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ON THE STATISTICAL EXAMINATION OF CONTINUOUS STATE MARKOV PROCESSES. I*

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Introduction

The statistical examination of stochastic processes has a history of only twenty years. The first significant results were those of Mann and Wald [1]; they dealt with the estimation of parameters of an n-dimensional discrete stationary Gaussian Markov process. In this context the name of John von Neumann must also be mentioned. He investigated one-dimensional discrete-time stationary Gaussian Markov processes, and he introduced the so-called serial correlation coefficient. In the fifties the first survey, written by Ulf Grenander [1], appeared. His work was of fundamental importance first of all for the theory of continuous-time processes; up to the present day it remains highly significant, which can be illustrated, for example, by its recent translation into Russian and its publication in book form. In 1948 A. N. Kolmogorov posed the problem of parameter estimation for stationary Gaussian Markov processes as one of the most important problems of the theory of stochastic processes. He wrote that the problem has a definite meaning and a specific relevance only when investigating the parameters as a whole and not one by one. A partial solution was provided in a paper by Linnik [1]. Further results in this direction were obtained by Luvsanceren [1], [2]. The complete solution of the problem as formulated by Kolmogorov and in the form in which he had assumed it could be solved was given by the author of the present paper in his dissertation [1].

In the fifties the theory of stochastic processes (or simply processes) went through a period of intensive development, above all due to its applications in radioelectronics and physics. It was during this period that the theory of processes developed within probability theory and become its nucleus. Recently a valuable book on the statistical examination of Markov processes, written by Patrick Billingsley [1], appeared; it deals mainly with discrete-time processes and has no relation to the present paper.

In the present paper knowledge of elements of mathematical statistics and the theory of stochastic processes is assumed, but nevertheless references will be

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provided whenever necessary. The restriction to Gaussian processes may at the beginning seem essential to the reader, but, as can be seen in the relevant literature, research undertaken so far is in fact concerned only with this type of processes. Although many of the results obtained are also valid in other types of processes, known applications usually meet conditions making the restriction only to Gaussian processes possible. On the other hand, we shall see that even problems concerning Gaussian processes are much more complicated than when independent observations are made.

The present paper and its sequels are in the nature of a survey; they aim at providing a complete description, and therefore they deal with parts of the theory which are commonly known together with new results. The aim of the present paper is to underline those special features of statistical problems in stochastic processes which do not appear when dealing with sequences of independent observations. The most fundamental in this investigation, for example in estimation of parameters, is the problem of asymptotic distribution of the parameter estimators. The main difficulty here is that the unknown parameters may have values not very different from those for which the process becomes singular (and in applications mostly this case occurs). For singular processes the central limit theorem is not valid, and therefore the distribution of estimators need not be, uniformly in the parameter values, asymptotically normal (even if each parameter separately has asymptotically normal distribution), which means that the limit distribution cannot be used for construction of confidence limits. This is, of course, only one way to formulate specific features of stochastic processes; other possibilities yield the language of information theory.

These problems were suggested to me by A. N. Kolmogorov during my postgraduate studies. His encouragement and constant help, together with valuable comments I received from Ja. G. Sinaĭ, made it possible for me to finish my thesis. I want to express here again my gratitude to both of them.

The present paper makes use of the basic results obtained in my thesis and contains detailed proofs of results which have been published in previous papers [2], [3] without proofs.

The Continuous-Time Stationary Normal One-Dimensional Case

§1. Characteristics of processes. Their physical meaning and mathematical description

A significant number of physical processes can be described not by the following differential equation:

$$\frac{dx(t)}{dt} = -\lambda x(t) \qquad (\lambda > 0)$$

(with a solution $x = x_0 e^{-\lambda t}$), but rather by a so-called stochastic differential equation, which is written as

$$d\xi(t) = -\lambda \xi(t)\,dt + d\zeta(t), \qquad (\mathsf{E}\zeta(t) = \mathsf{E}\xi(t) = 0), \tag{1.1}$$

where $\zeta(t)$ is a Wiener process (a Gaussian Markov process with independent increments); $\zeta(t+\tau)$ and $\xi(t)$ are independent for $\tau > 0$. Except when otherwise stated, we shall deal only with Gaussian processes. The meaning of the above differential equation can be given with the help of the following integral equation ($\xi(t)$ is its solution):

$$\xi(t) - \xi(t_0) = -\lambda \int_{t_0}^{t} \xi(s)\,ds + \zeta(t) - \zeta(t_0), \tag{1.1'}$$

where the integral is to be understood in the mean square sense; the existence and uniqueness of the solution follows from general theorems (Doob [1]; Itô [1]), of course in the mean square sense.

The stochastic process described by (1.1) differs from the process described by an ordinary differential equation in that damping is not steady but some perturbation is always present.

In the Wiener process, for example, the motion of a particle of a perfect gas or fluid is described without taking into account the velocity of the particle, but the Gaussian Markov process does take this velocity into account. Here it should be mentioned that a procedure which takes into account the term causing small perturbations has previously been applied in the theory of differential equations by Pontrjagin, Andronov and Vitt [1].

Equation (1.1) can be understood as an equation which describes the motion of a particle with velocity $\xi(t)$ under a stochastic external force with friction proportional to its velocity. In a similar way $\xi(t)$ can be interpreted as a stochastically variable potential whose growth is proportional to the potential itself.

Since we have assumed that the process is of the Gaussian type and $\mathsf{E}\xi(t) = 0$, the process itself is uniquely determined by its covariance function. We ~~shall~~ have ~~now prove the following known theorem:~~

THEOREM 1. *A one-dimensional Gaussian process* ($\mathsf{E}\xi(t) = 0$) *is of Markov type if and only if its function* $R(s, t)$ *satisfies the condition*

$$R(s, t) = R(s, u)\, R(u, t), \qquad s < u < t, \tag{1.2}$$

where $R(s, t) = \mathsf{E}\xi(s)\xi(t)/\mathrm{Var}\,\xi(s)$.

PROOF. Let the Gaussian process $\xi(t)$ be of Markov type. Then

$$\mathsf{E}\{\xi(t)|\xi(u),\xi(s)\} = \mathsf{E}\{\xi(t)|\xi(u)\} = R(u,t)\xi(u)$$

and $\xi(t) - \mathsf{E}\{\xi(t)|\xi(u), \xi(s)\}$ is orthogonal to $\xi(s)$ (if $s < u$), which we shall write as follows: $\xi(t) - \mathsf{E}\{\xi(t)|\xi(u), \xi(s)\} \perp \xi(s)$; therefore

$$\mathsf{E}\{\xi(t)\xi(s)\} = \mathsf{E}\{\xi(s)\xi(u)R(u,t)\} = R(u,t)\ \mathsf{E}\{\xi(s)\xi(u)\}.$$

After dividing by Var $\xi(s)$ we get (1.2).

Now let a real Gaussian process satisfy (1.2); then evidently

$$\mathsf{E}\{\xi(t)\xi(s)\} - R(u,t)\ \mathsf{E}\{\xi(s)\xi(u)\} = 0,$$

which means that $\xi(t) - R(u, t)\xi(u) \perp \xi(s)$ for $s < u$. Therefore almost surely

$$R(u,t)\xi(u) = \mathsf{E}\{\xi(t)|\xi(u)\} = \mathsf{E}\{\xi(t)|\xi(u), \xi(s_1), \ldots, \xi(s_n)\},$$

where $s_i < u$ ($i = 1, \ldots, n$), and thus the process is of Markov type.

In particular, when $\xi(t)$ is a stationary process and Var $\xi(s) = \sigma_\xi^2$, then $R(s, t) = R(t, -s)$ and

$$R(t_1 + t_2) = R(t_1)R(t_2), \tag{1.3}$$

which gives us

$$R(t) = \sigma_\xi^2 e^{-\lambda|t|}. \tag{1.4}$$

After some simple calculations it becomes evident that parameters λ in (1.1) and in the correlation function are identical. The solution of (1.1') is a stationary Gaussian Markov process; therefore its correlation function has the form (1.4). When $\mathsf{E}(d\zeta(t))^2 = \sigma_\zeta^2\,dt$, on the basis of (1.1) we obtain $\sigma_\zeta^2 = 2\lambda\sigma_\xi^2$.

It now becomes clear that a separable process $\xi(t)$ is continuous and non-differentiable with probability 1 (Kolmogorov's theorem; see Doob [1], Russian p. 576).*

The process $\int_{t_0}^{t}\xi(t)\,dt = \eta(t)$ exists in the mean square sense, and the following formula is valid:

$$\mathsf{E}\eta(t)\eta(s) = \frac{\sigma_\xi^2}{\lambda^2}\,[e^{-\lambda s} + e^{-\lambda t} + 2\lambda s - 1 - e^{-\lambda(t-s)}]. \tag{1.5}$$

Evidently

$$\mathsf{E}\eta(t)\eta(s) = \int_{t_0}^{t}\int_{t_0}^{s} \sigma_\xi^2 e^{-\lambda|t_1 - t_2|}\,dt_1\,dt_2$$

and our statement follows after simple calculations.

**Editor's note.* The citation is to the translator's appendix in the Russian edition of Doob's book. It refers to Theorem 4 of E. Slutsky's paper *Qualche proposizione relativa alla teoria delle funzione aleatorie* (Giorn. Ist. Ital. Attuari 8 (1937), 183–199).

We shall now prove the following assertion.

THEOREM 2. *The stationary Gaussian Markov process satisfies equation* (1.1), *i.e. the process* $\xi(t) - \xi(t_0) + \lambda\eta(t)$ *is a Wiener process.*

PROOF. From (1.5) we have

$$\mathsf{E}(\eta(t))^2 = \frac{2\sigma_\xi^2}{\lambda^2}\,[e^{-\lambda t} + \lambda t - 1]; \tag{1.6}$$

on the other hand, for $t_2 \geqslant t_1 \geqslant s_2 \geqslant s_1$ we have

$$\begin{aligned} \mathsf{E}(\eta(t_2) - \eta(t_1))(\eta(s_2) - \eta(s_1)) &= \frac{\sigma_\xi^2}{\lambda}\,(e^{\lambda s_2} - e^{\lambda s_1})(e^{-\lambda t_1} - e^{-\lambda t_2}) \\ &= -\frac{1}{\lambda^2}\,\mathsf{E}(\xi(t_2) - \xi(t_1))(\xi(s_2) - \xi(s_1)). \end{aligned} \tag{1.7}$$

The formula

$$\mathsf{E}(\eta(t)\xi(s)) = \begin{cases} \dfrac{\sigma_\xi^2}{\lambda}\,[2 - e^{-\lambda t} - e^{-\lambda(t-s)}] & \text{for } t > s \\ \dfrac{\sigma_\xi^2}{\lambda}\,[e^{-\lambda(s-t)} - e^{-\lambda s}] & \text{for } t \leqq s \end{cases} \tag{1.8}$$

is also true, and therefore

$$\begin{aligned} \mathsf{E}\,[(\eta(t_2) - \eta(t_1))(\xi(s_2) - \xi(s_1))] &= \frac{\sigma_\xi^2}{\lambda}\,(e^{\lambda s_2} - e^{\lambda s_1})(e^{-\lambda t_1} - e^{-\lambda t_2}) \\ &= \mathsf{E}\,[(\eta(s_2) - \eta(s_1))(\xi(t_2) - \xi(t_1))]. \end{aligned} \tag{1.9}$$

From (1.9) and (1.7) we see that the process $\lambda\eta(t) + \xi(t) - \xi(t_0)$ is one of independent increments and is normal, i.e. a Wiener process, as was to be proved.

From a theorem formulated by Baxter [1] it follows that

$$\lim_{\max(t_k - t_{k-1}) \to 0} \sum\,[\xi(t_k) - \xi(t_{k-1})]^2 = \sigma_\zeta^2 \cdot T \quad \text{(with probability 1)} \tag{1.10}$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ is a partition of the interval $[0, T]$. The one-dimensional stationary Gaussian Markov process can, in general, be determined by three parameters, m, σ_ξ^2 and λ (where $m = \mathsf{E}\xi(t)$); (1.10) expresses the fact that the "diffusion coefficient" is determined almost surely by one single realization, and therefore, because of the formula $\sigma_\zeta^2 = 2\lambda\sigma_\xi^2$, the number of unknown parameters is two (either m and λ, or m and σ_ξ^2). In this special case we need not recall general theorems to prove (1.10); elementary reasoning could also provide this result.

Simple calculations give us the following equation:

$$R(t) = \frac{\sigma_\zeta^2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iut}}{|\lambda + iu|^2}\,du;$$

therefore the spectral density function of the process has the form

$$f_\xi(u) = \frac{\sigma_\zeta^2}{2\pi} \frac{1}{|\lambda + iu|^2}.$$

§ 2. The likelihood ratio. Sufficient statistics and their distributions

In another interpretation, formula (1.10) expresses the fact that the measures P_1 and P_2 in the corresponding spaces of realizations of the process $\xi(t)$ $(0 \leq t \leq T)$, which are of the stationary Gaussian Markov type and have "diffusion coefficients" $\sigma_{\xi_1}^2 \neq \sigma_{\xi_2}^2$, are mutually singular.

The space R_ξ of realizations $\xi(t)$ $(0 \leq t \leq T)$ can be understood as a Cartesian product of the real line $\xi(0)$ and the space of realizations $\eta(t) = \xi(t) - \xi(0)$. Let W denote Wiener's well-known conditional measure with parameters $(0, \sigma_\xi)$ on the space of functions defined on the interval $0 < t \leq T$, and let L be the usual one-dimensional Lebesgue measure. Let $V = L \times W$. When P stands for the measure belonging to the stationary Gaussian Markov process with parameters m, λ and σ_ξ^2, then P is absolutely continuous with respect to V and its Radon-Nikodym derivative with respect to V is (see Striebel [1])

$$\frac{dP}{dV} = \sqrt{\frac{\lambda}{\pi}} \frac{1}{\sigma_\zeta} \exp\left\{-\frac{\lambda}{\sigma_\zeta^2}\left[s_{01}^2 - \frac{1}{2}\sigma_\zeta^2 T + \frac{1}{2}\varkappa s_{02}^2\right]\right\}, \tag{2.1}$$

where

$$\kappa = \lambda T, \tag{2.2}$$

$$s_{01}^2 = \frac{1}{2}\{[\xi(o) - m]^2 + [\xi(T) - m]^2\}, \qquad s_{02}^2 = \frac{1}{T}\int_0^T (\xi(t) - m)^2\, dt. \tag{2.3}$$

It is now clear that for a known m the quantities s_{01}^2 and s_{02}^2 form a sufficient statistic for λ or σ_ξ^2. When λ is known and m unknown, we can rewrite (2.1) as

$$\begin{aligned} \frac{dP}{dV} = \sqrt{\frac{\lambda}{\pi}} \cdot \frac{1}{\sigma_\zeta} \exp\Bigg\{-\frac{\lambda}{\sigma_\zeta^2}\Bigg[&-2m\left(\frac{\xi(o)+\xi(T)}{2} + \frac{\varkappa}{2}\frac{1}{T}\int_0^T \xi(t)\, dt\right) \\ &+ m^2\left(1 + \frac{\varkappa}{2}\right) - \frac{1}{2}\sigma_\zeta^2 T + \frac{\varkappa}{2}\frac{1}{T}\int_0^T \xi^2(t)\, dt + \frac{\xi^2(o)+\xi^2(T)}{2}\Bigg]\Bigg\} \end{aligned} \tag{2.4}$$

(see Grenander [1], p. 65) and thus see that the weighted average of statistics

$$m_1 = \frac{\xi(o) + \xi(T)}{2}, \qquad m_2 = \frac{1}{T}\int_0^T \xi(t)\, dt, \tag{2.5}$$

i.e. the statistic

$$m^* = \frac{m_1 + \frac{\varkappa}{2} m_2}{1 + \frac{\varkappa}{2}} \tag{2.6}$$

forms a sufficient statistic of the unknown parameter m.

When the parameters m and λ are unknown, we can rewrite (2.1) in the form

$$\frac{dP}{dV} = \sqrt{\frac{\lambda}{\pi}} \cdot \frac{1}{\sigma_\zeta} \exp\left\{-\frac{\lambda}{\sigma_\zeta^2}\left[s_1^2 - \frac{1}{2}\sigma_\zeta^2 T + \frac{1}{2}\varkappa s_2^2 + (m - m_1)^2 + \frac{\varkappa}{2}(m - m_2)^2\right]\right\}, \tag{2.7}$$

where

$$\begin{aligned} s_1^2 &= \frac{1}{2}\{[\xi(o) - m_1]^2 + [\xi(T) - m_1]^2\} = \frac{1}{4}[\xi(T) - \xi(o)]^2, \\ s_2^2 &= \frac{1}{T}\int_0^T [\xi(t) - m_2]^2\, dt. \end{aligned} \tag{2.8}$$

From (2.7) we conclude that the system m_1, m_2, s_1^2, s_2^2 forms a sufficient statistic.

The transformation

$$t = t' \cdot T, \qquad \xi = \xi' \sigma_\zeta \sqrt{T} \tag{2.9}$$

enables us to treat the special case $T = 1$ and $\sigma_\xi = 1$ only, instead of the general case; here $\lambda' = \lambda \cdot T = \kappa$ and therefore in the case of a known m the realizations of the process are characterized by only one parameter; this is independent of the choice of a time unit. In what follows we shall often assume that the transformation (2.9) has been made and instead of λ we shall simply write κ. In such cases (2.1) has the simpler form

$$\sqrt{\frac{\varkappa}{\pi}} \exp\left\{-\varkappa\left[s_{01}^2 - \frac{1}{2} + \frac{1}{2}\varkappa s_{02}^2\right]\right\}. \tag{2.1'}$$

Formally, (2.1) and (2.1') can be "obtained" simply as follows.

Let $\mathsf{E}\xi(t) = 0$; then $\xi(0)$ has the density function

$$f_{\xi(o)}(x_0) = \frac{1}{\sqrt{2\pi}\sigma_\xi} e^{-\frac{x_0^2}{2\sigma_\xi^2}} = \sqrt{\frac{\lambda}{\pi}} \frac{1}{\sigma_\zeta} e^{-\frac{\lambda x_0^2}{\sigma_\zeta^2}}. \tag{2.10}$$

On the other hand, from (1.1') we can write the following formula concerning the 'functionals' of densities of the processes $\zeta(t)$ and $\xi(t)$:

$$\exp\left\{-\frac{1}{2\sigma_\zeta^2}\int_0^T \frac{d\zeta^2}{dt}\right\} = \exp\left\{-\frac{1}{2\sigma_\zeta^2}\int_0^T \frac{(d\xi+\lambda\xi\, dt)^2}{dt}\right\}$$

$$(2.11)\qquad = \exp\left\{-\frac{1}{2\sigma_\zeta^2}\left[\int_0^T \frac{d\xi^2}{dt} + 2\lambda\int_0^T \xi\, d\xi + \lambda^2\int_0^T \xi^2(t)\, dt\right]\right\}.$$

When considering a diffusion process $\eta(t)$ with coefficients $m(x, t)$ and $b(x, t)$ we may approximate the density function of the variables $\eta_0, \eta_1, \dots, \eta_n$ by

$$p(x_0, \dots, x_n) = p_0(x_0)\prod_{i=0}^{n-1}[2\pi b(x_i, t_i)]^{-1/2}\exp\left\{-\frac{1}{2}\sum_{i=0}^{n-1}\left[\frac{x_{i+1}-x_i}{\Delta_i}-a_i\right]^2\frac{\Delta_i}{b_i}\right\},$$

where

$$t_{i+1}-t_i = \Delta_i,\ \eta_i=\eta(t_i),\ a_i=a(x_i, t_i),\ b_i=b(x_i, t_i),\ t_n=T,\ t_0=0.$$

The sum obtained reminds us of the integral

$$-\frac{1}{2}\int_0^T\left[\frac{d\eta}{dt}-a(x, t)\right]^2\frac{dt}{b(x, t)}.$$

It is known (see Doob [1], p. 494) that

$$\int_0^T \xi\, d\xi = \frac{1}{2}[\xi^2(T)-\xi^2(o)]-\frac{1}{2}T\sigma_\xi^2.$$

From (2.11), taking (2.10) into account, we get (2.1′). However, a rigorous proof of these simple 'calculations' is much more difficult; the reader can check this statement not only in this context with formula (2.1) (see Striebel [1]), but also in the case of other proofs (see, for example, Prohorov [1]).

In what follows we shall give the distribution of sufficient statistics

$$m_1 = \frac{\xi(o)+\xi(T)}{2},\quad m_2 = \frac{1}{T}\int_0^T \xi(t)\, dt,$$

$$s_{01}^2 = \frac{1}{2}\{[\xi(o)-m]^2+[\xi(T)-m]^2\},\quad s_{02}^2 = \int_0^T(\xi(t)-m)^2\, dt.$$

For the sake of simplicity let $m = 0$; then the characteristic function of the above random vector is

$$\mathsf{E}\exp\{i\alpha_1 m_1+i\alpha_2 s_{01}^2+i\alpha_3 m_2+i\alpha_4 s_{02}^2\} = \frac{2\sqrt{\lambda}\, e^{\frac{\varkappa}{2}}(\varkappa^2-2T\sigma_\zeta^2 i\alpha_4)^{\frac{1}{4}}}{\sqrt{T}\,[\varphi(\alpha_2, \alpha_4)]^{\frac{1}{2}}}\cdot$$

$$\cdot\exp\frac{1}{2}\left\{-\frac{\alpha_1\alpha_3\sigma_\zeta^2+\alpha_3^2\sigma_\zeta^2}{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}T+\left(\frac{i\alpha_1}{2}-T\frac{i\alpha_3\lambda+\alpha_2\alpha_3\sigma_\zeta^2}{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}\right)\cdot\right.$$

$$\cdot\left[\left(\frac{i\alpha_1\sigma_\zeta^2}{2}\left(1+e^{-\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}}\right)+i\alpha_3\sigma_\zeta^2\frac{1-e^{-\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}}}{\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}}\right)\cdot\right.$$

$$(2.13)\qquad\cdot\frac{\left(\varkappa-T\sigma_\zeta^2 i\alpha_2+\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}\right)e^{\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}}-\left(\varkappa-T\sigma_\zeta^2 i\alpha_2-\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}\right)}{T\cdot\varphi(\alpha_2, \alpha_4)}$$

$$+\left(\frac{i\alpha_1}{2}\sigma_\zeta^2\left(1+e^{\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}}\right)-i\alpha_3\sigma_\zeta^2\frac{1-e^{\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}}}{\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}}\right)$$

$$\left.\left.\cdot\frac{\left(\varkappa-T\sigma_\zeta^2 i\alpha_2+\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}\right)-\left(\varkappa-T\sigma_\zeta^2 i\alpha_2-\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}\right)e^{-\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}}}{T\varphi(\alpha_2, \alpha_4)}\right]\right\},$$

where

$$(2.14)\qquad \begin{aligned}\varphi(\alpha_2, \alpha_4) = {} & \frac{1}{T^2}e^{\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}}\left(\varkappa-T\sigma_\zeta^2 i\alpha_2+\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}\right)^2 \\ & -\frac{1}{T^2}e^{-\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}}\left(\varkappa-T\sigma_\zeta^2 i\alpha_2-\sqrt{\varkappa^2-2T\sigma_\zeta^2 i\alpha_4}\right)^2.\end{aligned}$$

From (2.13) and (2.14) it follows that the characteristic function of m_1 and m_2 is

$$(2.15)\qquad \exp-\frac{1}{2}\left\{\alpha_1^2\frac{\sigma_\zeta^2(1+e^{-\varkappa})}{4\lambda}+\alpha_1\alpha_3\frac{\sigma_\zeta^2(1-e^{-\varkappa})}{T\lambda^2}+\alpha_3^2\frac{\sigma_\zeta^2\left(T+\frac{e^{-\varkappa}-1}{\lambda}\right)}{\varkappa^2}\right\},$$

while the characteristic function of s_{01}^2 and s_{02}^2 is

$$(2.16)\qquad \frac{2\sqrt{\frac{\lambda}{T}}\, e^{\frac{\varkappa}{2}}(\varkappa^2-2T\sigma_\zeta^2 i\alpha_4)^{\frac{1}{4}}}{[\varphi(\alpha_2, \alpha_4)]^{\frac{1}{2}}}.$$

When $\kappa = \lambda T \longrightarrow 0$, the characteristic function of $(\sqrt{\lambda}m_1, \sqrt{\lambda}m_2, \lambda s_{01}^2, \lambda s_{02}^2)$ has the form

$$(2.17)\qquad \frac{\left(1+\frac{\varkappa}{2}\right)e^{-\frac{(\alpha_1+\alpha_3)^2}{2(1-\sigma_\zeta^2 i\alpha_2)}\sigma_\zeta^2-\frac{\varkappa\sigma_\zeta^2}{12}\alpha_3^2}}{\left\{1-\sigma_\zeta^2 i\alpha_2+\frac{\varkappa}{2}\left[(1-\sigma_\zeta^2 i\alpha_2)^2+1-2\sigma_\zeta^2\frac{i\alpha_4}{T}\right]\right\}^{\frac{1}{2}}}+o(\varkappa)$$

and therefore when $\kappa \longrightarrow 0$ the random variables m_1 and s_{01}^2 form an asymptotically sufficient statistic.

PROOF OF (2.13). Let

$$m_1^{(n)} = \frac{\xi_1+\xi_n}{2}, \quad s_{01}^{(n)} = \frac{\xi_1^2+\xi_n^2}{2}, \quad m_2^{(n)} = \sum_2^n \xi_i \cdot \Delta t, \quad s_{02}^{(n)} = \sum_2^n \xi_i^2 \cdot \Delta t,$$

where

$$\Delta t = \frac{T}{n}, \quad \xi_i = \xi\left(\frac{i-1}{n}T\right), \quad (i = 1, 2, \ldots, n) \text{ and } \beta = e^{-\lambda \Delta t} = 1-\lambda\Delta t + o(\Delta t).$$

Evidently

$$\mathsf{E}\, e^{i(\alpha_1 m_1^{(n)} + \alpha_2 s_{01}^{(n)} + \alpha_3 m_2^{(n)} + \alpha_4 s_{02}^{(n)})}$$

(2.18)
$$= (2\pi)^{-\frac{n}{2}} \sigma^{-n} (1-\beta^2)^{-\frac{n-1}{2}} \int \ldots \int e^{-\frac{1}{2}[XA_nX^* - XC^*]} dx_1 \ldots dx_n,$$

where

$$A_n = \frac{1}{\sigma^2(1-\beta^2)} \begin{pmatrix} a_1 & -\beta & 0 & 0 \cdots 0 \\ -\beta & a & -\beta & 0 \cdots 0 \\ 0 & -\beta & a & -\beta \cdots 0 \\ \vdots & \vdots & & \ddots \quad \vdots \\ 0 & 0 & \cdots & a \quad -\beta \\ 0 & 0 & \cdots & -\beta \quad a_1 \end{pmatrix} = \frac{1}{\sigma^2(1-\beta^2)} B_n,$$

$$a_1 = 1 - i\alpha_2 \sigma_\xi^2 \Delta t, \quad a = 2(1-\lambda\Delta t - i\alpha_4 \sigma_\xi^2 \Delta t + \lambda^2 (\Delta t)^2 + o(\Delta t)),$$

$$C^* = 2 \begin{pmatrix} \frac{i\alpha_1}{2} \\ i\alpha_3 \Delta t \\ \vdots \\ i\alpha_3 \Delta t \\ \frac{i\alpha_1}{2} \end{pmatrix}.$$

Meanwhile we made use of the fact that the density function of random variables $\xi_1, \ldots, \xi_n$ has the form

(2.19)
$$f_{\xi_1,\ldots,\xi_n}(x_1, \ldots, x_n) = (2\pi)^{-\frac{n}{2}} \sigma^{-n} (1-\beta^2)^{-\frac{n-1}{2}} \exp\left\{-\frac{1}{2\sigma^2(1-\beta^2)} \cdot \left[(1-\beta^2)x_1^2 + \sum_2^n (x_i - \beta x_{i-1})^2\right]\right\},$$

or, in matrix form,

(2.19′)
$$f_{\xi_1,\ldots,\xi_n}(x_1, \ldots, x_n) = (2\pi)^{-\frac{n}{2}} \sigma^{-n} (1-\beta^2)^{-\frac{n-1}{2}} \cdot \exp\left\{-\frac{1}{2\sigma^2(1-\beta^2)} XR_n^{-1}X^*\right\},$$

where

$$R_n^{-1} = \begin{pmatrix} 1 & -\beta & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ -\beta & 1+\beta^2 & -\beta & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -\beta & 1+\beta^2 & -\beta & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & & & \ddots & & & \\ 0 & 0 & & & & & 1+\beta^2 & -\beta \\ 0 & 0 & & & & & -\beta & 1 \end{pmatrix}$$

and

$$X = (x_1, \ldots, x_n), \quad X^* = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Fromula (2.19) can easily be obtained by taking into account that ξ_n satisfies the difference equation

(2.20)
$$\xi_{n+1} = \beta \xi_n + \zeta_{n+1},$$

where ζ_n is a sequence of independent normally distributed random variables (white noise) and $\sigma_\zeta^2 = (1-\beta^2)\sigma^2$, where σ_ξ^2 has been simply denoted as σ^2. The functional determinant of the transformation (2.20), $i = 1, \ldots, n$, equals 1, and therefore (2.19) is a simple consequence of the independence of the variables ζ_i $(i = 1, \ldots, n)$.

Let the numbers $d_1, \ldots, d_n$ satisfy

(2.21)
$$\begin{aligned} a_1 d_1 - \beta d_2 &= \frac{i\alpha_1}{2} \sigma_\xi^2 \cdot \Delta t, \\ -\beta d_1 + a d_2 - \beta d_3 &= i\alpha_3 \sigma_\xi^2 (\Delta t)^2, \\ &\vdots \\ -\beta d_{k-1} + a d_k - \beta d_{k+1} &= i\alpha_3 \sigma_\xi^2 (\Delta t)^2, \\ &\vdots \\ -\beta d_{n-1} + a_1 d_n &= \frac{i\alpha_1}{2} \sigma_\xi^2 \Delta t; \end{aligned}$$

then the system can be written as

(2.22)
$$XA_nX^* - XC^* = YA_nY^* - D_n,$$

where

$$D_n = a_1 d_1^2 + a(d_2^2 + \dots + d_{n-1}^2) + a_1 d_n^2 - 2\beta(d_1 d_2 + \dots + d_n d_{n-1})$$

$$(2.23)\quad = d_1(a_1 d_1 - \beta d_2) + d_2(a d_2 - \beta d_1 - \beta d_3) + \dots + d_{n-1}(a d_{n-1} - \beta d_{n-2} - \beta d_n)$$

$$+ d_n(a_1 d_n - \beta d_{n-1}) = \frac{i\alpha_1}{2}(d_1 + d_n)\sigma_\xi^2 \cdot \Delta t + i\alpha_3 \sigma_\xi^2 (\Delta t)^2 \sum_{2}^{n-1} d_i.$$

A particular solution of (2.21) is

$$d_i = d = \frac{i\alpha_3 \sigma_\xi^2}{\lambda^2 - 2\sigma_\xi^2 i\alpha_4} \qquad (i = 2, \dots, n-1),$$

while the general solution has the form

$$(2.24)\qquad d_i = d + \theta_1 u_1^i + \theta_2 u_2^i \qquad (i = 1, \dots, n),$$

where u_1 and u_2 are the two roots of the equation $\beta u^2 - au - \beta = 0$, i.e.

$$(2.25)\quad \begin{aligned} u_1 &= \frac{a - \sqrt{a^2 - 4\beta^2}}{2\beta} = 1 - \Delta t\sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4} + o(\Delta t), \\ u_2 &= \frac{a + \sqrt{a^2 - 4\beta^2}}{2\beta} = 1 + \Delta t\sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4} + o(\Delta t). \end{aligned}$$

θ_1 and θ_2 can be determined from the first and last equation of (2.21), which means that

$$(2.26)\quad \begin{aligned} \theta_1 &= \left(\frac{i\alpha_1}{2}\sigma_\xi^2 \Delta t - \frac{i\alpha_3 \sigma_\xi^2 \Delta t(\lambda - i\alpha_2 \sigma_\xi^2)}{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}\right) \cdot \\ &\cdot \frac{a_1 u_2^n - \beta u_2^{n-1} - (a_1 u_2 - \beta u_2^2)}{(a_1 u_1 - \beta u_1^2)(a_1 u_2^n - \beta u_2^{n-1}) - (a_1 u_2 - \beta u_2^2)(a_1 u_1^n - \beta u_1^{n-1})} \\ \theta_2 &= \left(\frac{i\alpha_1}{2}\sigma_\xi^2 \Delta t - \frac{i\alpha_3 \sigma_\xi^2 \Delta t(\lambda - i\alpha_2 \sigma_\xi^2)}{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}\right) \cdot \\ &\cdot \frac{a_1 u_1 - \beta u_1^2 - (a^1 u_1^n - \beta u_1^{n-1})}{(a_1 u_1 - \beta u_1^2)(a_1 u_2^n - \beta u_2^{n-1}) - (a_1 u_2 - \beta u_2^2)(a_1 u_1^n - \beta u_1^{n-1})}. \end{aligned}$$

According to Cramér's theorem (see Cramér [1], Chapter 24)

$$\int \dots \int \exp\left\{-\frac{1}{2} X A_n X^*\right\} dx_1 \dots dx_n = (2\pi)^{\frac{n}{2}} |A_n|^{-\frac{1}{2}};$$

therefore (2.18), by virtue of (2.22), will become

$$(2.27)\qquad (1 - \beta^2)^{\frac{1}{2}} |B_n|^{-\frac{1}{2}} e^{\frac{D_n}{2\sigma_\xi^2 \Delta t}}.$$

Let us first determine the limit of $\exp(D_n / 2\sigma_\xi^2 \Delta t)$ as $n \to \infty$. After simple calculations we obtain

$$a_1 u_1 - \beta u_1^2 = \Delta t(\lambda - i\sigma_\xi^2 \alpha_2 + \sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}) + o(\Delta t),$$

$$a_1 u_2 - \beta u_2^2 = \Delta t(\lambda - i\sigma_\xi^2 \alpha_2 - \sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}) + o(\Delta t),$$

$$a_1 u_1^n - \beta u_1^{n-1} = e^{-T\sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}}(\lambda - i\sigma_\xi^2 \alpha_2 - \sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}) + o(1),$$

$$a_1 u_2^n - \beta u_2^{n-1} = e^{T\sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}}(\lambda - i\sigma_\xi^2 \alpha_2 + \sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}) + o(1),$$

and therefore

$$\lim_{n\to\infty} \frac{D_n}{2\sigma_\xi^2 \Delta t} = \frac{1}{2}\left\{-\frac{\alpha_1 \alpha_3 \sigma_\xi^2 + \alpha_3^2 \sigma_\xi^2 T}{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}\right.$$

$$+\left(\frac{i\alpha_1}{2} - \frac{i\alpha_3 \lambda + \alpha_3 \alpha_2 \sigma_\xi^2}{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}\right)\left[\left(\frac{i\alpha_1}{2}\sigma_\xi^2 + \frac{i\alpha_1 \sigma_\xi^2}{2} e^{-T\sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}} + i\alpha_3 \sigma_\xi^2 \frac{1 - e^{-T\sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}}}{\sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}}\right) \cdot\right.$$

$$(2.28)\quad \left.\cdot \frac{(\lambda - i\sigma_\xi^2 \alpha_2 + \sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}) e^{T\sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}} - (\lambda - i\alpha_2 \sigma_\xi^2 - \sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4})}{\varphi(\alpha_2, T\alpha_4)}\right)$$

$$+\left(\frac{i\alpha_1}{2}\sigma_\xi^2 + \frac{i\alpha_1}{2}\sigma_\xi^2 e^{T\sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}} - i\alpha_3 \sigma_\xi^2 \frac{1 - e^{T\sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}}}{\sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}}\right) \cdot$$

$$\left.\left.\cdot \frac{(\lambda - i\sigma_\xi^2 \alpha_2 + \sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}) - e^{-T\sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4}}(\lambda - i\alpha_2 \sigma_\xi^2 - \sqrt{\lambda^2 - 2\sigma_\xi^2 i\alpha_4})}{\varphi(\alpha_2, T\alpha_4)}\right]\right\}.$$

When calculating $|B_n|$, it should be mentioned that

$$(2.29)\qquad |B_n| = a_1^2 |\tilde{B}_{n-2}| - 2\beta^2 a_1 |B_{n-3}| + \beta^4 |\tilde{B}_{n-4}|,$$

where $|\tilde{B}_n|$ satisfies the difference equation

$$(2.30)\qquad |\tilde{B}_n| = a|\tilde{B}_{n-1}| - \beta^2 |B_{n-2}|$$

and therefore

$$(2.31)\qquad |\tilde{B}_n| = \alpha_1 v_1^n + \alpha_2 v_2^n,$$

where v_1 and v_2 are the roots of equation $v^2 - av + \beta^2 = 0$, while α_1 and α_2 can be obtained from the conditions $|\tilde{B}_1| = a$ and $|\tilde{B}_2| = a^2 - \beta^2$. We thus have

$$(2.32)\qquad \alpha_1 = \frac{v_1}{v_1 - v_2}, \quad \alpha_2 = -\frac{v_2}{v_1 - v_2}.$$

Finally,

$$(2.33)\qquad |B_n|^{-1/2} = \left\{\frac{v_1^{n-3}}{v_1 - v_2}[a_1 v_1 - \beta^2]^2 - \frac{v_2^{n-3}}{v_1 - v_2}[a_1 v_2 - \beta^2]^2\right\}^{-1/2}.$$

Because

$$v_1 = 1-\lambda\Delta t+\Delta t\sqrt{\lambda^2-2\sigma_\xi^2 i\alpha_4}+o(\Delta t),$$

$$v_2 = 1-\lambda\Delta t-\Delta t\sqrt{\lambda^2-2\sigma_\xi^2 i\alpha_4}+o(\Delta t),$$

we get

$$a_1v_1-\beta^2 = \Delta t\left(\lambda-\sigma_\xi^2 i\alpha_2+\sqrt{\lambda^2-2\sigma_\xi^2 i\alpha_4}\right)+o(\Delta t),$$

$$a_1v_2-\beta^2 = \Delta t\left(\lambda-\sigma_\xi^2 i\alpha_2-\sqrt{\lambda^2-2\sigma_\xi^2 i\alpha_4}\right)+o(\Delta t),$$

$$v_1^{n-3} = e^{T(\sqrt{\lambda^2-2\sigma_\xi^2 i\alpha_4}-\lambda)}+o(1),$$

$$v_2^{n-3} = e^{-T(\sqrt{\lambda^2-2\sigma_\xi^2 i\alpha_4}-\lambda)}+o(1),$$

and therefore

$$(1-\beta^2)^{\frac{1}{2}}|B_n|^{-\frac{1}{2}} = \frac{2\sqrt{\lambda}(\lambda^2-2\sigma_\xi^2 i\alpha_4)e^{\frac{\lambda T}{2}}}{\varphi(\alpha_2, T\alpha_4)}(1+o(1)). \tag{2.34}$$

From (2.27), (2.28) and (2.34) follows (2.13), which is what we wished to prove.

This procedure represents one way of determining the characteristic functions of the functionals m_1, m_2, s_{01}^2 and s_{02}^2. Other possibilities could be suggested when considering the differential equation of the conditional characteristic function (under the condition that $\xi(0) = x$); this equation has a unique solution (as was demonstrated by Dynkin [1]), which can be found.

We can write

$$u(T, x) = \mathrm{E}\{e^{i(\alpha_1 m_1+\alpha_2 s_{01}^2+\alpha_3 T m_2+\alpha_4 T s_{02}^2)}|_{\xi(0)=x}\}. \tag{2.35}$$

It clearly follows that

$$u(T+\Delta T, x) = \frac{1}{\sqrt{2\pi\Delta T}\,\sigma_\xi}\int_{-\infty}^{\infty} e^{-\frac{(x_1-x+\lambda x\Delta T)^2}{2\Delta T\sigma_\xi^2}}\left[u(T, x)+\frac{\partial u}{\partial x_1}\bigg|_{x_1=x}(x-x_1)\right.$$

$$+\frac{\partial^2 u}{\partial x_1^2}\bigg|_{x_1=x}\frac{(x_1-x)^2}{2}+\ldots\bigg](1+i\alpha_4 x^2\Delta T)(1+i\alpha_3 x\Delta T)\left[1-\frac{i\alpha_1}{2}(x_1-x)\right.$$

$$\left.-\frac{\alpha_1^2}{8}(x_1-x)^2+\ldots\right]\left[1-\frac{i\alpha_2}{2}((x_1-x)^2+2x(x_1-x))\right. \tag{2.36}$$

$$\left.-\frac{\alpha_2^2}{8}(4x^2(x_1-x)^2+\ldots)+\ldots\right]dx_1.$$

For $\Delta T \longrightarrow 0$ we get the following partial differential equation for $u(T, x)$:

$$\frac{\partial u}{\partial T} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\left[-x(\lambda+i\alpha_2)-\frac{i\alpha_1}{2}\right]+u\left[x^2\left(\lambda i\alpha_2+i\alpha_4-\frac{\alpha_2^2}{2}\right)\right.$$

$$\left.-x\left(\lambda\frac{i\alpha_1}{2}+i\alpha_3-\frac{\alpha_1\alpha_2}{2}\right)-\frac{i\alpha_2}{2}-\frac{\alpha_1^2}{8}\right]. \tag{2.37}$$

Later in the paper we shall deal with the solution of a similar equation, but for now it will be omitted.

§3. Estimation of parameters and their distributions

If λ is known, then the maximum likelihood estimator of the expected value can be obtained from (2.4) as

$$\hat{m} = \frac{\xi(o)+\xi(T)+\lambda\int_0^T \xi(t)\,dt}{2+\lambda T}; \tag{3.1}$$

m is normally distributed with parameters (m, σ_1), where

$$\sigma_1^2 = \sigma_\xi^2(4\kappa+1-e^{-\kappa})/2(1+\kappa)^2$$

(see Grenander [1], p. 215).

This estimate is of minimum variance because the likelihood ratio dP/dV forms a complete system of functions in the sense of Lehmann and Scheffé [1] in the case of an unknown m.

Let $m = 0$, and let λ (or σ_ξ^2) be unknown. When considering (2.1) and

$$\log\frac{dP}{dv} = c+\frac{1}{2}\log\frac{\lambda}{\sigma_\xi^2}-\frac{\lambda}{\sigma_\xi^2}\left[s_{01}^2-\frac{1}{2}\sigma_\xi^2 T+\frac{1}{2}\lambda T s_{02}^2\right]$$

the maximum likelihood estimator of λ will be the solution of the following equation:

$$\frac{\sigma_\xi^2}{2\lambda}-\left(s_{01}^2-\frac{1}{2}\sigma_\xi^2 T\right)-\lambda T s_{02}^2 = 0, \tag{3.2}$$

or the following equation for σ_ξ^2 (we set $\sigma_\xi^2 = \sigma_\zeta^2/2\lambda$):

$$\sigma_\xi^4-\left(s_{01}^2-\frac{1}{2}\sigma_\zeta^2 T\right)\sigma_\xi^2-\frac{T}{2}\sigma_\zeta^2 s_{02}^2 = 0. \tag{3.3}$$

It can easily be shown that the only positive solution of (3.3) is given by

$$\hat{\sigma}_\xi^2 = 1/2(s_{01}^2-1/2\sigma_\zeta^2 T)+1/2\sqrt{(s_{01}^2-1/2\sigma_\zeta^2 T)^2+2T\sigma_\zeta^2 s_{02}^2}. \tag{3.4}$$

For simplicity let us denote $d = 1/2(s_{01}^2 - 1/2\sigma_\zeta^2 T)$. Now we can easily obtain

$$P\{\hat{\sigma}_\xi^2 < y\sigma_\xi^2\} = P\{d+\sqrt{d^2+1/2T\sigma_\xi^2 s_{02}^2} < y\sigma_\xi^2\}$$

(3.5) $$= P\left\{\frac{2\varkappa\lambda}{y^2\sigma_\xi^2}s_{02}^2 + \frac{2\lambda}{y\sigma_\xi^2}s_{01}^2 < \frac{\varkappa}{y}+1\right\}.$$

From (2.16) we can see that the random variable

$$\eta_y = \frac{2\varkappa^2}{y^2\sigma_\xi^2}s_{01}^2 + \frac{2\varkappa}{y\sigma_\xi^2}s_{01}^2$$

has asymptotically normal distribution as $\kappa \longrightarrow \infty$ and that the estimate $\hat{\sigma}_\xi^2$ is equivalent to the estimate s_{02}^2. The following theorem holds true (assuming that $\sigma_\xi^2 = 1$ and $T = 1$, i.e. that transformation (2.9) has been carried out).

THEOREM 3.1. *For $m = 0$ and $\kappa \longrightarrow \infty$ the estimate $s_{02}^2 \sim \sigma_\xi^2$ is asymptotically efficient, and the distribution of the ratio $(s_{02}^2 - \sigma_\xi^2)/s_{02}^2\sqrt{2/\kappa}$ tends to the* (0, 1) *normal distribution.*

PROOF. Straightforward calculations of the mean and variance yield the following:

$$\mathrm{E}s_{02}^2 = \sigma_\xi^2$$

(3.6) $$\mathrm{Var}\,s_{02}^2 = \frac{\sigma_\xi^2}{\varkappa^2}(2\varkappa + e^{-2\varkappa} - 1).$$

The characteristic function of s_{02}^2 is given by

(3.7) $$f(\alpha) = \frac{2\left(1-\dfrac{4i\alpha\sigma_\xi^2}{\varkappa}\right)^{1/4} e^{\varkappa/2}}{\left[\left(1+\sqrt{1-\dfrac{4i\alpha}{\varkappa}\sigma_\xi^2}\right)^2 e^{\varkappa\sqrt{1-\frac{4i\alpha}{\varkappa}\sigma_\xi^2}} - \left(1-\sqrt{1-\dfrac{4i\alpha}{\varkappa}\sigma_\xi^2}\right)^2 e^{-\varkappa\sqrt{1-\frac{4i\alpha}{\varkappa}\sigma_\xi^2}}\right]^{1/2}}$$

and (see (2.16)) the characteristic function of $(s_{02}^2 - \sigma_\xi^2)/s_{02}^2\sqrt{2/\kappa}$ tends to $e^{-\alpha^2/2}$ as $\kappa \longrightarrow \infty$, i.e. it tends to the characteristic function of the normal distribution. On the other hand, as $\kappa \longrightarrow \infty$ we have $s_{02}^2 \longrightarrow \sigma_\xi^2$ in probability, and therefore, according to Cramér's theorem (see Cramér [1], §33.3), the theorem holds. Since the likelihood system of functions (2.1′) is not complete (we shall later deal with this problem), an unbiased estimate of minimum variance for σ_ξ^2 (or for κ) does not exist. An estimate is called asymptotically efficient when its asymptotic distribution exists and coincides with the asymptotic distribution of the maximum likelihood estimate.

Next let $\sigma_\xi^2 = T = 1$. From (2.16) it follows that the characteristic functions of $2\kappa s_{02}^2$ and κs_{01}^2 have the following form as $\kappa \longrightarrow 0$:

(3.8) $$f(\alpha_1, \alpha_2) = \frac{1}{(1-i\alpha_1-2i\alpha_2)^{1/2}} + o(\varkappa),$$

which means that s_{01}^2 and s_{02}^2 are asymptotically equivalent. From (3.8) it follows that $s_{02}^2/\sigma_\xi^2 = 2\kappa s_{02}^2$ has the χ^2 distribution with one degree of freedom as $\kappa \longrightarrow 0$:

(3.9) $$P\left\{\frac{s_{02}^2}{\sigma_\xi^2} < x^2\right\} \to \frac{1}{\sqrt{2\pi}}\int_0^{x^2} e^{-y/2}y^{-1/2}\,dy = \sqrt{\frac{2}{\pi}}\int_0^x e^{-y^2/2}\,dy.$$

For values of κ which are neither too large nor too small we must use the statistics s_{01}^2 and s_{02}^2 to estimate κ (or σ_ξ^2). Confidence intervals for κ can be constructed by using (3.5) and determining the distribution of the random variable $\eta_y = \kappa^2 s_{02}^2/y^2 + \kappa s_{01}^2/y$. The characteristic function f of this variable is (see (2.16))

(3.10) $$f_{\eta_y}(\alpha) = \frac{2\left(1-\dfrac{2}{y^2}i\alpha\right)^{1/4} e^{\varkappa/2}}{\left[\left(1-\dfrac{i\alpha}{y}+\sqrt{1-\dfrac{2}{y^2}i\alpha}\right)^2 e^{\varkappa\sqrt{1-\frac{2}{y^2}i\alpha}} - \left(1-\dfrac{i\alpha}{y}-\sqrt{1-\dfrac{2}{y^2}i\alpha}\right)^2 e^{-\varkappa\sqrt{1-\frac{2}{y^2}i\alpha}}\right]^{1/2}}.$$

For an arbitrarilly chosen level α and for σ_ξ^2 the equation

(3.11) $$P_{\sigma_\xi^2}\{\hat{\sigma}^2 > x\} = \alpha$$

has a unique solution $x = \varphi(\sigma_\xi^2)$. Its inverse function

(3.12) $$\varphi^{-1}(x) = \psi_\alpha(x)$$

can also be uniquely determined and therefore gives the limits of a confidence interval, which means that identically in σ_ξ^2

(3.13) $$P_{\sigma_\xi^2}\{\sigma_\xi^2 \leqq \psi_\alpha(\hat{\sigma}^2)\} \equiv \alpha.$$

For $\kappa \longrightarrow \infty$ or $\kappa \longrightarrow 0$, these limits are determined by the corresponding distribution.

Effective determination of the limits of a confidence interval in the one-dimensional case has not yet been demonstrated. In the case of complex Gaussian Markov processes the relevant calculations for the 'damping coefficient' κ have been carried out in the Department of Probability Theory of Moscow State University under the guidance of A. N. Kolmogorov (see Arató, Rykova and Sinaĭ [1]).

When m and λ (or σ_ξ^2) are unknown, we obtain the maximum likelihood equations from (2.7) as follows:

$$(3.14)\quad \begin{aligned} &\frac{\sigma_\xi^2}{2\lambda}(s_1^2 - 1/2\sigma_\xi^2 T) - \lambda T s_2^2 - (m-m_1)^2 - \lambda T(m-m_2)^2 = 0,\\ &2(m-m_1) + \lambda T(m-m_2) = 0. \end{aligned}$$

The solution of (3.14) is very complicated, but it is nevertheless worth mentioning that the estimates $\hat{m}$ and $\hat{\lambda}$ are related by

$$(3.15)\quad \hat{m} = \frac{2m_1 + \hat{\varkappa} m_2}{2+\hat{\varkappa}}.$$

As we shall see later, when dealing with estimators for σ_ξ^2 (or λ), it is common to work with such statistics which do not depend on the initial point of the process; for example, s_1^2, s_2^2, $(m_1 - m_2)^2$ might be such a system. For a large κ is is not difficult to find a suitable estimate of the parameters m and λ.

If we again write $\sigma_\xi^2 = T = 1$, the following theorem can be proved.

THEOREM 3.2. *For $\kappa \longrightarrow \infty$ the estimates $m \sim m_2$ and $\sigma_\xi^2 \sim s_2^2$ are simultaneously asymptotically efficient, and the distribution function of the random vector*

$$\frac{m_2 - m}{2s_2^2},\quad \frac{s_2^2 - \sigma_\xi^2}{s_2^2\sqrt{2/\varkappa}}$$

tends to the normal distribution with parameters $(0, 0, \left|\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right|)$.

PROOF. Simple calculations yield

$$(3.16)\quad \begin{aligned} &\mathsf{E}m_2 = m,\ \mathrm{Var}\,m_2 = \frac{2\sigma_\xi^2(\varkappa + e^{-\varkappa} - 1)}{\varkappa^2},\\ &\mathsf{E}s_2^2 = \sigma_\xi^2 - \frac{2\sigma_\xi^2}{\varkappa}\left[1 + \frac{1}{\varkappa}(e^{-\varkappa} - 1)\right],\\ &\mathrm{Var}\,s_2^2 = \frac{\sigma_\xi^4}{\varkappa}\left\{2 + \frac{1}{\varkappa}(e^{-2\varkappa} - 1) + \frac{8}{\varkappa^3}(\varkappa + e^{-\varkappa} - 1)^2\right.\\ &\qquad\left. - \frac{4}{\varkappa^2}(4\varkappa + 2\varkappa e^{-\varkappa} - 7 + 8e^{-\varkappa} - e^{-2\varkappa})\right\}. \end{aligned}$$

From (2.13) it follows that the characteristic function of the random vector $m_2 - m$, s_{02}^2 has the form

$$(3.17)\quad f(\alpha_1, \alpha_2) = \exp\left\{-\frac{1}{2}\frac{\alpha_1^2 2\sigma_\xi^2}{\varkappa} + o\left(\frac{1}{\varkappa}\right)\right\}\exp\left\{i\alpha_2\sigma_\xi^2 - \frac{1}{2}\alpha_2^2\frac{2\sigma_\xi^2}{\varkappa} + o\left(\frac{1}{\varkappa}\right)\right\}$$

as $\kappa \longrightarrow \infty$, and therefore its coordinates are asymptotically normally distributed. According to Cramér's theorem mentioned above, and because $m_2 \longrightarrow m$ and $s_{02}^2 \longrightarrow \sigma_\xi^2$ in probability, the asymptotic distributions of the vectors

$$\frac{m_2 - m}{\sqrt{\frac{2\sigma_\xi^2}{\varkappa^2}(\varkappa + e^{-\varkappa} - 1)}},\quad \frac{s_{02}^2 - \sigma_\xi^2}{\sigma_\xi^2\sqrt{\frac{2}{\varkappa}}}$$

and

$$\frac{m_2 - m}{2s_2^2},\quad \frac{s_2^2 - \sigma_\xi^2}{s_2^2\sqrt{\frac{2}{\varkappa}}}$$

coincide, q.e.d.

For arbitrary values of T and σ_ξ^2—which is important in applications—the theorem has the following obvious corollary:

THEOREM 3.2′. *For $\kappa = \lambda T \longrightarrow \infty$ the estimates $m \sim m_2$ and $\lambda \sim \sigma_\xi^2/2s_2^2 = \hat{\lambda}$ are simultaneously efficient, and the distribution function of the random vector*

$$\frac{m_2 - m}{\sqrt{\frac{2\sigma_\xi^2}{\lambda^2 T^2}(\lambda T + e^{-\lambda T} - 1)}},\quad \frac{\hat{\lambda} - \lambda}{\sqrt{\frac{2\lambda}{T}}}\quad \left(\text{or}\quad \frac{\sigma_\xi^2 - s_2^2}{\sigma_\xi^2\sqrt{\frac{2}{\lambda T}}}\right)$$

tends to the normal distribution with parameters $(0, 0, \left|\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right|)$.

For $\kappa \longrightarrow 0$ the statistics m_1 and m_2 are asymptotically equivalent, which can be seen from their characteristic functions.

The statistics s_1^2 and s_2^2, given by

$$(3.17)\quad \begin{aligned} &s_1^2 = \frac{1}{4}[\xi(1) - \xi(0)]^2,\\ &s_2^2 = \int_0^1 (\xi(t) - \xi(0))^2\,dt - \left(\int_0^1 (\xi(t) - \xi(0))\,dt\right)^2 \end{aligned}$$

are asymptotically independent both of the parameters m and κ and of the statistics m_1 and m_2, when $\kappa \longrightarrow 0$. This means that in this case the parameter m is completely free, and we cannot set a lower limit on the parameter κ (or an upper limit on σ_ξ^2).

A well-known theorem in mathematical statistics says that if $\xi_1, \ldots, \xi_n$ $(n \geqslant 2)$ is a random sample from a normal population with parameters (m, σ), then, with an arbitrary degree of confidence, a finite confidence interval can be constructed (Cramér [1], p. 563). This means that there exist functions $\bar{h}(x_1, \ldots, x_n)$ and $h(x_1, \ldots, x_n)$, also for an arbitrary degree of confidence $\alpha > 1/2$, for which

$$P\{\bar{h}(\xi_1, \ldots, \xi_n) \geqq m\} \geqq \alpha,\quad P\{h(\xi_1, \ldots, \xi_n) < m\} > \alpha$$

holds uniformly with respect to m and σ. The functions $\bar{h}(\cdot)$ and $h(\cdot)$ are independent of σ. When $n = 1$, i.e. there is only one observation at our disposal, such finite functions h and $\bar{h}$ do not exist (naturally assuming that $h(\infty) > -\infty$ and $\bar{h}(-\infty) < \infty$).

Stationary Gaussian Markov processes satisfy the following assertion (assuming that $T = \sigma_\xi = 1$):

THEOREM 3.3. *Let $\alpha > 1/2$, and let $\mu(\xi)$ and $\bar{\mu}(\xi)$ be real-valued functionals (which may assume values $-\infty$ or $+\infty$) on the space R_ξ, which are continuous in the $C[0, 1]$ metric*[1] *and which satisfy the conditions*

$$P\{m \geqq \bar{\mu}(\xi)\} \geqq \alpha$$

$$P\{m < \bar{\mu}(\xi)\} \geqq \alpha$$

for any m and κ $(-\infty < m < \infty,\ \kappa > 0)$. Then

$$P\{\bar{\mu}(\xi) = -\infty\} \geqq f(\varkappa, \alpha)$$

$$P\{\bar{\mu}(\xi) = +\infty\} \geqq f(\varkappa, \alpha),$$

where $f(\kappa, \alpha)$ does not depend on the choice of these functionals (under the common assumption that $\inf_\mu \mu(\infty) > -\infty$, $\sup_{\bar{\mu}} \bar{\mu}(+\infty) < \infty$, or that in case of shifting the functional changes its value by a quantity equal to this shift), and $f(\kappa, \alpha) \longrightarrow 1/2$ as $\kappa \longrightarrow 0$.

THEOREM 3.4. *Let $\alpha > 0$, and let $\kappa(\xi)$ be a positive functional defined in the space R_ξ and continuous in the $C[0, 1]$ metric. Let it satisfy for any m and κ the condition $P\{\kappa \geqslant \kappa(\xi)\} \geqslant \alpha$. Then*

$$P\{\varkappa(\xi) = 0\} \leqq g(\varkappa, \alpha),$$

where the positive function g does not depend on the choice of the functional (under the common assumption that $\kappa(\infty) = \kappa(-\infty) = \infty$) and $g(\kappa, \alpha) \longrightarrow 1$ as $\kappa \longrightarrow 0$.

REMARK. For $\kappa \longrightarrow \infty$ the functions f and g tend to zero for any fixed $s < 1$. This is a consequence of Theorem 3.2.

PROOF OF THEOREM 3.3. Because of the symmetry it is sufficient to prove the theorem for $\bar{\mu}(\xi)$. For a bounded functional the inequality $P\{m < \bar{\mu}(\xi)\} \geqslant \alpha$ cannot hold true for all m and κ, because when $\bar{\mu}(\xi) \leqslant K < \infty$ we have

$$P_{K,\varkappa}\{K < \bar{\mu}(\xi)\} = 0.$$

For sufficiently large values of c there exist $\xi_0(t) \geqslant -k > -\infty$, independent of $\bar{\mu}$, so that $\bar{\mu}(\xi) \leqslant c$ when $\xi(t) \leqslant \xi_0(t)$ for all $0 \leqslant t \leqslant 1$. Let

[1] The continuity of functionals assuming infinite values is to be understood as continuity induced by the topology of the real line, closed by points $-\infty$ and $+\infty$.

$$\Gamma = \{\xi : \bar{\mu}(\xi) < c\}, \quad \Gamma_1 = \{\xi : -\varkappa^{-1+\delta} \leqq \xi \leqq \xi_0\},$$

where $0 < \delta < 1/2$. Evidently $\Gamma \supset \Gamma_1$, $P(\Gamma) \geqslant P(\Gamma_1)$ and

$$P_{c,\varkappa}\{c < \bar{\mu}(\xi)\} = 1 - P_{c,\varkappa}\{\Gamma\} \leqq 1 - P_{c,\varkappa}\{\Gamma_1\}. \tag{3.18}$$

By using

$$\frac{dP}{dV} = \sqrt{\frac{\varkappa}{\pi}} \exp\Big\{-\varkappa(\xi_0 - c)^2 - 1/2\big[\varkappa\{(\xi(1) - c)^2 - (\xi(0) - c)^2\} - \varkappa + \varkappa^2 \int_0^1 (\xi(t) - c)^2\, dt\big]\Big\}$$

we get

$$P_{c,\varkappa}\{\Gamma_1\} = \int_{\Gamma_1} \frac{dP}{dV}\, dV \geqq \left(1 - \frac{\varkappa^{2\delta}}{2}\right) \int_{\Gamma_1} \sqrt{\frac{\varkappa}{\pi}}\, e^{-\varkappa(x_0 - c) - \frac{\varkappa}{2}[(x_1 - c)^2 - (x_0 - c)^2]}\, dL \times dW. \tag{3.19}$$

Let

$$\Gamma_2 = \{\xi : -\varkappa^{-1+\delta} \leqq \xi \geqq \xi_0,\ 0 < t \leqq 1;\ -\varkappa^{-1+\delta} + \varkappa^{-\varepsilon} < \xi(0) \leqq \xi_0(0) - \varkappa^{-\varepsilon}\},$$

where $0 < \epsilon < \delta/2$, ϵ is arbitrary, and

$$\Gamma_3 = \{\xi : \sup_{0 \leqq t \leqq 1} |\xi(t) - \xi(0)| < \varkappa^{-\varepsilon},\ -\varkappa^{-1+\delta} + \varkappa^{-\varepsilon} < \xi(0) \leqq \xi_0(0) - \varkappa^{-\varepsilon}\}.$$

By using the formula

$$\int_{\Gamma_3} dW \geqq 1 - 2\varkappa^{\varepsilon} \sqrt{\frac{2}{\pi}}\, e^{-\frac{\varkappa^{-2\varepsilon}}{2}}, \tag{3.20}$$

which is valid for Wiener processes (see Doob [1], p. 392), we get the inequality

$$\int_{\Gamma} e^{-\frac{\varkappa}{2}[(x_1 - c)^2 - (x_0 - c)^2]}\, dW \geqq e^{-\varkappa^{-\varepsilon}(\varkappa^{\delta} + |c|\varkappa)} \int dW \geqq e^{-\varkappa^{-\varepsilon}(\varkappa^{\delta} + c|\varkappa|)} \cdot \left(1 - 2\varkappa^{\varepsilon} \sqrt{\frac{2}{\pi}}\, e^{-\frac{\varkappa^{-2\varepsilon}}{2}}\right). \tag{3.21}$$

Let $\Phi(x)$ denote the normal density function with parameters (0, 1); then

$$\int_{\Gamma_2} \sqrt{\frac{\varkappa}{\pi}}\, e^{-\varkappa(x_0 - c)^2}\, dx_0 = \Phi\{\sqrt{2\varkappa}(\xi_0(0) - c - \varkappa^{-\varepsilon})\} - \Phi\{\sqrt{2\varkappa}(-\varkappa^{-1+\delta} - c + \varkappa^{-\varepsilon})\}. \tag{3.22}$$

From (3.19), (3.21) and (3.22) we obtain

$$P_{c,\varkappa}\{\Gamma_1\} \geqq \left(1 - \frac{\varkappa^{2\delta}}{2}\right)(1 - \varkappa^{\delta - \varepsilon})\left(1 - 2\varkappa^{\varepsilon} \sqrt{\frac{2}{\pi}}\, e^{-\frac{\varkappa^{-2\varepsilon}}{2}}\right)\big[\Phi\{\sqrt{2\varkappa}(\xi_0(0) - c - \varkappa^{-\varepsilon})\} - \Phi\{\sqrt{2\varkappa}(-\varkappa^{-1+\delta} - c + \varkappa^{-\varepsilon})\}\big]. \tag{3.23}$$

Hence as $\kappa \longrightarrow 0$ we have

$$P_{c,\varkappa}\{c<\bar{\mu}(\xi)\}=1-P_{c,\varkappa}\{\Gamma_1\}\leq \frac{1}{2}+\varepsilon_0, \quad \text{for} \quad \varkappa<\varkappa_0(\varepsilon_0),$$

and for an arbitrarily small $\epsilon_0 > 0$, which proves the theorem.

Theorem 3.3 can be reworded as follows: *When the parameters m and κ (or σ_ξ^2) of a stationary Gaussian Markov process are unknown, it is impossible to construct finite confidence intervals for m using continuous functionals.* Later we shall deal with the problem of infinite confidence intervals (without any restriction concerning κ). Their construction is similar to the one used for κ (or σ_ξ^2) in those cases where m is known; nevertheless those special properties characterized by Theorem 3.3 must be used.

COROLLARY. *From the proof provided we can see that for any $\epsilon > 0$ there exists $\Lambda(\epsilon)$ such that for small values of κ*

$$\sup_{m,\varkappa<\varkappa_0} P_{m,\varkappa}\{\bar{\mu}(\xi)>m\}\leq \frac{1}{2}+\Lambda\varkappa_0^{\frac{1}{2}-\varepsilon}.$$

Thus we can make an estimate of the behavior of the function $f(\kappa, \alpha)$.

The proof of Theorem 3.4 is similar to that of Theorem 3.3. According to this theorem, no nonzero lower limit can be constructed for the parameter κ with any degree of confidence.

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