Extremes of the stochastic heat equation with additive Lévy noise

Carsten Chong* Péter Kevei†

Abstract

We analyze the spatial asymptotic properties of the solution to the stochastic heat equation driven by an additive Lévy space-time white noise. For fixed time $t > 0$ and space $x \in \mathbb{R}^d$ we determine the exact tail behavior of the solution both for light-tailed and for heavy-tailed Lévy jump measures. Based on these asymptotics we determine for any fixed time $t > 0$ the almost-sure growth rate of the solution as $|x| \to \infty$.

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1 Introduction

We consider the stochastic heat equation (SHE) on $\mathbb{R}^d$ driven by an additive Lévy space-time white noise $\dot{\Lambda}$, with zero initial condition, given by

$$\begin{aligned} \partial_t Y(t,x) &= \frac{\kappa}{2}\Delta Y(t,x) + \dot{\Lambda}(t,x), \qquad (t,x) \in (0,\infty)\times\mathbb{R}^d, \\ Y(0,\cdot) &= 0, \end{aligned} \tag{1.1}$$

where Δ stands for the Laplacian, $\kappa > 0$ is the diffusion constant, and the measure Λ is given by

$$\Lambda(\mathrm{d}t,\mathrm{d}x) = m\,\mathrm{d}t\mathrm{d}x + \int_{(1,\infty)} z\,\mu(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) + \int_{(0,1]} z\,(\mu-\nu)(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z). \tag{1.2}$$

Here, $m \in \mathbb{R}$ and μ is a Poisson random measure on $(0,\infty)\times\mathbb{R}^d\times(0,\infty)$ whose intensity measure ν takes the form $\nu(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) = \mathrm{d}t\,\mathrm{d}x\,\lambda(\mathrm{d}z)$, with a Lévy measure satisfying $\int_{(0,\infty)}(1\wedge z^2)\,\lambda(\mathrm{d}z) < \infty$. To exclude trivialities, we always assume that λ is not identically zero.

In this case the mild solution to (1.1) can be written explicitly in the form

$$Y(t,x) = \int_0^t \int_{\mathbb{R}^d} g(t-s,x-y)\,\Lambda(\mathrm{d}s,\mathrm{d}y), \tag{1.3}$$

*Department of Statistics, Columbia University, 1255 Amsterdam Avenue, New York, NY 10027, USA, e-mail: carsten.chong@columbia.edu
†Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, 6720 Szeged, Hungary, e-mail: kevei@math.u-szeged.hu

where

$$g(t,x) = \frac{1}{(2\pi\kappa t)^{d/2}} e^{-\frac{|x|^2}{2\kappa t}}, \qquad t > 0,\ x \in \mathbb{R}^d, \tag{1.4}$$

is the heat kernel. In our earlier paper [8] we obtained a complete description of the almost-sure growth behavior of $Y(t,x)$ at a fixed spatial point $x \in \mathbb{R}^d$ as $t \to \infty$. In particular, $t \mapsto Y(t,x)$ satisfies a weak law of large numbers but surprisingly violates the strong law of large numbers. In the present paper we continue these investigations and analyze the almost-sure behavior for fixed time $t > 0$, as $|x| \to \infty$.

To this end, we determine in Section 2 the exact tail asymptotics for $Y(t,x)$ both for light-tailed and for heavy-tailed Lévy measures. Note that since the heat kernel is singular at the origin, the results of [13, 14, 24] for moving-average processes with bounded kernels do not apply. In [7] we proved that for any jump measure λ, the $(1+\frac{2}{d})$-moment of $Y(t,x)$ is infinite, which suggests a power-law tail behavior. In Theorem 2 we show that this is indeed the case, regardless of whether the noise itself is light- or heavy-tailed. Section 3 contains the tail asymptotics for $\sup_{x\in A} Y(t,x)$, where A is a bounded Borel set. Based on these results, we determine in Section 4 the almost-sure growth behavior of $Y(t,x)$ as $|x| \to \infty$. The behavior is very different from the behavior of the Gaussian case, in which

$$\limsup_{|x|\to\infty} \frac{Y(t,x)}{(\log|x|)^{1/2}} = \left(\frac{4t}{\pi\kappa}\right)^{\frac{1}{4}} \tag{1.5}$$

almost surely; see [18, Eq. (6.3)]. All the proofs are gathered together in Section 5. In our companion paper [9] we address the same questions for the SHE with multiplicative Lévy noise.

Let us end this introductory section by stating necessary and sufficient conditions in terms of the jump measure λ for the existence of the integral (1.3). To our best knowledge, this result is new. While many works have studied sufficient conditions for existence [1, 4, 5, 26], necessary and sufficient conditions have only been derived for multiplicative noise [2] or for specific types of noises such as α-stable noise [11]. Introduce the measure η as

$$\eta(B) = \nu\Big(\{(s,y,z) :\ s \le t,\ g(s,y)z \in B\}\Big), \tag{1.6}$$

where $B \subseteq (0,\infty)$ is a Borel set.

Theorem 1. *Suppose that* Λ *is of the form* (1.2).

(i) The integral defining $Y(t,x)$ *in* (1.3) *exists if and only if (iff)*

$$\int_{(1,\infty)} (\log z)^{d/2}\,\lambda(\mathrm{d}z) < \infty \qquad \text{and} \qquad \begin{cases} \int_{(0,1]} z^2\,\lambda(\mathrm{d}z) < \infty & \text{if } d = 1, \\ \int_{(0,1]} z^2|\log z|\,\lambda(\mathrm{d}z) < \infty & \text{if } d = 2, \\ \int_{(0,1]} z^{1+2/d}\,\lambda(\mathrm{d}z) < \infty & \text{if } d \ge 3. \end{cases} \tag{1.7}$$

In this case, η *is a Lévy measure and* $Y(t,x)$ *is infinitely divisible with characteristic function*

$$\mathbf{E}[e^{\mathrm{i}\theta Y(t,x)}] = \exp\left\{\mathrm{i}\theta A + \int_{(0,\infty)} \left(e^{\mathrm{i}\theta u} - 1 - \mathrm{i}\theta u \mathbb{1}(u \le 1)\right)\eta(\mathrm{d}u)\right\}, \tag{1.8}$$

where $\mathbb{1}$ *stands for the indicator function and* $A \in \mathbb{R}$ *is some explicit constant.*

(ii) The integral

$$\int_0^t \int_{\mathbb{R}^d} \int_{(0,\infty)} g(t-s, x-y) z \, \mu(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) \tag{1.9}$$

exists iff

$$\int_{(1,\infty)} (\log z)^{d/2} \, \lambda(\mathrm{d}z) < \infty \qquad \textit{and} \qquad \int_{(0,1]} z \, \lambda(\mathrm{d}z) < \infty. \tag{1.10}$$

Remark 1. Note that (1.7) is identical to the necessary and sufficient condition found in [2] for the existence of solutions to the SHE with multiplicative noise in dimensions $d = 1, 2$ but is weaker than the necessary condition found in [2, Prop. 2.2] for $d \geq 3$. In other words, if $d \geq 3$, there are Lévy noises for which the SHE with additive noise has a solution but the SHE with multiplicative noise does not.

Whenever $\int_{(0,1]} z \, \lambda(\mathrm{d}z) < \infty$, there is no need for compensation, so we assume without loss of generality that $m = \int_{(0,1]} z \, \lambda(\mathrm{d}z)$. In this case,

$$Y(t,x) = \int_0^t \int_{\mathbb{R}^d} \int_{(0,\infty)} g(t-s, x-y) z \, \mu(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) = \sum_{\tau_i \leq t} g(t-\tau_i, x-\eta_i) \zeta_i, \tag{1.11}$$

where $(\tau_i, \eta_i, \zeta_i)$ are the points of the Poisson random measure μ. In what follows we always assume that (1.7) holds.

2 Tail asymptotics

Since $Y(t,x)$ is infinitely divisible, its tail behavior is the same as the tail behavior of its Lévy measure η, whenever the tail is subexponential. This result was proved by [12] for nonnegative infinitely divisible random variables and by [21, 22] in the general case. Therefore, we need to determine the tail of the Lévy measure η in (1.6).

For $\gamma > 0$ introduce the moments and truncated moments of λ as

$$m_\gamma(\lambda) = \int_{(0,\infty)} z^\gamma \, \lambda(\mathrm{d}z) \qquad \text{and} \qquad M_\gamma(x) = \int_{(0,x]} z^\gamma \, \lambda(\mathrm{d}z). \tag{2.1}$$

Lemma 1. *Let $D = (2\pi\kappa t)^{d/2}$. For any $r > 0$,*

$$\overline{\eta}(r) = \eta((r,\infty)) = r^{-(1+2/d)} \frac{d^{d/2}}{\pi\kappa(d+2)^{d/2+1}\Gamma(\frac{d}{2}+1)} \int_0^\infty e^{-u} u^{d/2} M_{1+2/d}(Dr e^{ud/(d+2)}) \, \mathrm{d}u. \tag{2.2}$$

From the representation above we immediately see that as soon as $m_{1+2/d}(\lambda) < \infty$, then $\overline{\eta}(r) \sim c \, r^{-1-2/d}$, for some $c > 0$. We can determine the tail even if this moment condition does not hold, provided that $\overline{\lambda}$ is regularly varying.

In the following, the class of regularly varying functions with index $\rho \in \mathbb{R}$ is denoted by $\mathcal{RV}_\rho$. For general theory on regular variation we refer to [3]. Write $\overline{\lambda}(r) = \lambda((r,\infty))$. By Karamata's theorem, for $\alpha > 0$, $\overline{\lambda} \in \mathcal{RV}_{-\alpha}$ iff the truncated moment $M_{1+2/d}$ in (2.1) is also regularly varying. However, for $\alpha = 0$, the latter holds iff $\overline{\lambda}$ belongs to the de Haan class (see e.g. [3, Thm. 3.7.1]). Therefore, it is more difficult to determine the asymptotics of $\overline{\eta}$ for $\alpha = 0$, and in fact the result itself is surprising.

Lemma 2. *Let λ satisfy* (1.7).

(i) Assume that $m_{1+2/d}(\lambda) < \infty$. Then

$$\overline{\eta}(r) \sim r^{-1-2/d} \frac{d^{d/2}}{\pi\kappa(d+2)^{d/2+1}} m_{1+2/d}(\lambda), \qquad r \to \infty.$$

(ii) Assume that $\overline{\lambda}(r) = \ell(r)r^{-\alpha}$ for $\alpha \in (0, 1+\frac{2}{d}]$, where ℓ is slowly varying, and if $\alpha = 1+\frac{2}{d}$, further assume that $\int_1^\infty \ell(u)u^{-1}\,\mathrm{d}u = \infty$. Then as $r \to \infty$,

$$\overline{\eta}(r) \sim \begin{cases} \ell(r)r^{-\alpha} \dfrac{D^{1+2/d-\alpha}}{d\pi\kappa\alpha^{d/2}(1+\frac{2}{d}-\alpha)} & \text{if } \alpha < 1+\frac{2}{d}, \\ L(r)r^{-1-2/d}(d\pi\kappa(1+\frac{2}{d})^{d/2})^{-1} & \text{if } \alpha = 1+\frac{2}{d}, \end{cases}$$

where

$$L(r) = \int_1^r \ell(u)u^{-1}\,\mathrm{d}u. \tag{2.3}$$

(iii) Assume that $\alpha = 0$ and $\overline{\lambda}(x) = \ell(x)$ is slowly varying. Then as $r \to \infty$,

$$\overline{\eta}(r) \sim L_0(r) \frac{D^{1+2/d}}{2\pi\kappa\Gamma(\frac{d}{2}+1)(1+\frac{2}{d})},$$

where

$$L_0(r) := \int_1^\infty \ell(ry)y^{-1}(\log y)^{d/2-1}\,\mathrm{d}y$$

is slowly varying and $L_0(r)/\ell(r) \to \infty$ as $r \to \infty$.

Example 1. Assume that $\overline{\lambda}(r) = (\log r)^{-\beta}$ for $r > e$. Then (1.7) holds iff $\beta > \frac{d}{2}$. By substituting $u = (1 + \log y/\log r)^{-1}$, we obtain

$$L_0(r) = (\log r)^{d/2-\beta}\,\mathrm{B}(\tfrac{d}{2}, \beta - \tfrac{d}{2}), \qquad r \to \infty,$$

where B is the usual beta function.

To determine the tail of the spatial supremum, we need the tail of the largest contribution to $Y(t,x)$ by a single atom. Without loss of generality, consider $x = 0$ and define

$$\overline{Y}(t) = \sup_{\tau_i \le t} g(t-\tau_i, \eta_i)\zeta_i. \tag{2.4}$$

For $r > 0$ large, let

$$S_r = \{(s,y,z) : s \in [0,t], g(s,y)z > r\}.$$

Clearly, $\overline{Y}(t) \le r$ iff $\mu(S_r) = 0$, which shows that

$$\mathbf{P}(\overline{Y}(t) \le r) = e^{-\nu(S_r)} = e^{-\overline{\eta}(r)}. \tag{2.5}$$

As a result we obtain the following.

Theorem 2. *Let $Y(t,x)$ be given in* (1.3) *and assume* (1.7).

(i) The tail function $\overline{\eta}$ has extended regular variation at infinity [3, p. 65], that is, there are $\theta_1, \theta_2 \in \mathbb{R}$ such that for any $c > 1$,

$$c^{\theta_1} \le \liminf_{x\to\infty} \frac{\overline{\eta}(cx)}{\overline{\eta}(x)} \le \limsup_{x\to\infty} \frac{\overline{\eta}(cx)}{\overline{\eta}(x)} \le c^{\theta_2}. \tag{2.6}$$

(ii) As $r \to \infty$,

$$\mathbf{P}(Y(t,x) > r) \sim \mathbf{P}(\overline{Y}(t) > r) \sim \overline{\eta}(r). \tag{2.7}$$

(iii) For $\alpha \in [0, 1+\frac{2}{d})$, $\overline{\eta} \in \mathcal{RV}_{-\alpha}$ iff $\overline{\lambda} \in \mathcal{RV}_{-\alpha}$. For $\alpha = 1 + \frac{2}{d}$, we have $\overline{\eta} \in \mathcal{RV}_{-1-2/d}$ iff $r \mapsto \int_0^r u^{2/d} \int_1^\infty \overline{\lambda}(uv)(\log v)^{d/2-1} v^{-1}\,\mathrm{d}v\,\mathrm{d}u$ is slowly varying. In particular, the latter holds if $m_{1+2/d}(\lambda) < \infty$.

3 Spatial supremum

Let $A \in \mathcal{B}(\mathbb{R}^d)$ be a Borel subset of $\mathbb{R}^d$ with finite and positive Lebesgue measure and define

$$X_A(t) = \begin{cases} \displaystyle\sum_{\eta_i \in \overline{A}, \tau_i \le t} (2\pi\kappa(t-\tau_i))^{-d/2} \zeta_i \mathbb{1}_{\{(2\pi\kappa(t-\tau_i))^{-d/2}\zeta_i > 1\}} & \text{if } d = 1, \\ \displaystyle\sum_{\eta_i \in \overline{A}, \tau_i \le t} (2\pi\kappa(t-\tau_i))^{-d/2} \zeta_i & \text{if } d \ge 2, \end{cases} \tag{3.1}$$

where $\overline{A}$ is the closure of A. Since $X_A(t)$ is a functional of a Poisson random measure, one easily obtains necessary and sufficient conditions for the existence.

Define the measure τ as

$$\tau(B) = (\mathrm{Leb} \times \lambda)\Big(\{(s,z) : (2\pi\kappa s)^{-d/2} z \in B \cap (\mathbb{1}_{\{d=1\}}, \infty),\ s \le t\}\Big), \qquad B \in \mathcal{B}(\mathbb{R}^d), \tag{3.2}$$

where Leb is the Lebesgue measure. For a Borel set A let $|A|$ be its Lebesgue measure.

Theorem 3. *Suppose that $|\overline{A}| \in (0,\infty)$. The sum $X_A(t)$ is finite a.s. iff*

$$\int_{(0,1)} z^{2/d} |\log z|^{\mathbb{1}_{\{d=2\}}}\, \lambda(\mathrm{d}z) < \infty. \tag{3.3}$$

Furthermore, if (3.3) *holds then*

$$\mathbf{E}[e^{\mathrm{i}\theta X_A(t)}] = \exp\left\{ |\overline{A}| \int_{(0,\infty)} (1 - e^{-\mathrm{i}\theta u})\, \tau(\mathrm{d}u) \right\}. \tag{3.4}$$

Note that (3.3) holds for any Lévy measure if $d = 1$. From (3.2) we obtain that for $r > 1$

$$\begin{aligned} \overline{\tau}(r) = \tau((r,\infty)) &= \int_{(0,\infty)} \Big((2\pi\kappa)^{-1} (z/r)^{2/d} \wedge t \Big)\, \lambda(\mathrm{d}z) \\ &= r^{-2/d} (2\pi\kappa)^{-1} M_{2/d}(rD) + t\,\overline{\lambda}(rD) \\ &= \frac{1}{\pi\kappa d} r^{-2/d} \int_0^{rD} u^{2/d-1} \overline{\lambda}(u)\,\mathrm{d}u. \end{aligned} \tag{3.5}$$

In specific cases, we can determine the asymptotic behavior of $\overline{\tau}$ explicitly.

Lemma 3. *Assume* (3.3).

(i) If $m_{2/d}(\lambda) < \infty$, then $\overline{\tau}(r) \sim (2\pi\kappa)^{-1} m_{2/d}(\lambda) r^{-2/d}$ as $r \to \infty$.

(ii) Assume that $\overline{\lambda}(r) = \ell(r) r^{-\alpha}$ for $\alpha \in [0, \frac{2}{d}]$, where ℓ is slowly varying, and further assume $\int_1^\infty \ell(u) u^{-1}\, \mathrm{d}u = \infty$ if $\alpha = \frac{2}{d}$. Recalling the definition of L from (2.3)*, we have as $r \to \infty$ that*

$$\overline{\tau}(r) \sim \begin{cases} \dfrac{2tD^{-\alpha}}{2 - d\alpha} \ell(r) r^{-\alpha} & \text{if } \alpha < \frac{2}{d}, \\ \frac{2}{d} (2\pi\kappa)^{-1} L(r) r^{-2/d} & \text{if } \alpha = \frac{2}{d}. \end{cases}$$

Introduce the notation

$$\overline{X}_A(t) = \sup\left\{ (2\pi\kappa(t - \tau_i))^{-d/2} \zeta_i \,:\, \tau_i \le t, \eta_i \in \overline{A} \right\}. \tag{3.6}$$

To determine the tail of $\overline{X}_A(t)$, let $T_r = \{(s,z) : s \le t,\ (2\pi\kappa s)^{-d/2} z > r\}$. Then $\overline{X}_A(t) \le r$ iff $\mu(A \times T_r) = 0$, thus

$$\mathbf{P}(\overline{X}_A(t) > r) = 1 - e^{-|\overline{A}| \overline{\tau}(r)}. \tag{3.7}$$

Theorem 4. *Assume* (3.3).

(i) The tail function $\overline{\tau}$ has extended regular variation at infinity.

(ii) For every bounded Borel set A,

$$\mathbf{P}(X_A(t) > r) \sim \mathbf{P}(\overline{X}_A(t) > r) \sim |\overline{A}|\, \overline{\tau}(r), \qquad r \to \infty. \tag{3.8}$$

(iii) For $\alpha \in [0, \frac{2}{d})$, $\overline{\tau} \in \mathcal{RV}_{-\alpha}$ iff $\overline{\lambda} \in \mathcal{RV}_{-\alpha}$. For $\alpha = \frac{2}{d}$, we have $\overline{\tau} \in \mathcal{RV}_{-2/d}$ iff $r \mapsto \int_0^r u^{2/d-1} \overline{\lambda}(u)\, \mathrm{d}u$ is slowly varying. In particular, the latter holds if $m_{2/d}(\lambda) < \infty$ or if $\overline{\lambda} \in \mathcal{RV}_{-2/d}$.

In order to determine the tail asymptotics of the local supremum of the solution, let us introduce for each $A \in \mathcal{B}(\mathbb{R}^d)$ the measure

$$\eta_A(B) = \nu\Big(\big\{ (s,y,z) :\ s \le t,\ (2\pi s)^{-d/2} e^{-\frac{\mathrm{dist}(y,A)^2}{2\kappa s}} z \in B \cap (\mathbb{1}_{\{d=1\}}, \infty) \big\} \Big), \qquad B \in \mathcal{B}(\mathbb{R}^d). \tag{3.9}$$

If $m_{2/d}(\lambda) < \infty$ or if $\overline{\lambda}$ is regularly varying with index $-\alpha$ for some $\alpha \in (0, \frac{2}{d}]$, one can express $\overline{\eta}_A$ in terms of $\overline{\tau}$ or $\overline{\lambda}$.

Lemma 4. *Let A be a bounded Borel set. Assume* (3.3) *and $\int_{(1,\infty)} (\log z)^{d/2} \lambda(\mathrm{d}z) < \infty$.*

(i) If $m_{2/d}(\lambda) < \infty$ or $\overline{\lambda}(r) = \ell(r) r^{-2/d}$ and ℓ is slowly varying with $\int_1^\infty \ell(u) u^{-1}\, \mathrm{d}u = \infty$, then

$$\overline{\eta}_A(r) \sim |\overline{A}|\, \overline{\tau}(r), \qquad r \to \infty.$$

(ii) If $\overline{\lambda}(r) = \ell(r) r^{-\alpha}$ for $\alpha \in (0, \frac{2}{d})$, where ℓ is slowly varying, then

$$\overline{\eta}_A(r) \sim \overline{\lambda}(r) \int_0^t \int_{\mathbb{R}^d} (2\pi\kappa s)^{-\alpha d/2} e^{-\frac{\alpha \mathrm{dist}(y,A)^2}{2\kappa s}}\, \mathrm{d}s\, \mathrm{d}y, \qquad r \to \infty.$$

Theorem 5. *Let A be a bounded Borel set. Assume* (3.3) *and* $\int_{(1,\infty)}(\log z)^{d/2}\lambda(\mathrm{d}z) < \infty$*. If* $d = 1$*, further assume that*

$$\exists q \in (0,2): \;\; M_q(1) < \infty. \tag{3.10}$$

Then under the assumptions of Lemma 4 (i) or (ii) we have that

$$\mathbf{P}\left(\sup_{x\in A} Y(t,x) > r\right) \sim \overline{\eta}_A(r), \qquad r \to \infty.$$

Remark 2. As the proof shows, even without the assumptions of Lemma 4 (i) and (ii), the statement of Theorem 5 continues to hold provided $\overline{\eta}_A$ is subexponential. We were not able to prove or disprove this in general.

4 Growth rate

In what follows we assume (3.3). For $r > 0$ and $0 \le r_1 < r_2$, we write

$$V(r) = \left\{(s,z)\colon \frac{z}{(2\pi\kappa s)^{d/2}} > r,\; s \le t\right\}, \quad V(r_1,r_2) = \left\{(s,z)\colon \frac{z}{(2\pi\kappa s)^{d/2}} \in (r_1,r_2],\; s \le t\right\}. \tag{4.1}$$

Recalling (3.2) we have for $r > 1$

$$(\mathrm{Leb}\times\lambda)(V(r)) = \overline{\tau}(r). \tag{4.2}$$

Note that $\overline{\tau}(r)$ is a continuous strictly decreasing function, with $\overline{\tau}(\infty) = 0$ and $\overline{\tau}(0+) = \infty$ whenever $\lambda((0,1)) = \infty$. If $m_{2/d}(\lambda) < \infty$, then by (3.5)

$$\overline{\tau}(r) \le r^{-2/d}(2\pi\kappa)^{-1}m_{2/d}(\lambda). \tag{4.3}$$

From (3.5) we further see that whenever $\int_{(0,1]} z^{2/d}\,\lambda(\mathrm{d}z) = \infty$ we have for any $r > 0$

$$\sup_{y:|x-y|\le r} Y(t,y) = \infty. \tag{4.4}$$

Therefore, our standing assumption (3.3) is optimal for $d \ge 3$ and almost optimal for $d = 2$. For a more general result in this direction, see [6, Thm. 3.3]. Furthermore, by [6, Thm. 3.1], if $\int_{(0,1]} z^p\,\lambda(\mathrm{d}z) < \infty$ for some $p < \frac{2}{d}$, then for any fixed t the function $x \mapsto Y(t,x)$ is a.s. continuous.

If $\int_0^\infty z^{2/d}\,\lambda(\mathrm{d}z) = m_{2/d}(\lambda) < \infty$, the non-Gaussian analogue of (1.5) (see also [10, Thm. 1.2]) reads as follows.

Theorem 6. *Let* $f : (0,\infty) \to (0,\infty)$ *be nondecreasing and assume that* (3.3) *holds. If* $d = 1$*, further assume* (3.10)*. Then almost surely*

$$\limsup_{x\to\infty}\frac{\sup_{|y|\le x} Y(t,y)}{f(x)} = \infty \qquad \text{or} \qquad \limsup_{x\to\infty}\frac{\sup_{|y|\le x} Y(t,y)}{f(x)} = 0,$$

according to whether the following integral diverges or converges:

$$\int_1^\infty r^{d-1}\overline{\tau}(f(r))\,\mathrm{d}r. \tag{4.5}$$

The result says that there is no proper normalization. If $m_{2/d}(\lambda) < \infty$, then almost surely there are infinitely many peaks in $B(x) = \{y : |y| \leq x\}$ that are larger than $x^{d^2/2}(\log x)^{d/2}$ but only finitely many that are larger than $x^{d^2/2}(\log x)^{d/2+\varepsilon}$.

Remark 3. If the Lévy measure is small in the sense that $m_{2/d}(\lambda) < \infty$, then the large peaks of $Y(t,x)$ are caused by points very close to the time t. (If we remove jumps close to (t,x), this is equivalent to removing the singularity of g in (1.3). The local spatial supremum of the resulting process would have a finite moment of order $\frac{2}{d}$. In particular, its tail probability would be $o(r^{-2/d})$, which by the arguments of the proof of the theorem implies that the peaks will be of smaller order.) However, if $\overline{\lambda} \in \mathcal{RV}_{-\alpha}$ with $\alpha < \frac{2}{d}$, then the peaks are caused by large jumps, which are not necessarily close to t. Indeed, assume that the integral in (4.5) diverges. For some $\delta \in (0,t)$ and large $K > 0$ define

$$\widetilde{A}_n = \{\mu(\{(s,y,z)\colon s \leq t-\delta,\ |y| \in [n,n+1],\ z > Kf(n)\}) \geq 1\}.$$

Then as $n \to \infty$,

$$\mathbf{P}(\widetilde{A}_n) \sim Cn^{d-1}\overline{\lambda}(f(n)) \sim Cn^{d-1}\overline{\tau}(f(n)),$$

showing that $\sum_{n=1}^{\infty} \mathbf{P}(\widetilde{A}_n) = \infty$. By the second Borel–Cantelli lemma $\widetilde{A}_n$ occurs infinitely often.

In line with the previous remark we show in our next and final result that the largest peaks of $x \mapsto Y(t,x)$ are typically *not* attained at integer locations if $m_{2/d}(\lambda) < \infty$. To this end, introduce the process

$$Y_0(t,x) = \begin{cases} \displaystyle\int_0^t \int_{\mathbb{R}} \int_{(0,\infty)} g(t-s,x-y) \mathbb{1}_{\{|x-y| \leq \frac{1}{2},\ g(t-s,x-y)z > 1\}}\, \mu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z), & \text{if } d = 1, \\ \displaystyle\int_0^t \int_{\mathbb{R}} \int_{(0,\infty)} g(t-s,x-y) \mathbb{1}_{\{|x-y| \leq \frac{1}{2}\}}\, \mu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z), & \text{if } d \geq 2, \end{cases} \tag{4.6}$$

which is infinitely divisible with Lévy measure

$$\eta_0(B) = \nu(\{(s,y,z) : s \leq t,\ |y| \leq \tfrac{1}{2},\ g(s,y)z \in B \cap (\mathbb{1}_{\{d=1\}}, \infty)\}).$$

Theorem 7. *Let $f : (0,\infty) \to (0,\infty)$ be nondecreasing and assume that* (1.7) *holds. If $d = 1$, further assume* (3.10)*. Then*

$$\int_1^{\infty} r^{d-1}\overline{\eta}(f(r))\,\mathrm{d}r < \infty \implies \limsup_{x\to\infty} \frac{\max_{y\in\mathbb{Z}^d, |y|\leq x} Y(t,y)}{f(x)} = 0,$$

$$\int_1^{\infty} r^{d-1}\overline{\eta}_0(f(r))\,\mathrm{d}r = \infty \implies \limsup_{x\to\infty} \frac{\max_{y\in\mathbb{Z}^d, |y|\leq x} Y(t,y)}{f(x)} = \infty.$$

The result above is optimal if $\overline{\eta}(r) \asymp \overline{\eta}_0(r)$ (i.e., $0 < \liminf_{r\to\infty} \frac{\overline{\eta}_0(r)}{\overline{\eta}(r)} \leq \limsup_{r\to\infty} \frac{\overline{\eta}_0(r)}{\overline{\eta}(r)} < \infty$). We end with a sufficient condition for the asymptotic equivalence of $\overline{\eta}$ and $\overline{\eta}_0$ and an example where they are not.

Lemma 5. *(i) If $m_{1+2/d}(\lambda) < \infty$, or if there exist $\delta > 0$ and $C > 0$ such that for $r, y > 1$ large enough*

$$\frac{\overline{\lambda}(ry)}{\overline{\lambda}(r)} \leq Cy^{-\delta}, \tag{4.7}$$

then $\overline{\eta}_0(r) \asymp \overline{\eta}(r)$ as $r \to \infty$.

(ii) Under the assumptions of Lemma 2 (iii) we have $\overline{\eta}_0(r) = o(\overline{\eta}(r))$ as $r \to \infty$.

5 Proofs

5.1 Proofs for Section 1

Proof of Theorem 1. We start with the part (ii). By standard results on Poisson integrals (see e.g. [19, Thm. 2.7]), the integral in (1.9) exists a.s. iff

$$\iiint (1 \wedge g(s,y)z) \,\mathrm{d}s \,\mathrm{d}y \,\lambda(\mathrm{d}z) < \infty,$$

where $\iiint = \int_0^t \int_{\mathbb{R}^d} \int_{(0,\infty)}$. For any $u > 0$

$$\begin{aligned} g(s,y) \le u \iff s \ge (2\pi\kappa u^{2/d})^{-1} =: H_1(u) \quad \text{or} \quad \Big\{ s \in (0, H_1(u)) \\ \text{and } |y| \ge \sqrt{-\kappa ds \log(2\pi\kappa s u^{2/d})} = \sqrt{\kappa ds \log(H_1(u)/s)} =: H_2(s,u) \Big\}. \end{aligned} \tag{5.1}$$

Note that if $z \le (2\pi\kappa t)^{d/2} =: D$, then $H_1(1/z) \le t$. Let

$$\begin{aligned} A_1 &= \{(s,y,z) : z \le D, s \le H_1(1/z), |y| \le H_2(s,1/z)\}, \\ A_2 &= \{(s,y,z) : z > D, s \le t, |y| \le H_2(s,1/z)\} \end{aligned}$$

and

$$\begin{aligned} B_1 &= \{(s,y,z) : t \ge s > H_1(1/z)\}, \\ B_{21} &= \{(s,y,z) : z \le D, s \le H_1(1/z), |y| > H_2(s,1/z)\}, \\ B_{22} &= \{(s,y,z) : z > D, s \le t, |y| > H_2(s,1/z)\}. \end{aligned}$$

Then $A_1, A_2, B_1, B_{21}, B_{22}$ form a partition of $(0,t] \times \mathbb{R}^d \times (0,\infty)$. Moreover, by (5.1), $1 \le g(s,y)z$ iff $(s,y,z) \in A_1 \cup A_2$.

Consider the upper incomplete gamma function $\Gamma(s,x) = \int_x^\infty u^{s-1} e^{-u} \,\mathrm{d}u$. For $r \le H_1(1/z)$, by a change of variables $v = \log(H_1(1/z)/s)$,

$$\begin{aligned} \int_0^r H_2(s,1/z)^d \,\mathrm{d}s &= \int_0^r \left(\kappa ds \log \frac{H_1(1/z)}{s} \right)^{d/2} \mathrm{d}s \\ &= (\kappa d)^{d/2} H_1(1/z)^{d/2+1} \int_{\log \frac{H_1(1/z)}{r}}^\infty e^{-v(d/2+1)} v^{d/2} \,\mathrm{d}v \\ &= (\kappa d)^{d/2} \left(\frac{2}{d+2} \right)^{d/2+1} H_1(1/z)^{d/2+1} \Gamma(\tfrac{d}{2}+1, (\tfrac{d}{2}+1)\log(H_1(1/z)/r)). \end{aligned} \tag{5.2}$$

Therefore, on A_1, after simplifying the constant,

$$\begin{aligned} \iiint_{A_1} (1 \wedge g(s,y)z) \,\mathrm{d}s \,\mathrm{d}y \,\lambda(\mathrm{d}z) &= \int_{(0,D]} \int_0^{H_1(1/z)} v_d H_2(s,1/z)^d \,\mathrm{d}s \,\lambda(\mathrm{d}z) \\ &= \frac{d^{d/2}}{\pi\kappa(d+2)^{d/2+1}} \int_{(0,D]} z^{1+2/d} \,\lambda(\mathrm{d}z), \end{aligned}$$

where $v_d = \pi^{d/2}/\Gamma(\frac{d}{2}+1)$ is the volume of the unit ball $B(1)$. We see that this integral is finite iff $\int_{(0,1]} z^{1+2/d}\,\lambda(\mathrm{d}z) < \infty$. On A_2,

$$\iiint_{A_2} (1 \wedge g(s,y)z)\,\mathrm{d}s\,\mathrm{d}y\,\lambda(\mathrm{d}z)$$
$$= \frac{d^{d/2}}{\Gamma(\frac{d}{2}+1)\pi\kappa(d+2)^{d/2+1}} \int_{(D,\infty)} z^{1+2/d}\Gamma\Big(\tfrac{d}{2}+1, (\tfrac{d}{2}+1)\log\frac{z^{2/d}}{2\pi\kappa t}\Big)\,\lambda(\mathrm{d}z).$$

Since $\Gamma(\frac{d}{2}+1,u) \sim e^{-u}u^{d/2}$ as $u \to \infty$,

$$z^{1+2/d}\Gamma\Big(\tfrac{d}{2}+1, (\tfrac{d}{2}+1)\log\frac{z^{2/d}}{2\pi\kappa t}\Big) \sim (2\pi\kappa t)^{d/2+1}(1+\tfrac{2}{d})^{d/2}\,(\log z)^{d/2},$$

as $z \to \infty$, which implies that

$$\iiint_{A_2} (1 \wedge g(s,y)z)\,\mathrm{d}s\,\mathrm{d}y\,\lambda(\mathrm{d}z) < \infty \iff \int_{(1,\infty)} (\log z)^{d/2}\,\lambda(\mathrm{d}z) < \infty.$$

On B_1,

$$\iiint_{B_1} g(s,y)z\,\mathrm{d}s\,\mathrm{d}y\,\lambda(\mathrm{d}z) = \int_{(0,D]} z(t - H_1(1/z))\,\lambda(\mathrm{d}z),$$

which is finite iff $\int_{(0,1]} z\,\lambda(\mathrm{d}z) < \infty$. For any $h > 0$,

$$\int_{|y|>h} g(s,y)\,\mathrm{d}y = dv_d(2\pi\kappa s)^{-d/2}\int_h^\infty e^{-\frac{r^2}{2\kappa s}} r^{d-1}\,\mathrm{d}r = \frac{\Gamma(\frac{d}{2},\frac{h^2}{2\kappa s})}{\Gamma(\frac{d}{2})}. \tag{5.3}$$

Furthermore, for any $a > 0$,

$$\int_0^a \Gamma(\tfrac{d}{2}, \tfrac{d}{2}\log\tfrac{a}{s})\,\mathrm{d}s = a\Gamma(\tfrac{d}{2})(1-(1+\tfrac{2}{d})^{-d/2}). \tag{5.4}$$

Therefore, by (5.3) and (5.4),

$$\iiint_{B_{21}} g(s,y)z\,\mathrm{d}s\,\mathrm{d}y\,\lambda(\mathrm{d}z) = (2\pi\kappa)^{-1}(1-(1+\tfrac{2}{d})^{-d/2})\int_{(0,D]} z^{1+2/d}\,\lambda(\mathrm{d}z).$$

Finally, on B_{22}, we use (5.3) and the asymptotics $\Gamma(\frac{d}{2},u) \sim e^{-u}u^{d/2-1}$ to obtain that

$$\iiint_{B_{22}} g(s,y)z\,\mathrm{d}s\,\mathrm{d}y\,\lambda(\mathrm{d}z) < \infty \iff \int_{(1,\infty)} (\log z)^{d/2-1}\,\lambda(\mathrm{d}z) < \infty.$$

By [19, Thm. 2.7 (ii)] the characteristic function of the integral in (1.9) is

$$\exp\Big\{-\iiint (1-e^{\mathrm{i}\theta g(s,y)z})\,\mathrm{d}s\,\mathrm{d}y\,\lambda(\mathrm{d}z)\Big\} = \exp\Big\{-\int_{(0,\infty)} (1-e^{\mathrm{i}\theta u})\,\eta(\mathrm{d}u)\Big\}. \tag{5.5}$$

To prove the existence of $Y(t,x)$ defined as a compensated integral, we use the stochastic integration theory of [23]. By writing

$$\begin{aligned} Y(t,x) &= mt + \int_0^t \int_{\mathbb{R}^d} \int_{(0,1]} g(t-s, x-y) z\, (\mu - \nu)(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \int_{(1,\infty)} g(t-s, x-y) z\, \mu(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) \\ &=: mt + Y_s(t,x) + Y_b(t,x) \end{aligned} \tag{5.6}$$

and the previously proved existence result for $Y_b(t,x)$, it is enough to deal with $Y_s(t,x)$, that is, we may assume that there are only small jumps. Spelling out [23, Thm. 2.7] to our setting, we obtain that $Y(t,x)$ exists iff

$$\int_0^t \int_{\mathbb{R}^d} \int_{(0,1]} g(s,y) z \mathbb{1}(g(s,y)z > 1)\, \mathrm{d}s\, \mathrm{d}y\, \lambda(\mathrm{d}z) < \infty, \tag{5.7}$$

and

$$\int_0^t \int_{\mathbb{R}^d} \int_{(0,1]} (1 \wedge g(s,y)^2 z^2)\, \mathrm{d}s\, \mathrm{d}y\, \lambda(\mathrm{d}z) < \infty. \tag{5.8}$$

To check (5.7), as in (5.3) write

$$\int_{|y| \le h} g(s,y)\, \mathrm{d}y = \frac{1}{\Gamma(\frac{d}{2})} \int_0^{\frac{h^2}{2\kappa s}} e^{-u} u^{d/2-1}\, \mathrm{d}u.$$

Thus, as in (5.4),

$$\int_0^{H_1(1/z)} \int_0^{H_2^2(s,1/z)/(2\kappa s)} e^{-u} u^{d/2-1}\, \mathrm{d}u\, \mathrm{d}s = H_1(1/z)(1 + \tfrac{2}{d})^{-d/2} \Gamma(\tfrac{d}{2}),$$

which gives that

$$\int_0^t \int_{\mathbb{R}^d} \int_{(0,1]} g(s,y) z \mathbb{1}(g(s,y)z > 1)\, \mathrm{d}s\, \mathrm{d}y\, \lambda(\mathrm{d}z) = (1 + \tfrac{2}{d})^{-d/2} \int_{(0,1]} z H_1(1/z)\, \lambda(\mathrm{d}z).$$

The latter integral exists iff $\int_{(0,1]} z^{1+2/d}\, \lambda(\mathrm{d}z) < \infty$.

For (5.8), by the previous calculations, we only have to deal with the integral on $B_1 \cup B_{21}$. As

$$\int_{\mathbb{R}^d} g(s,y)^2\, \mathrm{d}y = 2^{-d} (\pi\kappa)^{-d/2} s^{-d/2},$$

we obtain that

$$\iiint_{B_1} g(s,y)^2 z^2\, \mathrm{d}s\, \mathrm{d}y\, \lambda(\mathrm{d}z) < \infty$$

iff the second part of (1.7) holds. Finally, for $h > 0$,

$$\int_{|y| > h} g(s,y)^2\, \mathrm{d}y = \frac{\Gamma(\frac{d}{2}, \frac{h^2}{\kappa s})}{2^d (\pi\kappa s)^{d/2} \Gamma(\frac{d}{2})},$$

and for $a > 0$,

$$\int_0^a s^{-d/2}\Gamma(\tfrac{d}{2}, d\log\tfrac{a}{s})\,\mathrm{d}s = a^{1-d/2}\Gamma(\tfrac{d}{2})\frac{2(\frac{1}{2}+\frac{1}{d})^{-d/2}-2}{d-2},$$

where the last fraction is $\frac{1}{2}$ if $d = 2$. Thus,

$$\iiint_{B_{21}} g(s,y)^2 z^2\,\mathrm{d}s\,\mathrm{d}y\,\lambda(\mathrm{d}z) = 2^{-d}(\pi\kappa)^{-d/2}\frac{2(\frac{1}{2}+\frac{1}{d})^{-d/2}-2}{d-2}\int_{(0,1]} z^2 H_1(1/z)^{1-d/2}\,\lambda(\mathrm{d}z),$$

which is finite iff $\int_{(0,1]} z^{1+2/d}\,\lambda(\mathrm{d}z) < \infty$. In summary, (5.7) and (5.8) hold iff (1.7) holds.

By [23, Thm. 2.7 (iv)], the characteristic function of $Y_s(t,x)$ is

$$\mathbf{E}[e^{\mathrm{i}\theta Y_s(t,x)}] = \exp\Big\{-\mathrm{i}\theta\int_0^t\int_{\mathbb{R}^d}\int_{(0,1]}\mathbb{1}(g(s,y)z>1)g(s,y)z\,\mathrm{d}s\,\mathrm{d}y\,\lambda(\mathrm{d}z) + \int_0^\infty\Big(e^{\mathrm{i}\theta u}-1-\mathrm{i}\theta(u\wedge 1)\Big)\,\eta(\mathrm{d}u)\Big\}.$$

Combining with (5.5), we obtain (1.8). □

5.2 Proofs for Section 2

Proof of Lemma 1. By (5.1) and (5.2) and Fubini's theorem, we have

$$\begin{aligned}\overline{\eta}(r) &= \int_{(0,\infty)}\int_0^{H_1(r/z)\wedge t} v_d H_2(s,r/z)^d\,\mathrm{d}s\,\lambda(\mathrm{d}z)\\ &= \int_{(0,\infty)}\frac{d^{d/2}}{\pi\kappa(d+2)^{d/2+1}\Gamma(\frac{d}{2}+1)}\left(\frac{z}{r}\right)^{1+2/d}\Gamma\left(\tfrac{d}{2}+1,(\tfrac{d}{2}+1)\log\frac{H_1(r/z)}{H_1(r/z)\wedge t}\right)\lambda(\mathrm{d}z) \qquad (5.9)\\ &= r^{-(1+2/d)}\frac{d^{d/2}}{\pi\kappa(d+2)^{d/2+1}\Gamma(\frac{d}{2}+1)}\int_0^\infty e^{-u}u^{d/2}M_{1+2/d}(Dre^{ud/(d+2)})\,\mathrm{d}u,\end{aligned}$$

proving the exact formula for $\overline{\eta}(r)$. □

Proof of Lemma 2. (i) If $m_{1+2/d}(\lambda) < \infty$, the asymptotic result follows immediately from (2.2).

(ii) Integration by parts gives for any $\gamma > 0$ that

$$M_\gamma(r) = \int_{(0,r]} z^\gamma\,\lambda(\mathrm{d}z) = \int_0^r \gamma u^{\gamma-1}\overline{\lambda}(u)\,\mathrm{d}u - r^\gamma\overline{\lambda}(r). \qquad (5.10)$$

Thus, as $r \to \infty$, we have by Karamata's theorem (see [3, Prop. 1.5.8 and 1.5.9a]) that for $\gamma > \alpha$,

$$M_\gamma(r) \sim r^\gamma\overline{\lambda}(r)\frac{\alpha}{\gamma-\alpha} = r^{\gamma-\alpha}\ell(r)\frac{\alpha}{\gamma-\alpha}, \qquad (5.11)$$

while for $\gamma = \alpha$,

$$M_\alpha(r) \sim \alpha\int_1^r \ell(y)y^{-1}\,\mathrm{d}y = \alpha L(r), \qquad (5.12)$$

where L is slowly varying and $L(r)/\ell(r) \to \infty$ as $r \to \infty$.

By (5.11) with $\gamma = 1 + \frac{2}{d}$ (or (5.12) for $\alpha = 1 + \frac{2}{d}$) and properties of slowly varying functions,

$$M_{1+2/d}(r) \int_0^\infty e^{-u} u^{d/2} \frac{M_{1+2/d}(Dre^{ud/(d+2)})}{M_{1+2/d}(r)} \, \mathrm{d}u \sim M_{1+2/d}(r) \int_0^\infty e^{-u} u^{d/2} (De^{ud/(d+2)})^{1+2/d-\alpha} \, \mathrm{d}u$$
$$= M_{1+2/d}(r) D^{1+2/d-\alpha} \left(\frac{d+2}{\alpha d} \right)^{d/2+1} \Gamma(\tfrac{d}{2}+1),$$

where the use of Lebesgue's dominated convergence theorem is justified by Potter's bounds.

(iii) For $\alpha = 0$ the truncated moment $M_{1+2/d}$ is not necessarily regularly varying, therefore more care is needed. First we analyze L_0, which is finite by (1.7) and satisfies, for any large K,

$$L_0(r) \geq \int_1^K \ell(ry) y^{-1} (\log y)^{d/2-1} \, \mathrm{d}y \sim \ell(r) \int_1^K y^{-1} (\log y)^{d/2-1} \, \mathrm{d}y.$$

Since the latter integral goes to infinity as $K \to \infty$, we obtain that $L_0(r)/\ell(r) \to \infty$ as $r \to \infty$. Next, for $a > 1$,

$$L_0(ar) = \int_a^\infty \ell(ry) y^{-1} (\log y/a)^{d/2-1} \, \mathrm{d}y,$$

thus

$$L_0(r) - L_0(ar) = \int_1^a \ell(ry) y^{-1} (\log y)^{d/2-1} \, \mathrm{d}y + \int_a^\infty \ell(ry) y^{-1} \Big((\log y)^{d/2-1} - (\log y/a)^{d/2-1} \Big) \, \mathrm{d}y,$$

which implies

$$\lim_{r \to \infty} \frac{L_0(r) - L_0(ar)}{L_0(r)} = 0,$$

that is, $L_0(r)$ is indeed slowly varying. Furthermore, for any $a > 1$,

$$L_0(r) \sim \int_a^\infty \ell(ry) y^{-1} (\log y)^{d/2-1} \, \mathrm{d}y, \qquad r \to \infty. \tag{5.13}$$

Next we turn to $\overline{\eta}(r)$. Changing variables $y = Dre^{ud/(d+2)}$ in (2.2), we obtain

$$\overline{\eta}(r) = \frac{D^{1+2/d}}{d\pi\kappa\Gamma(\frac{d}{2}+1)} \int_{Dr}^\infty y^{-2-2/d} \left(\log \frac{y}{Dr} \right)^{d/2} M_{1+2/d}(y) \, \mathrm{d}y.$$

By Fubini's theorem,

$$\int_r^\infty y^{-2-2/d} (\log y/r)^{d/2} M_{1+2/d}(y) \, \mathrm{d}y = \int_{(0,r]} z^{1+2/d} \lambda(\mathrm{d}z) \, r^{-1-2/d} \int_1^\infty u^{-2-2/d} (\log u)^{d/2} \, \mathrm{d}u$$
$$+ \int_{(r,\infty)} z^{1+2/d} r^{-1-2/d} \int_{z/r}^\infty u^{-2-2/d} (\log u)^{d/2} \, \mathrm{d}u \, \lambda(\mathrm{d}z)$$
$$= r^{-1-2/d} \int_{(0,\infty)} z^{1+2/d} f(1 \vee z/r) \, \lambda(\mathrm{d}z),$$

where $a \vee b = \max\{a, b\}$ and $f(y) = \int_y^\infty u^{-2-2/d} (\log u)^{d/2}\,\mathrm{d}u$. Using the fundamental theorem of calculus to write $z^{1+2/d} f(1 \vee z/r)$ as an integral, exchanging the two resulting integrals by Fubini's theorem, and changing variables $y = z/r$, we obtain

$$\begin{aligned} &r^{-1-2/d} \int_{(0,\infty)} z^{1+2/d} f(1 \vee z/r)\,\lambda(\mathrm{d}z) \\ &\quad = \int_0^\infty \overline{\lambda}(ry) \Big[(1+\tfrac{2}{d}) y^{2/d} f(1 \vee y) - \mathbb{1}(y>1) (\log y)^{d/2} y^{-1} \Big]\,\mathrm{d}y \\ &\quad = \int_0^1 \overline{\lambda}(ry)(1+\tfrac{2}{d}) y^{2/d} f(1)\,\mathrm{d}y \\ &\qquad + \int_1^\infty \overline{\lambda}(ry)(1+\tfrac{2}{d}) y^{2/d} \int_y^\infty u^{-2-2/d} \Big[(\log u)^{d/2} - (\log y)^{d/2} \Big]\,\mathrm{d}u\,\mathrm{d}y. \end{aligned}$$

Using that

$$(1+\tfrac{2}{d}) y^{2/d} \int_y^\infty u^{-2-2/d} \Big[(\log u)^{d/2} - (\log y)^{d/2} \Big]\,\mathrm{d}u = \tfrac{d}{2} y^{2/d} \int_y^\infty v^{-2-2/d} (\log v)^{d/2-1}\,\mathrm{d}v,$$

we end up with

$$\begin{aligned} \overline{\eta}(r/D) = \frac{D^{1+2/d}}{d\pi\kappa\Gamma(\frac{d}{2}+1)} \Big[&\int_0^1 \overline{\lambda}(ry)(1+\tfrac{2}{d}) y^{2/d} f(1)\,\mathrm{d}y \\ &+ \int_1^\infty \overline{\lambda}(ry) \tfrac{d}{2} y^{2/d} \int_y^\infty v^{-2-2/d} (\log v)^{d/2-1}\,\mathrm{d}v\,\mathrm{d}y \Big]. \end{aligned} \tag{5.14}$$

As $y \to \infty$,

$$y^{2/d} \int_y^\infty v^{-2-2/d} (\log v)^{d/2-1}\,\mathrm{d}v \sim (1+\tfrac{2}{d})^{-1} y^{-1} (\log y)^{d/2-1},$$

so for K large enough,

$$\begin{aligned} \int_K^\infty \overline{\lambda}(ry) y^{2/d} \int_y^\infty v^{-2-2/d} (\log v)^{d/2-1}\,\mathrm{d}v\,\mathrm{d}y &\sim (1+\tfrac{2}{d})^{-1} \int_K^\infty \overline{\lambda}(ry) y^{-1} (\log y)^{d/2-1}\,\mathrm{d}y \\ &\sim (1+\tfrac{2}{d})^{-1} L_0(r), \end{aligned}$$

where the last asymptotic follows from (5.13). Since $\ell(r)/L_0(r) \to 0$ as $r \to \infty$,

$$\overline{\eta}(r/D) \sim \frac{D^{1+2/d}}{2\pi\kappa\Gamma(\frac{d}{2}+1)(1+\frac{2}{d})} L_0(r),$$

as claimed. □

Proof of Theorem 2. (i) Starting from the first line of (5.9), we can also write $\overline{\eta}(r)$ as

$$\begin{aligned} \overline{\eta}(r) &= v_d (\kappa d)^{d/2} \int_0^t s^{d/2} \int_{(r(2\pi\kappa s)^{d/2},\infty)} \left(\log \frac{z^{2/d}}{2\pi\kappa s r^{2/d}} \right)^{d/2} \lambda(\mathrm{d}z)\,\mathrm{d}s \\ &= \frac{(2t)^{1+d/2} \kappa^{d/2} v_d}{d} r^{-1-2/d} \int_0^r v^{2/d} \int_{(Dv,\infty)} \left(\log \frac{z}{Dv} \right)^{d/2} \lambda(\mathrm{d}z)\,\mathrm{d}v, \end{aligned} \tag{5.15}$$

where we changed variables $v = (s/t)^{d/2} r$ to go from the first to the second line. By the fundamental theorem of calculus we have $\log \overline{\eta}(r) = C + \int_1^r \xi(v) v^{-1}\,\mathrm{d}v$ with

$$C = \log \frac{(2t)^{1+d/2}\kappa^{d/2} v_d}{d} + \log \int_0^1 v^{2/d} \int_{(Dv,\infty)} \left(\log \frac{z}{Dv}\right)^{d/2} \lambda(\mathrm{d}z)\,\mathrm{d}v,$$

$$\xi(v) = \frac{v^{1+2/d} \int_{(Dv,\infty)} (\log \frac{z}{Dv})^{d/2}\,\lambda(\mathrm{d}z)}{\int_0^v u^{2/d} \int_{(Du,\infty)} (\log \frac{z}{Du})^{d/2}\,\lambda(\mathrm{d}z)\,\mathrm{d}u} - (1 + \tfrac{2}{d}).$$

Since $u \mapsto \int_{(Du,\infty)} (\log \frac{z}{Du})^{d/2}\,\lambda(\mathrm{d}z)$ is decreasing in u, we have $-(1+\frac{2}{d}) \le \xi(v) \le 0$. The claim now follows from [3, Thm. 2.2.6].

(ii) By (i) and [3, Thm. 2.0.7], $\overline{\eta}$ has dominated variation [3, p. 54] and $\overline{\eta}(r+s)/\overline{\eta}(r) \to 1$ as $r \to \infty$ for any $s > 0$. Hence $\overline{\eta}$ is subexponential [15, Thm. 1] and (2.7) follows from (2.5) and [21, Thm. 3.1] (see also [22, Thm. 5.1]).

(iii) By the representation theorem of regularly varying functions $\overline{\eta} \in \mathcal{RV}_{-\alpha}$ iff $\lim_{r\to\infty} \xi(r) = 1 + \frac{2}{d} - \alpha$. By Karamata's theorem ([3, Thm. 1.6.1]) this holds for $\alpha < 1 + \frac{2}{d}$ iff $f \in \mathcal{RV}_{-\alpha}$, where

$$f(r) = \int_{(r,\infty)} \left(\log \frac{z}{r}\right)^{d/2} \lambda(\mathrm{d}z) = \frac{d}{2} \int_r^\infty \frac{\overline{\lambda}(z)}{z} \left(\log \frac{z}{r}\right)^{d/2-1} \mathrm{d}z. \tag{5.16}$$

Consider the kernel $k(u) = (\log u^{-1})^{d/2-1} \mathbb{1}_{(0,1)}(u)$. Define the *Mellin convolution* of f_1 and f_2 by

$$f_1 \overset{\mathrm{M}}{*} f_2(r) = \int_0^\infty f_1(r/t) f_2(t) t^{-1}\,\mathrm{d}t,$$

see e.g. [3, Sect. 4]. With this notation $f(r) = k \overset{\mathrm{M}}{*} \overline{\lambda}(r)$. The *Mellin transform* of k, that is,

$$\breve{k}(z) = \int_0^1 t^{-z-1} (\log t)^{d/2-1}\,\mathrm{d}t = \mathrm{i}^d \sqrt{z}^{-d}\,\Gamma(\tfrac{d}{2}),$$

is defined and nonzero whenever $\Re z < 0$ and $\sqrt{z}$ is chosen such that $\arg(\sqrt{z}) \in (\frac{1}{4}\pi, \frac{3}{4}\pi)$. Therefore, we can apply [3, Thm. 4.8.3]. (It is easy to check that the Tauberian condition is satisfied since $\overline{\lambda}$ is decreasing; see also [3, Exercise 1.11.14].) Therefore, $k \overset{\mathrm{M}}{*} \overline{\lambda} \in \mathcal{RV}_{-\alpha}$ implies that $\overline{\lambda} \in \mathcal{RV}_{-\alpha}$, as claimed. The other direction was proved in Lemma 2.

If $\alpha = 1 + \frac{2}{d}$, then $\lim_{r\to\infty} \xi(r) = 0$ iff

$$\frac{r^{1+2/d} f(r)}{\int_0^r u^{2/d} f(u)\,\mathrm{d}u} \to 0,$$

which holds iff $\int_0^r u^{2/d} f(u)\,\mathrm{d}u$ is slowly varying, see [3, Thm. 8.3.1] or [16, Thm. 1.1]. Using the first identity in (5.16), we can easily verify that the latter holds if $m_{1+2/d}(\lambda) < \infty$. □

5.3 Proofs for Section 3

Proof of Theorem 3. Without loss of generality, assume $\kappa = \frac{1}{2\pi}$. If $d = 1$, then by [19, Thm. 2.7] $X_A(t)$ exists iff

$$\iiint \mathbb{1}(s^{-d/2} z > 1,\ y \in \overline{A})\,\mathrm{d}s\,\mathrm{d}y\,\lambda(\mathrm{d}z) = |\overline{A}| \int_{(0,\infty)} (z^2 \wedge t)\,\lambda(\mathrm{d}z) < \infty,$$

which is true for any Lévy measure. If $d \geq 2$, $X_A(t)$ exists iff

$$\iiint (1 \wedge s^{-d/2} z \mathbb{1}(y \in \overline{A}))\, \mathrm{d}s\, \mathrm{d}y\, \lambda(\mathrm{d}z) = |\overline{A}| \int_0^t \int_{(0,\infty)} (1 \wedge s^{-d/2} z)\, \mathrm{d}s\, \lambda(\mathrm{d}z) < \infty.$$

For $z \leq t^{d/2}$, we have $\int_0^t (1 \wedge s^{-d/2} z)\, \mathrm{d}s = z^{2/d} + z \int_{z^{2/d}}^t s^{-d/2}\, \mathrm{d}s$, while for $z > t^{d/2}$, we have $\int_0^t (1 \wedge s^{-d/2} z)\, \mathrm{d}s = t$. Thus,

$$\int_0^t \int_{(0,\infty)} (1 \wedge s^{-d/2} z)\, \mathrm{d}s\, \lambda(\mathrm{d}z) = \int_{(0,t^{d/2}]} z^{2/d}\, \lambda(\mathrm{d}z) + \int_{(0,t^{d/2}]} z \int_{z^{2/d}}^t s^{-d/2}\, \mathrm{d}s\, \lambda(\mathrm{d}z) + t\, \overline{\lambda}(t^{d/2}),$$

which is finite iff (3.3) holds. The identity (3.4) follows from [19, Thm. 2.7 (ii)]. □

Proof of Lemma 3. (i) is an immediate consequence of (3.5). (ii) follows from (3.5) combined with Karamata's theorem. □

Proof of Theorem 4. Recall (3.7). Then, as in Theorem 2, claims (i) and (ii) follow by writing $\log \overline{\tau}(r) = C + \int_1^r \xi(u) u^{-1}\, \mathrm{d}u$ with

$$C = -\log(\pi\kappa d) + \log \int_0^D u^{2/d-1} \overline{\lambda}(u)\, \mathrm{d}u, \qquad \xi(u) = \frac{(Du)^{2/d} \overline{\lambda}(Du)}{\int_0^{Du} v^{2/d-1} \overline{\lambda}(v)\, \mathrm{d}v} - \tfrac{2}{d},$$

where ξ satisfies $-\frac{2}{d} < \xi(u) \leq 0$. For (iii), we have as in Theorem 2 that $\overline{\tau} \in \mathcal{RV}_{-\alpha}$ iff $\lim_{r\to\infty} \xi(r) = 2/d - \alpha$, which for $\alpha < \frac{2}{d}$ holds iff $\overline{\lambda} \in \mathcal{RV}_{-\alpha}$, as claimed. If $\alpha = \frac{2}{d}$ then using [3, Thm. 8.3.1] $\lim_{r\to\infty} \xi(r) = 0$ iff $\int_0^r u^{2/d-1} \overline{\lambda}(u)\, \mathrm{d}u$ is slowly varying. This holds if $m_{2/d}(\lambda) < \infty$ or $\overline{\lambda} \in \mathcal{RV}_{-2/d}$. □

Proof of Lemma 4. (i) If $m_{2/d}(\lambda) < \infty$, choose $\varepsilon > 0$ and observe that for $r > 1$,

$$\begin{aligned} \overline{\eta}_A(r) &\leq \iiint \mathbb{1}((2\pi\kappa s)^{-d/2} z > r) \mathbb{1}_{\{y \in A^\varepsilon\}}\, \mathrm{d}s\, \mathrm{d}y\, \lambda(\mathrm{d}z) \\ &\quad + \iiint \mathbb{1}\Big((2\pi\kappa s)^{-d/2} e^{-\frac{\mathrm{dist}(y,A)^2}{2\kappa s}} z > r\Big) \mathbb{1}_{\{y \notin A^\varepsilon\}}\, \mathrm{d}s\, \mathrm{d}y\, \lambda(\mathrm{d}z) \\ &\leq |A^\varepsilon| \overline{\tau}(r) + \iiint \mathbb{1}\Big((2\pi\kappa s)^{-d/2} e^{-\frac{\mathrm{dist}(y,A)^2}{2\kappa s}} z > r\Big) \mathbb{1}_{\{y \notin A^\varepsilon\}}\, \mathrm{d}s\, \mathrm{d}y\, \lambda(\mathrm{d}z). \end{aligned} \tag{5.17}$$

where $A^\varepsilon = \{x \in \mathbb{R}^d : \mathrm{dist}(x, A) < \varepsilon\}$. Since $\iiint (2\pi\kappa s)^{-1} e^{-\frac{\mathrm{dist}(y,A)^2}{d\kappa s}} \mathbb{1}_{\{y \notin A^\varepsilon\}} z^{2/d}\, \mathrm{d}s\, \mathrm{d}y\, \lambda(\mathrm{d}z) < \infty$, the last term in the previous display is $o(r^{-2/d})$, which together with Lemma 3 (i) shows that $\limsup_{r\to\infty} \overline{\eta}_A(r)/\overline{\tau}(r) \leq |A^\varepsilon|$, which converges to $|\overline{A}|$ by letting $\varepsilon \to 0$. The opposite relation follows from the fact that

$$\overline{\eta}_A(r) \geq \iiint \mathbb{1}((2\pi\kappa s)^{-d/2} z > r) \mathbb{1}_{\{y \in \overline{A}\}}\, \mathrm{d}s\, \mathrm{d}y\, \lambda(\mathrm{d}z) = |\overline{A}| \overline{\tau}(r).$$

If $\overline{\lambda}(r) = r^{-2/d} \ell(r)$, one can use Potter's bounds, dominated convergence and Lemma 3 (ii) to show that the last integral in (5.17) is

$$\sim r^{-2/d} \ell(r) \iiint (2\pi\kappa s)^{-1} e^{-\frac{\mathrm{dist}(y,A)^2}{d\kappa s}} \mathbb{1}_{\{y \notin A^\varepsilon\}}\, \mathrm{d}s\, \mathrm{d}y = o(\overline{\tau}(r)).$$

The remaining proof is the same as in the case $m_{2/d}(\lambda) < \infty$.

(ii) If $\overline{\lambda}(r) = r^{-\alpha}\ell(r)$ for some $\alpha \in (0, \frac{2}{d})$, a direct calculation shows that for $r > 1$,

$$\overline{\eta}_A(r) = \iiint \mathbb{1}\Big((2\pi\kappa s)^{-d/2} e^{-\frac{\mathrm{dist}(y,A)^2}{2\kappa s}} z > r\Big)\,\mathrm{d}s\,\mathrm{d}y\,\lambda(\mathrm{d}z)$$
$$\sim r^{-\alpha}\ell(r)\int_0^t \int_{\mathbb{R}^d} (2\pi\kappa s)^{-\alpha d/2} e^{-\frac{\alpha \mathrm{dist}(y,A)^2}{2\kappa s}}\,\mathrm{d}s\,\mathrm{d}y. \qquad \square$$

Proof of Theorem 5. Note that for $d \geq 2$ condition (3.3) implies summable jumps, in which case we assume that $Y(t,x)$ has the form (1.11). For $d = 1$, note that $Y(t,x) = Y'_d(t,x) + Y'_s(t,x) + Y'_b(t,x)$, where

$$\begin{aligned} Y'_d(t,x) &= mt + \iiint g(t-s,x-y)z(\mathbb{1}_{\{(2\pi\kappa(t-s))^{-1/2}z\leq 1\}} - \mathbb{1}_{\{z\leq 1\}})\,\mathrm{d}s\,\mathrm{d}y\,\lambda(\mathrm{d}z), \\ Y'_s(t,x) &= \iiint g(t-s,x-y)z\mathbb{1}_{\{(2\pi\kappa(t-s))^{-1/2}z\leq 1\}}\,(\mu-\nu)(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z), \\ Y'_b(t,x) &= \iiint g(t-s,x-y)z\mathbb{1}_{\{(2\pi\kappa(t-s))^{-1/2}z> 1\}}\,\mu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z). \end{aligned} \tag{5.18}$$

A straightforward computation shows that $Y'_d(t,x) < \infty$ for all Lévy measures λ. Furthermore, by (3.10) and the proof of [9, Thm. 3.8] one can show that

$$\mathbf{P}\Big(\sup_{x\in A}|Y'_s(t,x)| < \infty\Big) = 1. \tag{5.19}$$

For completeness, we sketch the proof. We use [20, Thm. 1] (with $\alpha = p = 2$) and Minkowski's integral inequality to obtain

$$\begin{aligned} &\mathbf{E}[|Y'_s(t,x) - Y'_s(t,x')|^2] \\ &\quad \leq C\iiint |g(t-s,x-y) - g(t-s,x'-y)|^2 z^2 \mathbb{1}_{\{(2\pi\kappa(t-s))^{-1/2}z<1\}}\,\nu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) \end{aligned}$$

for all $x, x' \in \mathbb{R}$. We have on the set $(2\pi\kappa(t-s))^{-1/2}z < 1$

$$\begin{aligned} |g(t-s,x-y) - g(t-s,x'-y)|^2 z^2 &= C(t-s)^{-1}z^2\Big|e^{-\frac{|x-y|^2}{2(t-s)}} - e^{-\frac{|x'-y|^2}{2(t-s)}}\Big|^2 \\ &\leq C|g((t-s),x-y) - g((t-s),x'-y)|^q z^q, \end{aligned}$$

where q is the exponent from (3.10) (which satisfies $q < 2$). With this estimate and again [26, Lemme A2], we conclude that

$$\mathbf{E}[|Y'_s(t,x) - Y'_s(t,x')|^2] \leq C|x-x'|^{3-q}.$$

Since $3 - q > 1$, it follows from [17, Thm. 4.3] that

$$\mathbf{E}\Big[\sup_{x\in A} Y'_s(t,x)^2\Big] \leq \mathbf{E}[Y'_s(t,0)^2] + \mathbf{E}\Big[\sup_{x,x'\in A}|Y'_s(t,x) - Y'_s(t,x')|^2\Big] < \infty,$$

which shows (5.19).

Next, choose $r > 0$ such that $A \subseteq B(r) = \{x \in \mathbb{R}^d \colon |x| \le r\}$. Then

$$\sup_{x\in A} Y_b'(t,x) \le \iiint (2\pi\kappa(t-s))^{-d/2} z \mathbb{1}_{\{y\in B(r)\}} \mathbb{1}_{\{d\ge 2 \text{ or } (2\pi\kappa(t-s))^{-1/2}z>1\}}\, \mu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)$$
$$+ \iiint (2\pi\kappa(t-s))^{-d/2} e^{-\frac{\mathrm{dist}(y,B(r))^2}{2\kappa(t-s)}} z \mathbb{1}_{\{y\notin B(r),\ d\ge 2 \text{ or } (2\pi\kappa(t-s))^{-1/2}z>1\}}\, \mu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z).$$

The first term on the right-hand side is simply $X_{B(r)}(t)$, which is finite a.s. by Theorem 3.3. The second term has the same distribution as $Y_b'(t,0)$, which is finite a.s. as well. Therefore, $\sup_{x\in A} Y(t,x) < \infty$ a.s. for all d. The assertion of the theorem now follows from Lemma 4 (which implies that $\overline{\eta}_A$ is subexponential under the stated assumptions) and [25, Thm. 3.1]. ☐

5.4 Proofs for Section 4

For $0 < r < r'$ let $B(r,r') = \{x \in \mathbb{R}^d \colon r < |x| \le r'\}$.

Proof of Theorem 6. First assume that (4.5) converges and let $K > 0$. We start with $d \ge 2$. Since $B(n,n+1)$ can be covered with $O(n^{d-1})$ many unit cubes and Y is stationary in space, Theorem 4 shows that

$$\begin{aligned}\mathbf{P}\left(\sup_{y\in B(n,n+1)} Y(t,y) > \frac{f(n)}{K}\right) &\le Cn^{d-1}\mathbf{P}\left(\sup_{y\in[0,1]^d} Y(t,y) > \frac{f(n)}{K}\right)\\ &\le Cn^{d-1}\mathbf{P}\left(X_{[0,1]^d}(t) > \frac{f(n)}{K}\right)\\ &\le 2Cn^{d-1}\overline{\tau}(f(n)/K) \le C'n^{d-1}\overline{\tau}(f(n)),\end{aligned} \tag{5.20}$$

which is summable by hypothesis. In the last inequality we also used that $\overline{\tau}$ is extended regularly varying. So by the first Borel–Cantelli lemma,

$$\sup_{y\in B(n,n+1)} Y(t,y) > \frac{f(n)}{K}$$

only happens finitely often and hence

$$\limsup_{x\to\infty} \frac{\sup_{|y|\le x} Y(t,y)}{f(x)} \le K^{-1}$$

almost surely, proving the claim since K was arbitrary. If $d = 1$, recall the decomposition (5.18). We can apply (5.20) to $Y_b'(t,x)$, while $|Y_d'(t,x) + Y_s'(t,x)|$ has a smaller tail by (5.19) (in $d = 1$, the tail of $Y_b'(t,x)$ is no smaller than Cr^{-2} by Lemma 3 (i)). Therefore, the final bound in (5.20) remains true.

For the converse statement, assume that the integral in (4.5) diverges. If $d = 1$, we consider again the decomposition (5.18). As before, we let $Y_b'(t,x) = Y(t,x)$ if $d \ge 2$. For $K > 0$ large consider the events

$$A_n = \{\mu(\{(s,y,z) : (s,z) \in V(Kf(n+1)),\ y \in B(n,n+1)\}) \ge 1\}, \quad n \ge 1. \tag{5.21}$$

By (4.2) and Theorem 4 (i),

$$\mathbf{P}(A_n) \sim v_d((n+1)^d - n^d)\overline{\tau}(Kf(n+1)) \geq Cn^{d-1}\overline{\tau}(f(n+1)).$$

Since the integral in (4.5) diverges, we have that $\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty$. Noting that the A_n's are independent, the second Borel–Cantelli lemma implies that A_n occurs infinitely often. On A_n,

$$\sup_{y \in B(n,n+1)} Y'_b(t,y) \geq Kf(n+1).$$

Thus, almost surely,

$$\limsup_{x\to\infty} \frac{\sup_{|y|\leq x} Y'_b(t,y)}{f(x)} \geq K,$$

which proves the claim for $d \geq 2$ since $K > 0$ is arbitrarily large.

If $d = 1$, note that the proof above shows that $Y'_b(t,x)$ develops infinitely many peaks larger than $x^{1/2}$ on $B(x)$ (because $\overline{\tau}(r)$ decreases no faster than shown in Lemma 3 (i)). So if we show that $|Y'_d(t,x)| + |Y'_s(t,x)|$ from (5.18) can only have finitely many peaks of that size, then the proof in $d = 1$ will be complete. For $|Y'_d(t,x)|$, this is trivial. For $|Y'_s(t,x)|$, this is a simple consequence of (5.19) and the arguments in the first part of the proof. □

Proof of Theorem 7. The upper bound proof is essentially the same as for Theorem 6, except that (5.20) should be replaced by

$$\begin{aligned}\mathbf{P}\left(\max_{y\in\mathbb{Z}^d, y\in B(n,n+1)} Y(t,y) > \frac{f(n)}{K}\right) &\leq Cn^{d-1}\mathbf{P}\left(Y(t,0) > \frac{f(n)}{K}\right)\\ &\leq Cn^{d-1}\overline{\eta}(f(n)/K) \leq C'n^{d-1}\overline{\eta}(f(n)).\end{aligned}$$

For the lower bound proof, if $d = 1$, we consider the decomposition $Y(t,x) = At + Y''_s(t,x) + Y''_b(t,x)$, where A is the same constant as in Theorem 1 and

$$\begin{aligned}Y''_s(t,x) &= \iiint g(t-s,x-y)z\mathbb{1}_{\{g(t-s,x-y)z\leq 1\}}\,(\mu-\nu)(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z),\\ Y''_b(t,x) &= \iiint g(t-s,x-y)z\mathbb{1}_{\{g(t-s,x-y)z> 1\}}\,\mu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z).\end{aligned}$$

If $d \geq 2$, we let $Y''_b(t,x) = Y(t,x)$. Clearly, $Y''_b(t,x) \geq Y_0(t,x)$ from (4.6) and $\mathbf{P}(Y_0(t,x) > r) \sim \overline{\eta}_0(r)$ similarly to Theorem 2. Because the $(Y_0(t,x))_{x\in\mathbb{Z}^d}$ are independent and

$$\sum_{n=1}^{\infty} \sum_{y\in\mathbb{Z}^d\cap B(n,n+1)} \mathbf{P}(Y_0(t,y) > Kf(n+1)) \geq C\sum_{n=1}^{\infty} n^{d-1}\overline{\eta}_0(f(n+1)K) = \infty,$$

the second Borel–Cantelli lemma shows that $Y''_b(t,x)/f(x) \geq Y_0(t,x)/f(x) \geq K$ for infinitely many $x \in \mathbb{Z}^d$. If $d = 1$, then as in the proof of Theorem 6 one can show that the peaks of $|Y''_s(t,x)|$ are of lower order. □

Proof of Lemma 5. Recall H_1 and H_2 from (5.1). For $r > 1$

$$\overline{\eta}_0(r) = \eta_0((r,\infty)) = \int_{(0,\infty)} \int_0^{H_1(r/z)\wedge t} v_d(\tfrac{1}{2} \wedge H_2(s,r/z))^d \,\mathrm{d}s\, \lambda(\mathrm{d}z).$$

For fixed $u > 0$ the map $s \mapsto H_2(s,u)$ is increasing on $[0,(2\pi\kappa e u^{2/d})^{-1}]$, and decreasing on $[(2\pi\kappa e u^{2/d})^{-1}, H_1(u)]$, with global maximum $H_2((2\pi\kappa e u^{2/d})^{-1}, u) = \sqrt{d/(2\pi e)}u^{-1/d}$. In particular, $H_2(s,u) \le \frac{1}{2}$ whenever $u \ge (2d/(\pi e))^{d/2}$. Therefore, as in the proof of Lemma 1,

$$\begin{aligned}
&\int_{(0,(\pi e/(2d))^{d/2}r]} \int_0^{H_1(r/z)\wedge t} v_d(\tfrac{1}{2} \wedge H_2(s,r/z))^d \,\mathrm{d}s\, \lambda(\mathrm{d}z) \\
&\quad = \frac{d^{d/2}}{\pi\kappa(d+2)^{d/2+1}\Gamma(\frac{d}{2}+1)} \int_{(0,(2\pi e/d)^{d/2}r]} \left(\frac{z}{r}\right)^{1+2/d} \Gamma\Big(\frac{d}{2}+1, (\frac{d}{2}+1)\log\frac{H_1(r/z)}{H_1(r/z)\wedge t}\Big)\, \lambda(\mathrm{d}z) \\
&\quad \ge c_1 r^{-1-2/d} M_{1+2/d}(c_2 r).
\end{aligned}$$

At the same time, if $u > 0$ is small enough, then

$$\int_0^{H_1(u)\wedge t} (\tfrac{1}{2} \wedge H_2(s,u))^d \,\mathrm{d}s \ge \frac{t}{3}.$$

Thus there exists c_3 such that

$$\int_{(c_3 r,\infty)} \int_0^{H_1(r/z)\wedge t} v_d(\tfrac{1}{2} \wedge H_2(s,r/z))^d \,\mathrm{d}s\, \lambda(\mathrm{d}z) \ge c_4 \overline{\lambda}(c_3 r).$$

It follows that there are finite constants $c_1, c_2, C_1, C_2 > 0$ depending only on d and t such that

$$c_1 r^{-1-2/d} M_{1+2/d}(c_2 r) + c_1\overline{\lambda}(c_2 r) \le \overline{\eta}_0(r) \le C_1 r^{-1-2/d} M_{1+2/d}(C_2 r) + C_1 \overline{\lambda}(C_2 r). \tag{5.22}$$

(The second inequality is an easy consequence of the first two displays in this proof.)

From (5.22) and Lemma 2 (i) we see that $\overline{\eta}_0(r) \asymp \overline{\eta}(r)$ whenever $m_{1+2/d}(\lambda) < \infty$. If (4.7) holds, using $\Gamma(\frac{d}{2}+1, r) \sim e^{-r} r^{d/2}$ as $r \to \infty$, we have

$$\begin{aligned}
&\int_{(rD,\infty)} \left(\frac{z}{r}\right)^{1+2/d} \Gamma\Big(\frac{d}{2}+1, (\frac{d}{2}+1)\log\frac{H_1(r/z)}{H_1(r/z)\wedge t}\Big)\, \lambda(\mathrm{d}z) \\
&\quad \le C \int_{(rD,\infty)} (\log(z/r))^{d/2}\, \lambda(\mathrm{d}z) = C \int_{rD}^\infty (\log(z/r))^{d/2-1} \overline{\lambda}(z) z^{-1} \,\mathrm{d}z \\
&\quad \le C\overline{\lambda}(r) \int_D^\infty (\log y)^{d/2-1} y^{-1-\delta} \,\mathrm{d}y = C\overline{\lambda}(r),
\end{aligned}$$

which implies $\overline{\eta}_0(r) \asymp \overline{\eta}(r)$.

On the other hand, (ii) follows from Lemma 2 (iii) and (5.22). □

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