

# A QUANTITATIVE HELLY-TYPE THEOREM: CONTAINMENT IN A HOMOTHET

GRIGORY IVANOV AND MÁRTON NASZÓDI

ABSTRACT. We introduce a new variant of quantitative Helly-type theorems: the minimal “homothetic distance” of the intersection of a family of convex sets to the intersection of a subfamily of a fixed size. As an application, we establish the following quantitative Helly-type result for the *diameter*. If  $K$  is the intersection of finitely many convex bodies in  $\mathbb{R}^d$ , then one can select  $2d$  of these bodies whose intersection is of diameter at most  $(2d)^3 \text{diam}(K)$ . The best previously known estimate, due to Brazitikos, is  $cd^{11/2}$ . Moreover, we confirm that the multiplicative factor  $cd^{1/2}$  conjectured by Bárány, Katchalski and Pach cannot be improved.

## 1. INTRODUCTION

In [BKP82] (see also [BKP84]), Bárány, Katchalski and Pach proved the following two statements. According to the **Quantitative Volume Theorem**, *if the intersection of a family of convex sets in  $\mathbb{R}^d$  is of volume one, then the intersection of some subfamily of size at most  $2d$  is of volume at most  $v(d)$ , a constant depending only on  $d$* . The **Quantitative Diameter Theorem** states that *if the intersection of a family of convex sets in  $\mathbb{R}^d$  is of diameter one, then the intersection of some subfamily of size at most  $2d$  is of diameter at most  $\delta(d)$ , a constant depending only on  $d$* .

In [BKP82], Bárány, Katchalski and Pach established an upper bound of roughly  $d^{d^2}$  on  $v(d)$  and conjectured that  $v(d) \leq (d)^{cd}$  holds for a constant  $c > 0$ . Naszódi confirmed this conjecture in [Nas16] using contact points of the John ellipsoid [Joh14] of the intersection of the family of convex sets. The current best bound,  $v(d) \leq (cd)^{3d/2}$ , is due to Brazitikos [Bra17].

In [BKP82], the authors obtained a bound on  $\delta(d)$  which is exponential in the dimension, and formulated the following conjecture.

**Conjecture 1.1** (Bárány, Katchalski, Pach [BKP82]).

$$\delta(d) \leq c\sqrt{d}$$

with a universal constant  $c > 0$ .

Brazitikos [Bra18] established the first polynomial bound on  $\delta(d)$  (see also [Bra16]): using a sparsification result from [BSS14] (see also [Bar14, Lemma 3.1]) related to contact points of John’s ellipsoid, he showed  $\delta(d) \leq cd^{11/2}$  with an absolute constant  $c > 0$ . Recently, Dillon and Soberón [DS20, Theorem 1.2] showed that a fractional version of Conjecture 1.1 holds.

Since  $v(1) = \delta(1) = 1$ , we will assume that  $d \geq 2$  throughout the paper. We use  $[n]$  and  $\binom{[n]}{k}$  to denote the sets  $\{1, \dots, n\}$  and the set of all  $k$ -element subsets of  $[n]$ , respectively; for a family of sets  $\{K_1, \dots, K_n\}$  and  $\sigma \subset [n]$ ,  $K_\sigma$  denotes the intersection  $\bigcap_{i \in \sigma} K_i$ .

We prove that  $v(d) \leq (2d)^{3d}$  and  $\delta(d) \leq (2d)^3$ .

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**Theorem 1.** *Let  $\{K_1, \dots, K_n\}$  be a family of closed convex sets in  $\mathbb{R}^d$  such that their intersection  $K = K_1 \cap \dots \cap K_n$  is a convex body. Then there is a  $\mu \in \binom{[n]}{\leq 2d}$  such that*

$$\text{vol}_d K_\mu \leq (2d)^{3d} \text{vol}_d K \quad \text{and} \quad \text{diam } K_\mu \leq (2d)^3 \text{diam } K.$$

The bound on  $v(d)$  is not the best, as both [Nas16] and [Bra17] provide stronger estimates. The method that yields it is new and quite simple as it does not require the use of the John ellipsoid. The bound on  $\delta(d)$ , on the other hand, is currently the best.

As we will see, Theorem 1 follows from our main result which concerns a very rough approximation of a convex polytope by the convex hull of  $2d$  of its well-chosen vertices.

**Theorem 2.** *Let  $\lambda > 0$  and let  $Q \subset \mathbb{R}^d$  be a convex body satisfying the inclusion  $Q \subset -\lambda Q$ . Then*

(1) *there is a subset  $Q'$  of at most  $2d + 1$  extreme points of  $Q$  such that*

$$Q \subset -(\lambda + 1)(d + 1) \text{conv } Q',$$

*and*

(2) *there is a subset  $Q''$  of at most  $2d$  extreme points of  $Q$  such that*

$$Q \subset -(\lambda + 1)(2d^2 + 2d + 1) \text{conv } Q''.$$

By shifting the origin, one can guarantee  $\lambda = d$  for any convex body  $Q$  in  $\mathbb{R}^d$ , see Lemma 3.1. Recall that the polar  $K^\circ$  of a convex set  $K \subset \mathbb{R}^d$  is defined by

$$K^\circ = \{p \in \mathbb{R}^d : \langle p, x \rangle \leq 1 \quad \text{for all } x \in K\}.$$

By a standard polarity (duality) argument, Theorem 2 yields the following.

**Theorem 3.** *Let  $\{K_1, \dots, K_n\}$  be a family of closed convex sets in  $\mathbb{R}^d$  such that their intersection  $K = K_1 \cap \dots \cap K_n$  is a convex body. Then there is a point  $z$  in the interior of  $K$  such that*

(1)  *$(K - z)^\circ \subset -\lambda(K - z)^\circ$ , where  $1 \leq \lambda \leq d$ ;*

(2) *there is a  $\sigma \in \binom{[n]}{\leq 2d+1}$  such that*

$$K_\sigma - z \subset -(\lambda + 1)(d + 1)(K - z) \subset -4d^2(K - z);$$

(3) *there is a  $\mu \in \binom{[n]}{\leq 2d}$  such that*

$$K_\mu - z \subset -(\lambda + 1)(2d^2 + 2d + 1)(K - z) \subset -8d^3(K - z).$$

The containments in Theorem 3 immediately yield Theorem 1.

There are several ways to find a point  $z$  in the interior of a convex body  $K$  such that  $(K - z)^\circ \subset -d(K - z)^\circ$ , see Lemma 3.1.

We may interpret Theorem 3 as a new type of quantitative Helly-type theorem. First, for a set  $A$  in  $\mathbb{R}^d$ , we call  $\tilde{A} = \mu A + x$  a *homothet* of  $A$ , if  $x$  is a vector in  $\mathbb{R}^d$  and  $\mu \in \mathbb{R}$ ,  $\mu \neq 0$ . Next, for two convex bodies  $K$  and  $L$  in  $\mathbb{R}^d$  with  $K \subset L$ , we define their *homothetic distance* as the quantity

$$\inf\{|\lambda| : K - z \subset L - z \subset \lambda(K - z), z \in \mathbb{R}^d, \lambda \in \mathbb{R}\}.$$

Note that  $\lambda$  may be negative. This definition is motivated by Grünbaum's extension of the Banach–Mazur distance to non-symmetric convex bodies [Grü63] (see also [GLM<sup>+</sup>04, JN11]). Now, Theorem 3 can be rephrased as finding a subfamily of a family of convex bodies such that the intersection of the subfamily is at a bounded homothetic distance from the intersection of the entire family.

As a lower bound on  $\delta(d)$ , we show that the  $\sqrt{d}$  in Conjecture 1.1 cannot be replaced by a lower power of  $d$ .

**Theorem 4.** *For every  $i \in [n]$ , let  $K_i = \{x \in \mathbb{R}^d : \langle x, u_i \rangle \leq 1\}$ , where  $u_i$  is a unit vector. Then for any  $\sigma \subset [n]$ , there is a point in  $K_\sigma$  with norm  $\frac{d}{\sqrt{|\sigma|}}$ .*

It follows that if the  $u_i$  form a sufficiently dense subset of the unit sphere (with a large  $n$ ), then  $K = K_{[n]}$  is almost the unit ball, while for any  $\sigma \subset [n]$  of size  $|\sigma| = 2d$ , we have that  $\text{diam}(K_\sigma) \geq \sqrt{d/2}$ .

We mention the following conjecture which is closely related to Theorem 4. It can be found in a different formulation in [Bör04, p.194].

**Conjecture 1.2.** *Let  $\{u_1, \dots, u_{2d}\}$  be unit vectors in  $\mathbb{R}^d$ . There is a point in the set*

$$\bigcap_{i=1}^{2d} \{x \in \mathbb{R}^d : \langle u_i, x \rangle \leq 1\}$$

with norm  $\sqrt{d}$ .

## 2. PROOF OF THEOREM 2

Clearly, there is a largest volume simplex in  $Q$ , whose vertices are extreme points of  $Q$ . Let  $S$  be such a simplex. It is well known that

$$(2.1) \quad -d(S - v) + v \supseteq Q,$$

where  $v$  is the centroid of  $S$ . If  $v$  coincides with the origin, then by this inclusion, there is nothing to prove. Thus, we assume that  $v$  is not origin.

To prove assertion (1), we set  $\ell$  to be the ray in  $\mathbb{R}^d$  emanating from the origin in the direction  $-v$ , and let  $y$  be the point of intersection of  $\ell$  with the boundary of  $Q$ . By Carathéodory's theorem, there are extreme points  $q_1, \dots, q_d$  of  $Q$  on a support hyperplane to  $Q$  at  $y$  such that  $y \in \text{conv}\{q_1, \dots, q_d\}$ .

We set  $Q'$  to be the union of the vertex set of  $S$  and the set  $\{q_1, \dots, q_d\}$ . Since  $v \in Q$  and by  $Q \subset -\lambda Q$ , one has  $-v \in \lambda Q$ , which, by the choice of  $Q'$ , yields  $-v \in \lambda \text{conv } Q'$ , that is,

$$(2.2) \quad v \in -\lambda \text{conv } Q'.$$

Consequently,

$$Q \subseteq -d(S - v) + v \subseteq -d(\text{conv } Q' - v) + v = -d \text{conv } Q' + (d + 1)v \subseteq -(\lambda + 1)(d + 1) \text{conv } Q'.$$

This completes the proof of assertion (1) of the theorem.

We proceed with assertion (2) of the theorem. Our goal is to find a vertex of  $S$  that can be omitted.

Consider the ray in  $\mathbb{R}^d$  emanating from  $v$  in the direction  $v$ , and let  $q$  denote the intersection point of this ray and the boundary of  $S$ . Let  $q$  be in a facet  $F$  of  $S$ . Denote the simplex  $\text{conv}\{v, F\}$  by  $S_1$  and its centroid by  $w$ . Clearly,  $S \subseteq (d + 1)(S_1 - w) + w = (d + 1)S_1 - dw$ , and hence,

$$\begin{aligned} -d(S - v) + v &= -dS + (d + 1)v \subseteq -d((d + 1)S_1 - dw) + (d + 1)S_1 \subseteq \\ &= -d(d + 1)S_1 + d^2w + (d + 1)(-d(S_1 - w) + w) = \\ &= -2d(d + 1)S_1 + (2d^2 + 2d + 1)w. \end{aligned}$$

Thus, by (2.1), we have

$$(2.3) \quad Q \subseteq -2d(d + 1)S_1 + (2d^2 + 2d + 1)w.$$

Let  $\ell_2$  be the ray in  $\mathbb{R}^d$  emanating from the origin in the direction  $-w$ , and let  $y_2$  be the point of intersection of  $\ell_2$  with the boundary of  $Q$ . By Carathéodory's theorem, there are extreme points  $p_1, \dots, p_d$  of  $Q$  on a support hyperplane to  $Q$  at  $y_2$  such that  $y_2 \in \text{conv}\{p_1, \dots, p_d\}$ .

We set  $Q''$  to be the union of the vertex set of  $F$  and the set  $\{p_1, \dots, p_d\}$ , and claim that  $v \in \text{conv } Q''$ . Indeed, consider the simplex  $\text{conv}\{o, F\}$ . By construction,  $S_1 \subset \text{conv}\{o, F\}$ . It follows that the ray emanating from the origin in the direction of  $w$  intersects the facet  $F$  of  $S$ . Thus, the origin is in  $\text{conv } Q''$ , and hence,  $v \in \text{conv } Q''$ , see Figure 1.

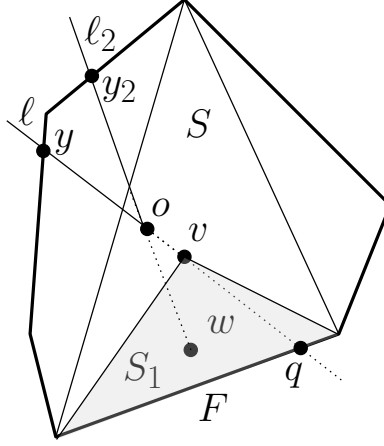


FIGURE 1.

Since  $q \in \text{conv } Q''$ , and  $v$  is on the line segment  $[o, q]$ , we have  $v \in \text{conv } Q''$ . It follows that

$$(2.4) \quad S_1 \subset \text{conv } Q''.$$

Again, since  $Q \subset -\lambda Q$ , we have  $-w \in \lambda Q$ , which, by the choice of  $Q''$ , yields  $-w \in \lambda \text{conv } Q''$ , that is,

$$(2.5) \quad w \in -\lambda \text{conv } Q''.$$

Finally, it follows from (2.3), (2.4) and (2.5) that

$$Q \subseteq -2d(d+1) \text{conv } Q'' - \lambda(2d^2 + 2d + 1) \text{conv } Q'' \subseteq -(\lambda+1)(2d^2 + 2d + 1) \text{conv } Q''.$$

This completes the proof of Theorem 2.

### 3. PROOF OF THEOREM 3

Recall that the *centroid*  $c(K)$  of a body  $K \subset \mathbb{R}^d$  is defined by

$$c(K) = \frac{1}{\text{vol}_d K} \int_K x \, dx.$$

**Lemma 3.1.** *Let  $K$  be a convex body in  $\mathbb{R}^d$ . Then the inclusion  $(K - z)^\circ \subset -d(K - z)^\circ$  holds if  $z$  is*

- (1) the centroid of  $K$ ;
- (2) the Santaló point of  $K$ ;
- (3) the center of John ellipsoid of  $K$ ,
- (4) the center of Löwner ellipsoid of  $K$ .

We note that other “centers” may be found using [GLM<sup>+</sup>04, Theorem 5.1].

*Proof.* It is well known (see [Grü60, Grü63]) that if  $c(K)$  is the origin, then  $K \subseteq -dK$ . By taking the polar of this containment, we obtain assertion (1).

Recall that the *Santaló point* of a convex body  $K \subset \mathbb{R}^d$  is the unique point  $z$  minimizing  $\text{vol}_d(K - z) \text{vol}_d(K - z)^\circ$ . Next, (2) follows from [AS17, Proposition D.2] which states that for any convex body  $K$  whose Santaló point is the origin, the centroid of  $K^\circ$  is the origin.

Recall that if  $E$  is the John ellipsoid of  $K$ , that is, the largest volume ellipsoid in  $K$ , and  $E$  is centered at  $z$ , then  $K \subseteq d(E - z) + z$ . A similar statement holds for the Löwner ellipsoid of  $K$ , that is, the smallest volume ellipsoid containing  $K$ . These containments then easily yield assertions (3) and (4), we leave the details to the reader.  $\square$

*Proof of Theorem 3.* By Lemma 3.1, there is a point  $z$  in  $K$  such that the inclusion  $(K - z)^\circ \subset -\lambda(K - z)^\circ$  holds with  $1 \leq \lambda \leq d$ . Fix such  $z$  and set  $Q = (K - z)^\circ$ .

Since the polar of the intersection of convex bodies containing the origin is the convex hull of the polars of these bodies, each extreme point of  $Q$  belongs to a set  $(K_i - z)^\circ$  for some  $i \in [n]$ . That is, if  $p$  is an extreme point of  $Q$ , then there exists  $i$  such that

$$K_i - z \subset \{x \in \mathbb{R}^d : \langle p, x \rangle \leq 1\}.$$

We will say in this case that  $i$  corresponds to  $p$ .

Let  $Q'$  and  $Q''$  be as in the assertions of Theorem 2. For every  $p \in Q'$ , we find one index  $i \in [n]$  that corresponds to  $p$ , and set  $\sigma$  to be the set of these indexes. Clearly,  $K_\sigma$  satisfies assertion (2) of Theorem 3. Analogously, for every  $p \in Q''$ , we find one index  $i \in [n]$  that corresponds to  $p$ , and set  $\mu$  to be the set of these indexes. Clearly,  $K_\mu$  satisfies assertion (3) of Theorem 3.  $\square$

#### 4. LOWER BOUND FOR DIAMETER

In this section, we prove Theorem 4. The result follows from the following lemma due to K. Ball and M. Prodromou.

**Lemma 4.1** ([BP09], Theorem 1.4). *Let vectors  $\{v_1, \dots, v_n\} \subset \mathbb{R}^d$  satisfying  $\sum_1^n v_i \otimes v_i = \text{Id}$ . Then for any positive semi-definite operator  $T: \mathbb{R}^d \rightarrow \mathbb{R}^d$ , there is a point  $p$  in the intersection of the strips  $\{x \in \mathbb{R}^d : |\langle x, v_i \rangle| \leq 1\}$  satisfying  $\langle p, Tp \rangle \geq \text{trace } T$ .*

*Proof of Theorem 4.* We prove a bit stronger statement. We will find a point with the desired large norm in the subset

$$K'_\sigma = \bigcap_{i \in \sigma} \{x : |\langle u_i, x \rangle| \leq 1\}$$

of  $K_\sigma$ . If  $\{u_i : i \in \sigma\}$  does not span the space, then  $K'_\sigma$  is unbounded. Thus, we assume  $\{u_i : i \in \sigma\}$  spans  $\mathbb{R}^d$ . Consider  $A = \sum_{i \in \sigma} u_i \otimes u_i$ . Since the vectors span the space,  $A$  is positive definite. Using Lemma 4.1 with  $v_i = A^{-1/2}u_i, i \in \sigma$ , and  $T = A^{-1}$ , we find a point  $p$  in

$$\bigcap_{i \in \sigma} \{x : |\langle v_i, x \rangle| \leq 1\}.$$

such that

$$\langle p, A^{-1}p \rangle \geq \text{trace } A^{-1}.$$

Denote  $q = A^{-1/2}p$ . Then, by the choice of  $p$ ,

$$1 \geq |\langle p, A^{-1/2}u_i \rangle| = |\langle A^{-1/2}p, u_i \rangle| = |\langle q, u_i \rangle|.$$

That is,  $q \in K'_\sigma$ . On the other hand,

$$|q|^2 = \langle A^{-1/2}p, A^{-1/2}p \rangle = \langle p, A^{-1}p \rangle \geq \text{trace } A^{-1}.$$

Finally, since  $\text{trace } A = |\sigma|$  and by the Cauchy–Schwarz inequality, one sees that  $\text{trace } A^{-1}$  is at least  $\frac{d^2}{|\sigma|}$ . Thus,  $|q| \geq \frac{d}{\sqrt{|\sigma|}}$ .  $\square$

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