

A LINK BETWEEN BOUGEROL'S IDENTITY AND A FORMULA DUE TO DONATI-MARTIN, MATSUMOTO AND YOR

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ABSTRACT. We point out an easy link between two striking identities on exponential functionals of the Wiener process and the Wiener bridge originated by Bougerol, and Donati-Martin, Matsumoto and Yor, respectively. The link is established using a continuous one-parameter family of Gaussian processes known as α -Wiener bridges or scaled Wiener bridges, which in case $\alpha = 0$ coincides with a Wiener process and for $\alpha = 1$ is a version of the Wiener bridge.

1. INTRODUCTION

Our starting point is Bougerol's identity in [5] which states that

$$(1.1) \quad \sinh(B_t) \stackrel{d}{=} W_{A_t} \quad \text{for every fixed } t \geq 0,$$

where $(B_t)_{t \geq 0}$ and $(W_t)_{t \geq 0}$ are independent standard Wiener processes, $\stackrel{d}{=}$ denotes equality in distribution, and

$$A_t = \int_0^t \exp(2 B_s) ds \quad \text{for } t \geq 0.$$

In fact there is also a generalization of Bougerol's identity with equality in law for stochastic processes due to Alili, Dufresne and Yor [1, Proposition 2]; cf. also [13, formula (69)] or [15, page 200]. Recently, there has been a renewed interest in generalizations of Bougerol's identity (1.1). Bertoin et al. [3] presented a two-dimensional extension of (1.1) that involves some exponential functional and the local time at 0 of a standard Wiener process. For another two-dimensional extension of (1.1), and even a three-dimensional one we refer to Vakeroudis [13, Sections 4.2 and 4.3].

We are only interested in the following particular case of the identity (1.1) presented in [13, 14]. Bougerol's identity (1.1) is equivalent to the equality of the corresponding

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continuous Lebesgue densities, which yields

$$\frac{1}{\sqrt{(1+x^2)t}} \exp\left(-\frac{\operatorname{Arsinh}^2(x)}{2t}\right) = \mathbb{E}\left[\frac{1}{\sqrt{A_t}} \exp\left(-\frac{x^2}{2A_t}\right)\right]$$

for all $t > 0$ and $x \in \mathbb{R}$, see, e.g., [14, formula (1.e)]. Especially, for $x = 0$, by the 1/2-self-similarity of a standard Wiener process and a change of variables $r = (4/\beta^2)s$ for some $\beta > 0$ we get

$$\begin{aligned} (1.2) \quad t^{-1/2} &= \mathbb{E}\left[\left(\int_0^t \exp(2B_s) \, ds\right)^{-1/2}\right] = \mathbb{E}\left[\left(\int_0^t \exp(\beta B_{(4/\beta^2)s}) \, ds\right)^{-1/2}\right] \\ &= \frac{2}{\beta} \cdot \mathbb{E}\left[\left(\int_0^{(4/\beta^2)t} \exp(\beta B_r) \, dr\right)^{-1/2}\right]. \end{aligned}$$

Hence, setting $t = \beta^2/4$ we get for every $\beta > 0$

$$\mathbb{E}\left[\left(\int_0^1 \exp(\beta B_s) \, ds\right)^{-1/2}\right] = 1.$$

This formula is a consequence of Bougerol's identity (1.1) which obviously holds for $\beta = 0$ and also remains true for $\beta < 0$, since $(-B_t)_{t \geq 0}$ is a Wiener process, i.e.,

$$(1.3) \quad \mathbb{E}\left[\left(\int_0^1 \exp(\beta B_s) \, ds\right)^{-1/2}\right] = 1 \quad \text{for every } \beta \in \mathbb{R}.$$

A similar identity due to Donati-Martin, Matsumoto and Yor [7, 8] holds when replacing the Wiener process $(B_t)_{t \geq 0}$ by a Wiener bridge $(B_t^\circ = B_t - tB_1)_{t \in [0,1]}$, a zero mean Gaussian process with covariance function $\operatorname{Cov}(B_s^\circ, B_t^\circ) = s(1-t)$ for $0 \leq s \leq t \leq 1$. Namely, this identity states that

$$(1.4) \quad \mathbb{E}\left[\left(\int_0^1 \exp(\beta B_s^\circ) \, ds\right)^{-1}\right] = 1 \quad \text{for every } \beta \in \mathbb{R}.$$

Hobson [9] provides a simple proof of (1.4) using a relationship between a Wiener bridge and a Wiener excursion obtained by Biane [4]. A further elementary proof of (1.4) is given in [7, Proposition 2.1].

Donati-Martin et al. [7] already pointed out how to obtain a link between the two identities (1.3) and (1.4) in the sense that the identity (1.3) follows from the identity (1.4) as a consequence of a formula combining exponential functionals of the Wiener process and the Wiener bridge, for details we refer to [7, Proposition 3.2].

Our aim is to give a different link between the two identities (1.3) and (1.4) using so-called α -Wiener bridges (also known as scaled Wiener bridges). These processes build a one-parameter family of Gaussian processes for parameter $\alpha \in \mathbb{R}$. They have been first considered by Brennan and Schwartz [6] and later have been investigated by Mansuy [11] and Barczy and Pap [2]. For our purposes an α -Wiener bridge $(X_t^{(\alpha)})_{t \in [0,1]}$ can be defined as a (weak) solution of the stochastic differential equation (SDE)

$$(1.5) \quad dX_t^{(\alpha)} = -\frac{\alpha}{1-t} X_t^{(\alpha)} dt + dB_t, \quad t \in [0, 1),$$

with initial condition $X_0^{(\alpha)} = 0$. Barczy and Pap [2] have shown that $(X_t^{(\alpha)})_{t \in [0,1]}$ is a bridge in the sense that $X_t^{(\alpha)} \rightarrow 0 =: X_1^{(\alpha)}$ as $t \uparrow 1$ almost surely if and only if $\alpha > 0$. Moreover, for $\alpha \geq 0$ it is shown in [2] that $(X_t^{(\alpha)})_{t \in [0,1]}$ is a zero mean Gaussian process with covariance function

$$(1.6) \quad \text{Cov}(X_s^{(\alpha)}, X_t^{(\alpha)}) = \begin{cases} \frac{(1-s)^\alpha(1-t)^\alpha}{1-2\alpha} (1 - (1-s)^{1-2\alpha}) & \text{if } \alpha \neq \frac{1}{2} \\ \sqrt{(1-s)(1-t)} \log\left(\frac{1}{1-s}\right) & \text{if } \alpha = \frac{1}{2} \end{cases}$$

for $0 \leq s \leq t \leq 1$. Note that for fixed $0 \leq s \leq t \leq 1$, (1.6) is continuous in $\alpha \geq 0$, which for $\alpha \rightarrow \frac{1}{2}$ can be easily seen by l'Hospital's rule. The unique strong solution of the SDE (1.5) with initial condition $X_0^{(\alpha)} = 0$ is given by

$$(1.7) \quad X_t^{(\alpha)} = \int_0^t \left(\frac{1-t}{1-s}\right)^\alpha dB_s \quad \text{for } t \in [0, 1),$$

and shows that $(X_t^{(0)})_{t \in [0,1]} = (B_t)_{t \in [0,1]}$ and $(X_t^{(1)})_{t \in [0,1]} \stackrel{d}{=} (B_t^\circ)_{t \in [0,1]}$. The latter is due to the fact that both sides of the equation are zero mean Gaussian processes with the same covariance function. Hence, variation of the parameter $\alpha \in [0, 1]$ continuously connects the Wiener process for $\alpha = 0$ with the Wiener bridge for $\alpha = 1$ in the sense that for $\alpha, \alpha_0 \geq 0$, the finite dimensional distributions of $(X^{(\alpha)})_{t \in [0,1]}$ converge weakly to those of $(X^{(\alpha_0)})_{t \in [0,1]}$ as $\alpha \rightarrow \alpha_0$. This follows directly from the continuity in α of the covariance function (1.6) and is the key observation for our link between the identities (1.3) and (1.4).

The paper is organized as follows. We will first show that certain space-time rescalings of an α -Wiener bridge either coincide in law with a usual Wiener bridge for $\alpha > \frac{1}{2}$ or with the Wiener process for $0 \leq \alpha < \frac{1}{2}$, see Proposition 2.1. Then an application of these space-time rescalings to the identity (1.4) and (1.3), respectively, yields two new

identities for certain transformations of exponential functionals of α -Wiener bridges which coincide when $\alpha = \frac{1}{2}$, see Theorem 2.2. We further show that a $\frac{1}{2}$ -Wiener bridge can be scaled to both, a Wiener bridge and a standard Wiener process, see Proposition 2.4. As a consequence, we present another two identities for certain transformations of exponential functionals of $\frac{1}{2}$ -Wiener bridges in Theorem 2.5.

2. LINK BETWEEN THE IDENTITIES

In the sequel, $\stackrel{\mathcal{D}}{=}$ denotes equality in law for stochastic processes on the space of continuous functions $C([0, 1])$ or $C([0, \infty))$, respectively.

Proposition 2.1. (a) For $\alpha > \frac{1}{2}$ we have

$$\left(\sqrt{2\alpha - 1} t^{\frac{\alpha-1}{2\alpha-1}} X_{1-t^{1/(2\alpha-1)}}^{(\alpha)} \right)_{t \in [0,1]} \stackrel{\mathcal{D}}{=} (X_t^{(1)})_{t \in [0,1]}.$$

(b) For $0 \leq \alpha < \frac{1}{2}$ we have

$$\left(\sqrt{1 - 2\alpha} (1-t)^{-\frac{\alpha}{1-2\alpha}} X_{1-(1-t)^{1/(1-2\alpha)}}^{(\alpha)} \right)_{t \in [0,1]} \stackrel{\mathcal{D}}{=} (X_t^{(0)})_{t \in [0,1]}.$$

Proof. We will first prove that the processes under consideration are zero mean Gaussian processes having almost surely continuous trajectories, which is not obvious for the left-hand sides as $t \downarrow 0$ for $\alpha \in (\frac{1}{2}, 1)$ in (a), and as $t \uparrow 1$ in (b), respectively. Once we know this, it remains to show the equality of covariance functions.

(a) Let $(M_t)_{t \in [0,1]}$ be the continuous martingale part of the process $X^{(\alpha)}$ given by (1.7)

$$M_t := \frac{X_t^{(\alpha)}}{(1-t)^\alpha} = \int_0^t \frac{1}{(1-s)^\alpha} dB_s \quad \text{for } t \in [0, 1]$$

with quadratic variation $\langle M \rangle_t = (1 - (1-t)^{1-2\alpha}) / (1-2\alpha) \rightarrow \infty$ as $t \uparrow 1$ for $\alpha > \frac{1}{2}$ as obtained in [2, formula (3.1)]. Then, similarly to the proof of [2, Lemma 3.1], for the increasing function $[1, \infty) \ni x \mapsto f(x) = x^{3/4}$ with $\int_1^\infty (f(x))^{-2} dx < \infty$, an application of [10, Theoreme 1] or Exercise 1.16 in Chapter V of [12] gives $M_t / f(\langle M \rangle_t) \rightarrow 0$ a.s. as $t \uparrow 1$. Letting $t = 1 - s^{1/(2\alpha-1)} \uparrow 1$ as $s \downarrow 0$ this shows

$$\frac{s^{\frac{-\alpha}{2\alpha-1}} X_{1-s^{1/(2\alpha-1)}}^{(\alpha)}}{\left((1-s^{-1}) / (1-2\alpha) \right)^{3/4}} \rightarrow 0 \quad \text{a.s. as } s \downarrow 0.$$

To obtain $s^{\frac{\alpha-1}{2\alpha-1}} X_{1-s^{1/(2\alpha-1)}}^{(\alpha)} \rightarrow 0$ a.s. as $s \downarrow 0$ it suffices to see that for $s \downarrow 0$ we have

$$s^{\frac{\alpha-1}{2\alpha-1}} s^{\frac{\alpha}{2\alpha-1}} (s^{-1} - 1)^{3/4} = s(s^{-1} - 1)^{3/4} = s^{1/4} (1-s)^{3/4} \rightarrow 0.$$

Hence the centered Gaussian processes under consideration almost surely have continuous sample paths on $[0, 1]$ starting in the origin. Thus it remains to show the equality of their covariance functions for $0 < s \leq t \leq 1$. Using (1.6) and the fact that the function $(0, 1] \ni t \mapsto 1 - t^{1/(2\alpha-1)}$ is decreasing, we get for $0 < s \leq t \leq 1$

$$\begin{aligned} \text{Cov} \left(X_{1-s^{1/(2\alpha-1)}}^{(\alpha)}, X_{1-t^{1/(2\alpha-1)}}^{(\alpha)} \right) &= \frac{s^{\frac{\alpha}{2\alpha-1}} t^{\frac{\alpha}{2\alpha-1}}}{1-2\alpha} (1-t^{-1}) \\ &= \frac{s^{\frac{\alpha}{2\alpha-1}} t^{\frac{\alpha}{2\alpha-1}-1}}{2\alpha-1} (1-t) = \frac{s^{\frac{1-\alpha}{2\alpha-1}} t^{\frac{1-\alpha}{2\alpha-1}}}{2\alpha-1} s(1-t) \end{aligned}$$

from which the assertion easily follows.

(b) In case $\alpha = 0$ the identity is trivially fulfilled. For $0 < \alpha < \frac{1}{2}$ it is shown in the proof of [2, Lemma 3.1] that $\lim_{t \uparrow 1} (1-t)^{-\alpha} X_t^{(\alpha)}$ exists in \mathbb{R} almost surely and has a normal distribution as a limit of normally distributed random variables. Letting $t = 1 - (1-s)^{1/(1-2\alpha)} \uparrow 1$ as $s \uparrow 1$ we have

$$\lim_{s \uparrow 1} (1-s)^{-\frac{\alpha}{1-2\alpha}} X_{1-(1-s)^{1/(1-2\alpha)}}^{(\alpha)} \text{ exists a.s.,}$$

which shows that the centered Gaussian processes under consideration almost surely have continuous sample paths on $[0, 1]$ starting in the origin. Thus it remains to show the equality of their covariance functions. Using (1.6) and the fact that the function $[0, 1] \ni t \mapsto 1 - (1-t)^{1/(1-2\alpha)}$ is increasing, we get for $0 \leq s \leq t \leq 1$

$$\text{Cov} \left(X_{1-(1-s)^{1/(1-2\alpha)}}^{(\alpha)}, X_{1-(1-t)^{1/(1-2\alpha)}}^{(\alpha)} \right) = \frac{(1-s)^{\frac{\alpha}{1-2\alpha}} (1-t)^{\frac{\alpha}{1-2\alpha}}}{1-2\alpha} s$$

from which again the assertion easily follows. \square

Theorem 2.2. (a) For $\alpha > \frac{1}{2}$ and any $\beta \in \mathbb{R}$ we have

$$\mathbb{E} \left[\left(\int_0^1 \exp \left(\frac{\beta}{(1-s)^{1-\alpha}} X_s^{(\alpha)} \right) \frac{ds}{(1-s)^{2(1-\alpha)}} \right)^{-1} \right] = 2\alpha - 1.$$

(b) For $0 \leq \alpha < \frac{1}{2}$ and any $\beta \in \mathbb{R}$ we have

$$\mathbb{E} \left[\left(\int_0^1 \exp \left(\frac{\beta}{(1-s)^\alpha} X_s^{(\alpha)} \right) \frac{ds}{(1-s)^{2\alpha}} \right)^{-1/2} \right] = \sqrt{1-2\alpha}.$$

(c) For $\alpha = \frac{1}{2}$ and any $\beta \in \mathbb{R}$ both identities in (a) and (b) hold.

Remark 2.3. For the $\frac{1}{2}$ -Wiener bridge the two identities in (a) and (b) of Theorem 2.2 are valid by part (c) and are in fact equivalent, since both identities show that

for any $\beta \in \mathbb{R}$ the non-negative random variable

$$Y(\beta) := \left(\int_0^1 \exp\left(\frac{\beta}{\sqrt{1-s}} X_s^{(1/2)}\right) \frac{ds}{1-s} \right)^{-1/2} = 0 \quad \text{almost surely.}$$

Hence the version of the Bougerol identity in (b) represents the mean $\mathbb{E}[Y(\beta)] = 0$, whereas the formula (a), as a version of the identity due to Donati-Martin, Matsumoto and Yor, represents the second moment $\mathbb{E}[(Y(\beta))^2] = 0$.

Proof of Theorem 2.2. (a) An application of Proposition 2.1 (a) to (1.4) together with a change of variables $s = 1 - t^{\frac{1}{2\alpha-1}}$ yields for any $\beta \in \mathbb{R}$

$$\begin{aligned} 1 &= \mathbb{E} \left[\left(\int_0^1 \exp(\beta X_t^{(1)}) dt \right)^{-1} \right] \\ &= \mathbb{E} \left[\left(\int_0^1 \exp\left(\beta \sqrt{2\alpha-1} t^{\frac{\alpha-1}{2\alpha-1}} X_{1-t^{1/(2\alpha-1)}}^{(\alpha)}\right) dt \right)^{-1} \right] \\ &= \mathbb{E} \left[\left(\int_0^1 \exp\left(\frac{\tilde{\beta}}{(1-s)^{1-\alpha}} X_s^{(\alpha)}\right) \cdot (2\alpha-1) \frac{ds}{(1-s)^{2(1-\alpha)}} \right)^{-1} \right], \end{aligned}$$

where $\tilde{\beta} = \beta \sqrt{2\alpha-1} \in \mathbb{R}$ is arbitrary.

(b) For $\alpha = 0$ the identity is a restatement of (1.3). For $0 < \alpha < \frac{1}{2}$ an application of Proposition 2.1 (b) to (1.3) together with a change of variables $s = 1 - (1-t)^{1/(1-2\alpha)}$ yields for any $\beta \in \mathbb{R}$

$$\begin{aligned} 1 &= \mathbb{E} \left[\left(\int_0^1 \exp(\beta X_t^{(0)}) dt \right)^{-1/2} \right] \\ &= \mathbb{E} \left[\left(\int_0^1 \exp\left(\beta \sqrt{1-2\alpha} (1-t)^{-\frac{\alpha}{1-2\alpha}} X_{1-(1-t)^{1/(1-2\alpha)}}^{(\alpha)}\right) dt \right)^{-1/2} \right] \\ &= \mathbb{E} \left[\left(\int_0^1 \exp\left(\frac{\tilde{\beta}}{(1-s)^\alpha} X_s^{(\alpha)}\right) \cdot (1-2\alpha) \frac{ds}{(1-s)^{2\alpha}} \right)^{-1/2} \right], \end{aligned}$$

where $\tilde{\beta} = \beta \sqrt{1-2\alpha} \in \mathbb{R}$ is arbitrary.

(c) For $\alpha = \frac{1}{2}$ the process $(M_t)_{t \in [0,1]}$ with $M_t = (1-t)^{-1/2} X_t^{(1/2)} = \int_0^t (1-s)^{-1/2} dB_s$ is a centered continuous martingale with quadratic variation $\langle M \rangle_t = -\log(1-t) \rightarrow \infty$ as $t \uparrow 1$; see formulas (3.1) and (3.2) in [2]. Hence by the Dambis, Dubins-Schwarz theorem there exists a Wiener process $(\tilde{B}_t)_{t \geq 0}$ such that $(M_t)_{t \in [0,1]} = (\tilde{B}_{\langle M \rangle_t})_{t \in [0,1]}$

almost surely; see Theorem 1.6 in Chapter V of [12]. It follows by a change of variables $t = \langle M \rangle_s = -\log(1-s)$ and monotone convergence that for $\beta \neq 0$

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^1 \exp \left(\frac{\beta}{\sqrt{1-s}} X_s^{(1/2)} \right) \frac{ds}{1-s} \right)^{-1/2} \right] \\ &= \mathbb{E} \left[\left(\int_0^1 \exp (\beta \tilde{B}_{-\log(1-s)}) \frac{ds}{1-s} \right)^{-1/2} \right] = \mathbb{E} \left[\left(\int_0^\infty \exp (\beta \tilde{B}_t) dt \right)^{-1/2} \right] \\ &= \lim_{T \rightarrow \infty} \mathbb{E} \left[\left(\int_0^T \exp (\beta \tilde{B}_t) dt \right)^{-1/2} \right] = \lim_{T \rightarrow \infty} T^{-1/2} = 0, \end{aligned}$$

where the last but one equality follows by setting $t = \beta^2 T/4$ in (1.2). Since in case $\beta = 0$ the expectation is obviously vanishing, this shows that the identity in (b) is fulfilled for $\alpha = \frac{1}{2}$. In particular it shows that a non-negative random variable has zero expectation and thus is equal to zero almost surely. Hence also its second moment vanishes, which proves the identity in (a) for $\alpha = \frac{1}{2}$. \square

In case $\alpha = \frac{1}{2}$ it is possible to link the $\frac{1}{2}$ -Wiener bridge $(X_t^{(1/2)})_{t \in [0,1]}$ to both identities (1.4) and (1.3) with non-vanishing expectation by either introducing an additional log-term in the integrand or by integrating over a smaller domain as follows. We first present the corresponding space-time scalings, which might be of independent interest.

Proposition 2.4. *We have*

$$(2.1) \quad \left(t \sqrt{\exp(t^{-1}-1)} X_{1-\exp(1-t^{-1})}^{(1/2)} \right)_{t \in [0,1]} \stackrel{\mathcal{D}}{=} (X_t^{(1)})_{t \in [0,1]}.$$

$$(2.2) \quad \left(e^{t/2} X_{1-\exp(-t)}^{(1/2)} \right)_{t \geq 0} \stackrel{\mathcal{D}}{=} (X_t^{(0)})_{t \geq 0}.$$

Proof. We first show that as $t \downarrow 0$ we have

$$(2.3) \quad t \sqrt{\exp(t^{-1}-1)} X_{1-\exp(1-t^{-1})}^{(1/2)} \rightarrow 0 \quad \text{a.s.}$$

From the proof of part (c) of Theorem 2.2 we know that there exists a Wiener process $(\tilde{B}_t)_{t \geq 0}$ such that $((1-s)^{-1/2} X_s^{(1/2)})_{s \in [0,1]} = (\tilde{B}_{-\log(1-s)})_{s \in [0,1]}$ almost surely. Letting $s = 1 - \exp(1-t^{-1})$ we get

$$\left(\sqrt{\exp(t^{-1}-1)} X_{1-\exp(1-t^{-1})}^{(1/2)} \right)_{t \in (0,1]} = \left(\tilde{B}_{t^{-1}-1} \right)_{t \in (0,1]} \quad \text{a.s.}$$

from which (2.3) follows by the strong law of large numbers for Brownian motion, since almost surely

$$t \sqrt{\exp(t^{-1} - 1)} X_{1-\exp(1-t^{-1})}^{(1/2)} = t \tilde{B}_{t^{-1}-1} = (1-t) \frac{t}{1-t} \tilde{B}_{\frac{1-t}{t}} \rightarrow 0$$

as $t \downarrow 0$. Hence the centered Gaussian processes under consideration in (2.1) almost surely have continuous sample paths on $[0, 1]$ starting in the origin. Thus it remains to show the equality of their covariance functions for $0 < s \leq t \leq 1$. Using (1.6) and the fact that the function $(0, 1] \ni t \mapsto 1 - \exp(1 - t^{-1})$ is decreasing, we get for any $0 < s \leq t \leq 1$,

$$\begin{aligned} \text{Cov} \left(X_{1-\exp(1-s^{-1})}^{(1/2)}, X_{1-\exp(1-t^{-1})}^{(1/2)} \right) &= \sqrt{\exp(1-s^{-1})} \sqrt{\exp(1-t^{-1})} (t^{-1} - 1) \\ &= \frac{\sqrt{\exp(1-s^{-1})} \sqrt{\exp(1-t^{-1})}}{s \cdot t} s(1-t), \end{aligned}$$

from which (2.1) easily follows. Similarly, for any $0 \leq s \leq t$ we get using (1.6)

$$\text{Cov} \left(X_{1-\exp(-s)}^{(1/2)}, X_{1-\exp(-t)}^{(1/2)} \right) = e^{-s/2} e^{-t/2} s$$

from which (2.2) easily follows. \square

Theorem 2.5. *For any $\beta \in \mathbb{R}$ we have*

$$\mathbb{E} \left[\left(\int_0^1 \exp \left(\frac{\beta}{\sqrt{1-s} (1 - \log(1-s))} X_s^{(1/2)} \right) \frac{ds}{(1-s) (1 - \log(1-s))^2} \right)^{-1} \right] = 1$$

and

$$\mathbb{E} \left[\left(\int_0^{1-e^{-1}} \exp \left(\frac{\beta}{\sqrt{1-s}} X_s^{(1/2)} \right) \frac{ds}{1-s} \right)^{-1/2} \right] = 1.$$

Proof. Applying (2.1) to (1.4) together with a change of variables $s = 1 - e^{-(t^{-1}-1)}$ yields for any $\beta \in \mathbb{R}$

$$\begin{aligned} 1 &= \mathbb{E} \left[\left(\int_0^1 \exp(\beta X_t^{(1)}) dt \right)^{-1} \right] \\ &= \mathbb{E} \left[\left(\int_0^1 \exp \left(\beta t \sqrt{\exp(t^{-1} - 1)} X_{1-\exp(1-t^{-1})}^{(1/2)} \right) dt \right)^{-1} \right] \\ &= \mathbb{E} \left[\left(\int_0^1 \exp \left(\frac{\beta}{\sqrt{1-s} (1 - \log(1-s))} X_s^{(1/2)} \right) \frac{ds}{(1-s) (1 - \log(1-s))^2} \right)^{-1} \right] \end{aligned}$$

which proves the first identity. Similarly, an application of (2.2) to (1.3) together with a change of variables $s = 1 - e^{-t}$ yields for any $\beta \in \mathbb{R}$

$$\begin{aligned} 1 &= \mathbb{E} \left[\left(\int_0^1 \exp(\beta X_t^{(0)}) dt \right)^{-1/2} \right] \\ &= \mathbb{E} \left[\left(\int_0^1 \exp \left(\beta e^{t/2} X_{1-e^{-t}}^{(1/2)} \right) dt \right)^{-1/2} \right] \\ &= \mathbb{E} \left[\left(\int_0^{1-e^{-1}} \exp \left(\frac{\beta}{\sqrt{1-s}} X_s^{(1/2)} \right) \frac{ds}{1-s} \right)^{-1/2} \right] \end{aligned}$$

which proves the second identity. □

Remark 2.6. Motivated by the identities (1.3) and (1.4), one can formulate the open question whether there exists a (continuous) function $p : [0, 1] \rightarrow (-\infty, 0)$ such that

$$\mathbb{E} \left[\left(\int_0^1 \exp \left(\beta X_t^{(\alpha)} \right) dt \right)^{p(\alpha)} \right] = 1 \quad \text{for every } \beta \in \mathbb{R}.$$

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