u \in S^1$ we choose $t_0(u) > 0$ to be the largest value such that $\mathrm{Area}(D(u, t)) < \delta_0$ for $t \leq t_0(u)$, with δ_0 from Lemma 3. Thus

$$D(u, t) = D_-(x_1, x_2),$$

and

$$(x_1, x_2) = (x_u - (1+t)u + u_1, x_u - (1+t)u + u_2).$$

Note that u_1 and u_2 are the unique outer unit normal vectors of $\partial B + x_u - (1+t)u$ at x_1 and x_2, respectively. This yields that, for fixed u and t, both u_1 and u_2 are in the

same arc of length $\ell(u,t)$ in S^1. We denote this arc by $L(u,t)$. Since $A_-(x_1,x_2) < \delta_0$, $D_-(x_1,x_2)$ is uniquely determined by Lemma 3. Now, the uniqueness of the vertex and height of a disc-cap guarantees that Φ is well-defined, bijective, and differentiable on a suitable domain of (u,t,u_1,u_2), cf. [6].

Let $v_0,\ldots,v_{k-1}$ denote the vertices of P labelled cyclically on ∂P in the positive direction, and let $\mathcal{N}(v_i) = \widehat{n_i m_i} \subset S^1$, which is a closed arc of S^1. Let $r : [0,2\pi) \to S^1$ be the usual parametrisation of the unit circle, and we introduce $\alpha_i = r^{-1}(n_i)$, $\beta_i = r^{-1}(m_i)$, for an arbitrary $u \in S^1$ we use $\beta = r^{-1}(u)$, and for simplicity we write $D(\beta,t) = D(r(\beta),t)$, etc. accordingly. Put

$$N_1 = \cup_{i=0}^{k-1} \mathcal{N}(v_i) \subset S^1, \quad N_2 = S^1 \backslash N_1,$$

and

$$B_1 = \{(x_1,x_2) \in P^2 : u \in N_1, \text{ where } D(u,t) = D(x_1,x_2)\}, \quad B_2 = P^2 \backslash B_1.$$

The same calculation as in the Appendix of [6] yields that the Jacobian $|J\Phi|$ of Φ satisfies

$$(16) \qquad |J\Phi(u,t,u_1,u_2)| = \begin{cases} (1+t)|u_1 \times u_2|, & \text{if } u \in N_1, \\ t|u_1 \times u_2|, & \text{if } u \in N_2. \end{cases}$$

We note that $|u_1 \times u_2|$ equals the sine of the length of the unit circular arc between x_1 and x_2 on the boundary of $D(u,t)$.

First, we show that the part of the integral in (14) on B_2 is negligible. Notice that if $u \in N_2$, then for any $t > 0$ the length $\ell(u,t)$ is large, say $\ell(u,t) > \frac{1}{2}\min\{e_{\min}, \frac{\pi}{2}\} = c$, with $e_{\min}$ being the length of the shortest edge of P. Here and later on, c, C are strictly positive generic constant, whose exact value is not important and can be different at each appearance. Therefore,

$$(17) \qquad A(u,t) > tc, \quad u \in N_2, t > 0.$$

Note that the height $t_0(u)$, defined after (15), is uniformly bounded: $0 < t_1 \le t_0(u) \le t_2$. Using (16) and (17) we have

$$\iint_{B_2} \left(1 - \frac{A_-(x_1,x_2)}{A}\right)^{n-2} \mathbf{1}(A_-(x_1,x_2) < \delta)\mathrm{d}x_1\mathrm{d}x_2$$

$$= \int_{N_2} \int_0^{t_0(u)} \int_{L(u,t)} \int_{L(u,t)} \left(1 - \frac{A(u,t)}{A}\right)^{n-2} t|u_1 \times u_2| \, \mathrm{d}u_1\mathrm{d}u_2 \, \mathrm{d}t\mathrm{d}u$$

$$= \int_{N_2} \mathrm{d}u \int_0^{t_0(u)} \left(1 - \frac{A(u,t)}{A}\right)^{n-2} t(\ell(u,t) - \sin\ell(u,t))\mathrm{d}t$$

$$\le C \int_0^{t_2} (1 - ct)^{n-2} t\mathrm{d}t = O(n^{-2}).$$

We also used that t_2 can be chosen sufficiently small to guarantee that $ct_2 < 1$. Furthermore, the variables u_1 and u_2 appear only in the $|u_1 \times u_2|$ term, thus the inner double integral can be evaluated explicitly. In summary, the integral on B_2 is negligible.

Next we deal with the part of the integral in (14) on B_1. We have that

$$\iint_{B_1} \left(1 - \frac{A_-(x_1, x_2)}{A}\right)^{n-2} \mathbf{1}(A_-(x_1, x_2) < \delta) \mathrm{d}x_1 \mathrm{d}x_2$$

$$= \int_{N_1} \mathrm{d}u \int_0^{t_0(u)} \left(1 - \frac{A(u, t)}{A}\right)^{n-2} (1 + t)(\ell(u, t) - \sin \ell(u, t)) \mathrm{d}t.$$

Replacing $t_0(u)$ with t_1 we lose a negligible part of the integral, as before.

We split the integral further according to the vertices. Fix $\varepsilon > 0$ small enough. If $\beta \in [\alpha_i + \varepsilon, \beta_i - \varepsilon]$, $i = 0, 1 \ldots, k-1$, then by (10) it follows that

$$\ell(\beta, t) \sim t(\cot(\beta - \alpha_i) + \cot(\beta_i - \beta)).$$

Since $\beta \in [\alpha_i + \varepsilon, \beta_i - \varepsilon]$, thus $A(\beta, t) \sim t\ell(\beta, t)/2 \geq ct^2$ as $t \to 0+$, uniformly in β. Therefore, for a fixed $\varepsilon > 0$, for each $i = 0, 1, \ldots, k-1$, it holds that

$$\int_{\alpha_i + \varepsilon}^{\beta_i - \varepsilon} \mathrm{d}\beta \int_0^{t_1} \left(1 - \frac{A(\beta, t)}{A}\right)^{n-2} (1 + t)(\ell(\beta, t) - \sin \ell(\beta, t)) \mathrm{d}t$$

$$\leq C \int_0^{t_1} (1 - ct^2)^n t^3 \mathrm{d}t = O(n^{-2}).$$

Therefore, the main contribution of the integral (14) comes from the corners.

For simplicity, choose the vertex v_0 and assume that $\alpha_0 = 0$. We determine the contribution of the integral on $\beta \in (0, \varepsilon)$. Introduce the notation

$$I = \int_0^\varepsilon \mathrm{d}\beta \int_0^{t_1} \left(1 - \frac{A(\beta, t)}{A}\right)^{n-2} (1 + t)(\ell(\beta, t) - \sin \ell(\beta, t)) \mathrm{d}t.$$

Let $\delta > 0$ be a fixed small number, to be determined later. We split I as follows

$$(18) \qquad I_1 = \int_0^\varepsilon \mathrm{d}\beta \int_{\delta\beta}^{t_1} \left(1 - \frac{A(\beta, t)}{A}\right)^{n-2} (1 + t)(\ell - \sin \ell) \, \mathrm{d}t,$$

$$(19) \qquad I_2 = \int_0^\varepsilon \mathrm{d}\beta \int_0^{\delta\beta} \left(1 - \frac{A(\beta, t)}{A}\right)^{n-2} (1 + t)(\ell - \sin \ell) \, \mathrm{d}t.$$

First we show that I_1 is negligible for any $\delta > 0$ and $\varepsilon > 0$.

To simplify notation, put $\ell_1 = \ell_1(\beta, t)$ and $\ell_2 = \ell - \ell_1$ (as in Lemma 5), and let A_i be the area corresponding to ℓ_i, $i = 1, 2$ (see Figure 2).

We note that ℓ_2 is small, since it follows from (10) that

$$(20) \qquad \ell_2(\beta, t) \sim t \cot(\beta_0 - \beta) \quad \text{as } t \to 0+,$$

uniformly in $\beta \leq \varepsilon$.

Now assume that $t > \delta\beta$ and $\ell_1 < \delta/2$. Then

$$\frac{\sin \ell_1 \sin \beta}{t} < \frac{\frac{\delta}{2}\beta}{\delta\beta} = \frac{1}{2},$$

which contradicts Lemma 5 if δ is sufficiently small. Therefore

$$(21) \qquad \ell_1(\beta, t) \geq \delta/2, \quad \text{if } t \geq \delta\beta.$$

Thus $D(\beta, t)$ contains a triangle with base of length t and height at least $\delta/4$ (see Figure 2), implying

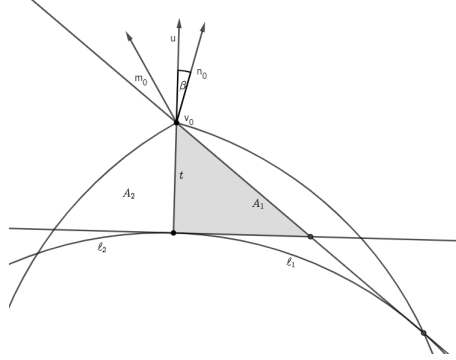$$(22) \qquad A(\beta, t) \geq A_1(\beta, t) > ct.$$

FIGURE 2.

We obtain that

$$I_1 \leq 2 \int_0^\varepsilon \mathrm{d}\beta \int_{\delta\beta}^{t_1} \left(1 - \frac{A(\beta,t)}{A}\right)^{n-2} \mathrm{d}t$$

$$\leq 2 \int_0^{t_1} \frac{t}{\delta} (1-ct)^{n-2} \mathrm{d}t = O(n^{-2}),$$

which proves that I_1 is negligible.

Finally, we estimate I_2, which carries the weight of the integral in (13). Let $\varepsilon_1 > 0$ be fixed. We apply (20) and Lemmas 5 and 6, and we choose $\delta > 0$ and $\varepsilon > 0$ small enough such that

$$(1 - \varepsilon_1)\frac{t^3}{6} \leq t - \sin t \leq (1 + \varepsilon_1)\frac{t^3}{6}, \quad t \in [0, \delta],$$

$$(1 - \varepsilon_1)\frac{t}{\beta} \leq \ell(\beta,t) \leq (1 + \varepsilon_1)\frac{t}{\beta}, \quad t/\delta \leq \beta \leq \varepsilon,$$

$$(1 - \varepsilon_1)\frac{t^2}{2\beta} \leq A(\beta,t) \leq (1 + \varepsilon_1)\frac{t^2}{2\beta}, \quad t/\delta \leq \beta \leq \varepsilon.$$

Substituting $y = t^2(1 - \varepsilon_1)/(2A\beta) =: d_1 t^2/\beta$ and changing the order of integration yield

$$I_2 \leq \int_0^\varepsilon \mathrm{d}\beta \int_0^{\delta\beta} \left(1 - \frac{(1-\varepsilon_1)t^2}{2\beta A}\right)^{n-2} \frac{t^3(1+\varepsilon_1)^3}{6\beta^3}(1+\delta\varepsilon)\mathrm{d}t$$

$$= \frac{(1+\delta\varepsilon)(1+\varepsilon_1)^3}{12\,d_1^2} \int_0^\varepsilon \mathrm{d}\beta \int_0^{\delta^2 d_1 \beta} (1-y)^{n-2}\frac{y}{\beta}\mathrm{d}y$$

$$= \frac{(1+\delta\varepsilon)(1+\varepsilon_1)^3}{12\,d_1^2\,n^2} \int_0^{n\delta^2\varepsilon d_1} \left(1 - \frac{x}{n}\right)^{n-2} x \left(\ln(\varepsilon\delta^2 d_1 n) - \ln x\right) \mathrm{d}x$$

$$\sim \frac{\ln n\,(1+\delta\varepsilon)(1+\varepsilon_1)^3}{12\,d_1^2\,n^2} \quad \text{as } n \to \infty.$$

Since $\varepsilon_1 > 0$ is arbitrary, and the lower bound can be obtained by an analogous argument, we have obtained that

(23) $$I_2 \sim \frac{A^2 \ln n}{3n^2} \quad \text{as } n \to \infty.$$

Since at each vertex we have twice the contribution of I_2, the statement follows when $f_0(P) \geq 3$.

To finish the proof we need to deal with the case in which $f_0(P) = 2$. By Lemma 4, if both $A_-(x_1, x_2)$ and $A_+(x_1, x_2)$ are small, then ℓ is larger than an absolute constant, and this part of the integral can be estimated similarly to I_1. The rest of the argument remains valid in this case as well.

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References

[1] I. Bárány, *Intrinsic volumes and f-vectors of random polytopes*, Math. Ann. **285** (1989), no. 4, 671–699.

[2] I. Bárány, *Random points and lattice points in convex bodies*, Bulletin of The American Mathematical Society **45** (2008), 339-365.

[3] I. Bárány and D. G. Larman, *Convex bodies, economic cap coverings, random polytopes*, Mathematika **35** (1988), no. 2, 274–291.

[4] K. Bezdek, Z. Lángi, M. Naszódi, and P. Papez, *Ball-polyhedra*, Discrete Comput. Geom. **38** (2007), no. 2, 201–230.

[5] F. Fodor, *Random ball polytopes in smooth convex bodies*, arXiv:1906.11480.

[6] F. Fodor, P. Kevei, and V. Vígh, *On random disc polygons in smooth convex discs*, Adv. in Appl. Probab. **46** (2014), no. 4, 899–918.

[7] A. E. Mayer, *Eine Überkonvexität*, Math. Z. **39** (1935), no. 1, 511–531.

[8] M. Reitzner, *Random polytopes and the Efron–Stein jackknife inequality*, Ann. Probab. **31** (2003), 2136–2166.

[9] A. Rényi and R. Sulanke, *Über die konvexe Hülle von n zufällig gewählten Punkten*, Z. Wahrscheinlichkeitsth. verw. Geb. **2** (1963), 75–84.

[10] A. Rényi and R. Sulanke, *Über die konvexe Hülle von n zufällig gewählten Punkten, II.*, Z. Wahrscheinlichkeitsth. verw. Geb. **3** (1964), 138–147.

[11] L. A. Santaló, *On plane hyperconvex figures*, Summa Brasil. Math. **1** (1946), 221–239.

[12] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Second expanded edition, Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014.

[13] R. Schneider, *Discrete aspects of stochastic geometry*, Handbook of Discrete and Computational Geometry, 3rd ed., CRC Press, Boca Raton, 2017.

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