

LEIBNIZ SEMINORMS IN PROBABILITY SPACES

ÁDÁM BESENYEI AND ZOLTÁN LÉKA

ABSTRACT. In this paper we study the (strong) Leibniz property of centered moments of bounded random variables. We shall answer a question raised by M. Rieffel on the non-commutative standard deviation.

1. INTRODUCTION

We say that a seminorm L on a unital normed algebra $(\mathcal{A}, \|\cdot\|)$ is strongly Leibniz if (i) $L(1_{\mathcal{A}}) = 0$, (ii) the Leibniz property

$$L(ab) \leq \|a\|L(b) + \|b\|L(a)$$

holds for every $a, b \in \mathcal{A}$ and, furthermore, (iii) for every invertible a ,

$$L(a^{-1}) \leq \|a^{-1}\|^2 L(a)$$

follows. Primary sources of strongly Leibniz seminorms are normed first-order differential calculi, see [8]. It is said that the couple (Ω, δ) is a normed first-order differential calculus over \mathcal{A} if Ω is a normed bimodule over \mathcal{A} and δ is a derivation from \mathcal{A} to Ω . Now let us assume that Ω is acting boundedly over \mathcal{A} ; that is, the inequalities

$$\|\omega a\| \leq \|\omega\|_{\Omega} \|a\| \quad \text{and} \quad \|\omega a\| \leq \|\omega\|_{\Omega} \|a\|$$

hold for every $\omega \in \Omega$ and for every $a \in \mathcal{A}$. From the derivation rule

$$\delta(ab) = \delta(a)b + a\delta(b),$$

the Leibniz property of the seminorm $L(a) = \|\delta(a)\|_{\Omega}$ simply follows. Furthermore, we clearly have that

$$\delta(a^{-1}) = -a^{-1}\delta(a)a^{-1},$$

whenever a is invertible, hence (iii) follows as well. For instance, if we choose a (real or complex) Banach space X and $\mathcal{B}(X)$ denotes the normed algebra of its bounded linear operators, practically, we can easily get a first-order differential calculus. Actually, with the choice of $\Omega = \mathcal{B}(X)$, which acts naturally over $\mathcal{B}(X)$ via the left and right multiplications, the commutator $\delta(A) = [D, A] = DA - AD$ for some fixed $D \in \mathcal{B}(X)$ defines the required calculus.

Consider a unital C^* -algebra \mathcal{A} and denote \mathcal{B} a C^* -subalgebra of \mathcal{A} with a common unit. Rieffel pointed out in [7, Theorem] that the factor norm $\inf_{b \in \mathcal{B}} \|a - b\|$ obeys the strong Leibniz property, since it equals to a commutator norm. To get connection with the standard deviation, notice that K. Audenaert provided sharp estimate for different types of non-commutative (or quantum) deviations determined by matrices [1]. Not long ago Rieffel extended these results to C^* -algebras with a completely different approach [8]. His theorem reads as follows: for any $a \in \mathcal{A}$,

$$\max_{\omega \in \mathcal{S}(\mathcal{A})} \omega(|a - \omega(a)|^2)^{1/2} = \min_{\lambda \in \mathbb{C}} \|a - \lambda \mathbf{1}_{\mathcal{A}}\|,$$

2000 *Mathematics Subject Classification.* Primary 46L53, 60E15 ; Secondary 26A51, 60B99.
Key words and phrases. standard deviation, Leibniz seminorm, central moments, C^* -algebra.
 This study was partially supported by the Hungarian NSRF (OTKA) grant no. K104206 .

where $\mathcal{S}(\mathcal{A})$ denotes the state space of \mathcal{A} ; i.e. the set of positive linear functionals of \mathcal{A} with norm 1. For a short proof of this theorem, exploiting the Birkhoff–James orthogonality in operator algebras, the reader might see [2]. The factor norm on the left-hand side above indicates that ‘the largest standard deviation’ is a strongly Leibniz seminorm. Surprisingly, the standard deviation itself is a strongly Leibniz seminorm. Precisely, whenever $\sigma_2^\omega(a) = \omega(|a - \omega(a)|^2)^{1/2}$, the seminorm σ_2^ω on \mathcal{A} is strongly Leibniz if ω is tracial [8, Proposition 3.4]. Moreover, if one defines the non-commutative standard deviation by the formula

$$\tilde{\sigma}_2^\omega(a) = \omega(|a - \omega(a)|^2)^{1/2} \vee \omega(|a^* - \omega(a^*)|^2)^{1/2},$$

then $\tilde{\sigma}_2^\omega$ is strongly Leibniz for any $\omega \in \mathcal{S}(\mathcal{A})$, see [8, Theorem 3.5] (without assuming that ω is tracial). Quite recently, the equality

$$\max_{\omega \in \mathcal{S}(\mathcal{A})} \omega(|a - \omega(a)|^k)^{1/k} = 2B_k^{1/k} \min_{\lambda \in \mathbb{C}} \|a - \lambda \mathbf{1}_{\mathcal{A}}\|$$

was proved in [4] for the k th central moments of normal elements, where k is even and B_k denotes the largest k th centered moment of the Bernoulli distribution. From this result it follows that ‘the largest k th moments’ in commutative C^* -algebras are strongly Leibniz as well.

The aim of the paper is to study whether general or higher-ordered centered moments possess the (strong) Leibniz property in ordinary probability spaces, or not. In the next section we shall give a rough estimate of the centered moments of products of bounded random variables which gives back Rieffel’s statement on the standard deviation. After that we shall present some scattered Leibniz-type result for different moments on different (discrete, general) probability spaces. We leave open the question whether all centered moments in general probability spaces define a strongly Leibniz seminorm. Lastly, in Section 3, we shall answer affirmatively Rieffel’s question on the standard deviation in non-commutative probability spaces.

2. LEIBNIZ SEMINORMS IN FUNCTION SPACES

In this section we shall study the Leibniz property and similar estimates in ordinary probability spaces. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. For any $f: \Omega \rightarrow \mathbb{C} \in L^\infty(\Omega, \mu)$ and $1 \leq p < \infty$, let us define

$$\sigma_p(f; \mu) = \left(\int_{\Omega} \left| f - \int_{\Omega} f d\mu \right|^p d\mu \right)^{1/p}$$

and

$$\sigma_\infty(f; \mu) = \operatorname{ess\,sup} \left| f - \int_{\Omega} f d\mu \right|.$$

If no confusion can arise, we simply use the notation $\sigma_p(f)$. Relying on [8], we know that the standard deviation is a strongly Leibniz seminorm; that is, the inequalities

$$\sigma_2(fg) \leq \|g\|_\infty \sigma_2(f) + \|f\|_\infty \sigma_2(g)$$

for $f, g \in L^\infty(\Omega, \mu)$, and

$$\sigma_2(1/f) \leq \|1/f\|_\infty^2 \sigma_2(f)$$

whenever $1/f \in L^\infty(\Omega, \mu)$ hold. For the non-commutative analogues of the result, see [8].

We begin with an observation which shows that one can reduce the problem of the strongly Leibniz property to that of the discrete uniform distributions.

Proposition 2.1. *Fix $1 \leq p < \infty$. The following statements are equivalent:*

- (i) *For any probability space $(\Omega, \mathcal{F}, \mu)$, σ_p is a strongly Leibniz seminorm on $L^\infty(\Omega, \mu)$.*

- (ii) For every $n \in \mathbb{Z}_+$, σ_p is a strongly Leibniz seminorm on ℓ_n^∞ endowed with the uniform distribution.

Proof. Obviously, (i) implies (ii). To see the reverse implication, choose pairwise disjoint sets $S_k \in \mathcal{F}$ ($1 \leq k \leq n$). As usual χ_{S_k} denotes the characteristic function of the set S_k . Let us consider the measurable simple functions $f_n = \sum_{k=1}^n a_k \chi_{S_k}$ and $g_n = \sum_{k=1}^n b_k \chi_{S_k}$ on Ω . Let us assume that $\bigcup_{k=1}^n S_k = \Omega$, so that the constants $\mu(S_k)$ define a probability measure μ_n on the set $\mathbb{Z}_n = \{1, \dots, n\}$. Then for any $\varepsilon > 0$ we can readily find a probability measure $\nu_n = (p_1, \dots, p_n)$ such that $p_i \in \mathbb{Q}$ ($1 \leq i \leq n$) and the inequalities

$$\begin{aligned} |\sigma_p(f_n; \mu_n) - \sigma_p(f_n; \nu_n)| &\leq \varepsilon \\ |\sigma_p(g_n; \mu_n) - \sigma_p(g_n; \nu_n)| &\leq \varepsilon \\ |\sigma_p(f_n g_n; \mu_n) - \sigma_p(f_n g_n; \nu_n)| &\leq \varepsilon \end{aligned}$$

hold. Now let us choose the integers m and r_i such that $p_i = r_i/m$ for every $1 \leq i \leq n$. Then the map

$$\Phi: (c_1, \dots, c_n) \mapsto (\underbrace{c_1, \dots, c_1}_{r_1}, \dots, \underbrace{c_n, \dots, c_n}_{r_n})$$

defines an isometric algebra homomorphism from ℓ_n^∞ into ℓ_m^∞ . Let λ_m denote the uniform distribution on the set \mathbb{Z}_m . We clearly have, for instance, $\sigma_p(f_n; \nu_n) = \sigma_p(\Phi(f_n); \lambda_m)$, hence

$$\sigma_p(f_n g_n; \nu_n) \leq \|f_n\|_\infty \sigma_p(g_n; \nu_n) + \|g_n\|_\infty \sigma_p(f_n; \nu_n)$$

follows as well. Since ε can be arbitrary small, we obtain that σ_p is a Leibniz seminorm on $\ell_n^\infty(\mu_n)$. Now if we choose sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ of measurable simple functions such that $f_n \rightarrow f$ and $g_n \rightarrow g$ in L^p norm, furthermore, $\|f_n\|_\infty = \|f\|_\infty$ and $\|g_n\|_\infty = \|g\|_\infty$ hold for every n , we infer that σ_p has the Leibniz property. A very similar reasoning on the invertible elements gives that σ_p is actually strongly Leibniz on $L^\infty(\Omega, \mu)$. \square

Despite of the above equivalence, in arbitrary measure spaces we do not know whether σ_p is strongly Leibniz or not. But later we will prove this property for σ_∞ in the real Banach space $L^\infty(\Omega, \mu; \mathbb{R})$ (see Theorem 2.6 below). Actually, the second part of the section deals with only real-valued functions. In the general situation, we have only a rough Leibniz-type estimate as we shall see below.

In any $L^p(\Omega, \mu)$ ($1 \leq p \leq \infty$) space, the projection P is given by the map

$$f \mapsto \mathbb{E}f = \int_\Omega f d\mu.$$

Then we are able to prove a slight generalization of Rieffel's statement [8, Proposition 3.4] in probability spaces.

Proposition 2.2. For any $1 \leq p \leq \infty$ and $f, g \in L^\infty(\Omega, \mu)$, we have that

$$\frac{2}{\|I - P\|_p + 1} \|fg - \mathbb{E}(fg)\|_p \leq \|g\|_\infty \|f - \mathbb{E}f\|_p + \|f\|_\infty \|g - \mathbb{E}g\|_p.$$

Proof. First, note that $\|I - P\|_p \geq 1$ (except for the trivial case $I = P$). Hence, without loss of generality, we can assume that

$$\|fg - \mathbb{E}(fg)\|_p \geq \max(\|f\|_\infty \|g - \mathbb{E}g\|_p, \|g\|_\infty \|f - \mathbb{E}f\|_p),$$

otherwise the proof is done. Obviously,

$$\|f(g - \mathbb{E}g) - \mathbb{E}(f(g - \mathbb{E}g))\|_p \leq \|I - P\|_p \|f(g - \mathbb{E}g)\|_p.$$

From the reversed triangle inequality we obtain that

$$\|fg - \mathbb{E}(fg)\|_p - \|\mathbb{E}f\mathbb{E}g - f\mathbb{E}g\|_p \leq \|f(g - \mathbb{E}g) - \mathbb{E}(f(g - \mathbb{E}g))\|_p,$$

which implies that

$$\|fg - \mathbb{E}(fg)\|_p \leq \|I - P\|_p \|f\|_\infty \|g - \mathbb{E}g\|_p + \|g\|_\infty \|f - \mathbb{E}f\|_p.$$

Changing the variables f, g and summing up the inequalities, we get the statement of the proposition. \square

Remark 2.3. One can find a non-trivial upper estimate of the constant $\|I - P\|_p$. For instance, if $\Omega = \{1, \dots, n\}$ and μ is the uniform distribution on Ω , from the definition of the matrix p -norms one can easily see that $\|I - P\|_1 = \|I - P\|_\infty = 2 - \frac{2}{n}$ and $\|I - P\|_2 = 1$. As another example, let us consider the Banach spaces $L^p[0, 1]$ endowed with the Lebesgue measure. Then a simple calculation shows that $\|I - P\|_1 = \|I - P\|_\infty = 2$. Moreover, $I - P$ is clearly an orthogonal projection in $L^2[0, 1]$; that is, $\|I - P\|_2 = 1$. Now a straightforward application of the Riesz-Thorin interpolation theorem gives that (see [6])

$$\|I - P\|_p \leq 2^{|1 - \frac{1}{2p}|}.$$

The projection $I - P$ is actually the minimal projection to the hyperplane $X_p = \{f \in L^p[0, 1] : \mathbb{E}f = 0\}$; i.e. it has the minimal norm among the projections of range X_p . C. Franchetti showed in his paper [3] that

$$\|I - P\|_p = \max_{0 \leq x \leq 1} (x^{p-1} + (1-x)^{p-1})^{1/p} (x^{q-1} + (1-x)^{q-1})^{1/q},$$

where $1/p + 1/q = 1$.

Remark 2.4. One can apply a derivation approach mentioned in the Introduction to obtain Leibniz-type estimates of the moments of invertible functions. To do this, let us renorm the space $L^p(\Omega, \mu)$, $2 \leq p < \infty$, so that

$$\|x\|_{p,\nu} := |\mathbb{E}x| + \|x - \mathbb{E}x\|_p.$$

Let X denote the renormed space. Define the multiplication operator $M_f : x \mapsto fx$ and the derivation $\delta(M_f) = [P, M_f] = PM_f - M_fP$. A straightforward calculation yields that

$$\|M_fx\|_{p,\nu} \leq \|f\|_\infty \|x\|_p + \|I - P\|_p \|f\|_\infty \|x\|_p \leq (1 + \|I - P\|_p) \|f\|_\infty \|x\|_{p,\nu};$$

that is, $\|M_f\| \leq (1 + \|I - P\|_p) \|f\|_\infty$. Moreover, $\delta(M_f)\mathbb{E}x = \mathbb{E}x(\mathbb{E}f - f) \in (I - P)X$, thus $\|\delta(M_f)|_{PX}\| = \sigma_p(f)$. On the other hand, $\delta(M_f)(x - \mathbb{E}x) = \mathbb{E}(fx) - \mathbb{E}f\mathbb{E}x = \mathbb{E}((f - \mathbb{E}f)(x - \mathbb{E}x))$. From Hölder's inequality we get that

$$\|\delta(M_f)(x - \mathbb{E}x)\|_{p,\nu} \leq \|f - \mathbb{E}f\|_q \|x - \mathbb{E}x\|_p \quad (1/q + 1/p = 1)$$

hence $\|\delta(M_f)|_{(I-P)X}\| \leq \sigma_p(f)$ follows. Since the operator $\delta(M_f)$ interchanges the subspaces PX and $(I - P)X$, we have

$$\|\delta(M_f)\| = \sigma_p(f).$$

An application of the derivation rules tells us that

$$\sigma_p(1/f) \leq (1 + \|I - P\|_p)^2 \|1/f\|_\infty^2 \sigma_p(f)$$

holds whenever $1/f \in L^\infty(\Omega, \mu)$.

For any $1 \leq p \leq \infty$, we can get a different estimate from the equality

$$(I - P)M_{1/f}(I - P)f = (1/f - \mathbb{E}(1/f))\mathbb{E}f.$$

Hence we conclude that

$$|\mathbb{E}f| \sigma_p(1/f) \leq \|I - P\|_p \|1/f\|_\infty \sigma_p(f).$$

Much of the rest of the section is devoted to a study of the optimality of the above proposition. We begin with the following observation.

Proposition 2.5. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. For any real-valued f and $x \in L^\infty(\Omega, \mu)$, the inequality*

$$\|f\mathbb{E}x - \mathbb{E}(fx)\|_\infty \leq \|x\|_\infty \|f - \mathbb{E}f\|_\infty$$

holds.

Proof. Without loss of generality, we can assume that $\mathbb{E}f = 0$ holds and $\|f\|_\infty = 1$. Note that the function $f \mapsto f\mathbb{E}x - \mathbb{E}(fx)$ is convex on the weak-* compact, convex set

$$L_0^\infty(\Omega) := \{f \in L^\infty(\Omega, \mu) : \|f\|_\infty \leq 1 \text{ and } \mathbb{E}f = 0\} \subseteq (L^1(\Omega, \mu))^*.$$

Hence, from the Krein–Milman theorem, it is enough to prove the statement if f is an extreme point of $L_0^\infty(\Omega)$. We claim that the extreme points of $L_0^\infty(\Omega)$ are the functions with essential range $\{-1, 1, c\}$ for some $-1 < c < 1$, ($\mu(\{f = c\}) = 0$ might be possible) and

$$(2.1) \quad \mathbb{E}f = \mu(\{f = 1\}) - \mu(\{f = -1\}) + c\mu(\{f = c\}) = 0.$$

Let us choose a measurable subset A of Ω such that $\|f\chi_A\|_\infty \leq 1 - \varepsilon < 1$. If μ is non-atomic (A is not a singleton), we can find a function $g \in L_0^\infty(\Omega)$ satisfying $\|g\|_\infty \leq \varepsilon$ and $g = 0$ a.e. on $\Omega \setminus A$. Since

$$f = \frac{1}{2}(f + g) + \frac{1}{2}(f - g),$$

f is an extreme point if and only if $\mu(A) = 0$. When μ is atomic, the set A might be a singleton, hence our claim follows.

Now let f be an extreme point of $L_0^\infty(\Omega)$. Obviously, $\|f - \mathbb{E}f\|_\infty = 1$. Furthermore, we have

$$\begin{aligned} \|f\mathbb{E}x - \mathbb{E}(fx)\|_\infty &= \max(|\mathbb{E}x - \mathbb{E}(fx)|, |\mathbb{E}x + \mathbb{E}(fx)|, |c\mathbb{E}x - \mathbb{E}(fx)|) \\ &= \max(|\mathbb{E}(x(1-f))|, |\mathbb{E}(x(1+f))|, |\mathbb{E}(x(c-f))|) \\ &\leq \|x\|_\infty \max(\|1-f\|_1, \|1+f\|_1, \|c-f\|_1). \end{aligned}$$

It remains to show that $\max(\|1-f\|_1, \|1+f\|_1, \|c-f\|_1) = 1$. Clearly, from (2.1)

$$\begin{aligned} \|1-f\|_1 &= 2\mu(\{f = -1\}) + |1-c|\mu(\{f = c\}) \\ &= 1 - \mu(\{f = 1\}) + \mu(\{f = -1\}) - c\mu(\{f = c\}) = 1. \end{aligned}$$

Similarly,

$$\begin{aligned} \|1+f\|_1 &= 2\mu(\{f = 1\}) + |1+c|\mu(\{f = c\}) \\ &= 1 + \mu(\{f = 1\}) - \mu(\{f = -1\}) + c\mu(\{f = c\}) = 1, \end{aligned}$$

and lastly we infer that

$$\begin{aligned} \|c-f\|_1 &= |c-1|\mu(\{f = 1\}) + |c+1|\mu(\{f = -1\}) \\ &= \mu(\{f = 1\}) + \mu(\{f = -1\}) + c^2\mu(\{f = c\}) \leq 1. \end{aligned}$$

The proof is complete. \square

For the real Banach space $L^\infty(\Omega, \mu; \mathbb{R})$, we can simply prove that the seminorm σ_∞ is strongly Leibniz as we have seen before for the standard deviation.

Theorem 2.6. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. For the real Banach space $L^\infty(\Omega, \mu; \mathbb{R})$,*

$$\sigma_\infty(f) = \|f - \mathbb{E}f\|_\infty$$

is a strongly Leibniz seminorm.

Proof. From Proposition 2.5, it follows that

$$\begin{aligned} \|fg - \mathbb{E}(fg)\|_p &= \|f(g - \mathbb{E}g) + (f\mathbb{E}g - \mathbb{E}(fg))\|_p \\ &\leq \|f\|_\infty \|g - \mathbb{E}g\|_p + \|g\|_\infty \|f - \mathbb{E}f\|_p, \end{aligned}$$

and

$$\left\| \frac{1}{f} - \mathbb{E} \frac{1}{f} \right\|_p = \left\| \frac{1}{f} \left(\mathbb{E} \left(f \cdot \frac{1}{f} \right) - f \mathbb{E} \frac{1}{f} \right) \right\|_p \leq \left\| \frac{1}{f} \right\|_\infty \cdot \left\| \frac{1}{f} \right\|_\infty \cdot \|f - \mathbb{E}f\|_p,$$

which is what we intended to have. \square

Regarding the case of the uniform distributions seen above in Proposition 2.1, we are able to prove the analogue of Proposition 2.5 in very particular cases. Let λ_n stand for the uniform distribution on \mathbb{Z}_n .

Proposition 2.7. *Fix $1 \leq n \leq 4$. For $1 \leq p < \infty$, and any real-valued $f, x \in \ell_n^\infty(\lambda_n)$, we have*

$$\|f\mathbb{E}x - \mathbb{E}(fx)\|_p \leq \|x\|_\infty \|f - \mathbb{E}f\|_p.$$

Proof. First note that the case $\Omega = \mathbb{Z}_1$ is trivial. On the other hand, in case of $\Omega = \mathbb{Z}_2$, one can have arbitrary distribution. Indeed, let $\mu(1) = p_1$ and $\mu(2) = p_2 = 1 - p_1$. Then by simple calculation we obtain

$$f - \mathbb{E}f = (f_1 - f_2) \cdot (p_2, -p_1)$$

and

$$f\mathbb{E}x - \mathbb{E}(fx) = (f_1 - f_2) \cdot (p_2x_2, -p_1x_1)$$

so the desired inequality follows immediately.

To prove the remaining cases $\Omega = \mathbb{Z}_3$ and $\Omega = \mathbb{Z}_4$, let us rescale the inequality and assume that $\|x\|_\infty = 1$. Notice that the function

$$x \mapsto \|f\mathbb{E}x - \mathbb{E}(fx)\|_p$$

is convex on the closed unit ball $\{x \in L^\infty(\Omega, \mu) : \|x\|_\infty \leq 1\}$, therefore it suffices to check the inequality only for its extreme points.

First, we turn to the case $\Omega = \mathbb{Z}_3$. Clearly, for $x = (1, 1, 1)$ even equality holds, so after possible rearrangement and multiplication by constants we may assume that $x = (1, 1, -1)$. Then

$$f - \mathbb{E}f = \frac{1}{3}(2f_1 - f_2 - f_3, -f_1 + 2f_2 - f_3, -f_1 - f_2 + 2f_3)$$

and

$$f\mathbb{E}x - \mathbb{E}(fx) = \frac{1}{3}(f_3 - f_1, f_3 - f_2, 2f_3 - f_1 - f_2).$$

By using the notation $a_1 = 2f_1 - f_2 - f_3$ and $a_2 = 2f_2 - f_1 - f_3$, the inequality reduces to the form

$$\left| \frac{2a_1 + a_2}{3} \right|^p + \left| \frac{a_1 + 2a_2}{3} \right|^p \leq |a_1|^p + |a_2|^p,$$

which is obviously true from the convexity of the function $t \mapsto |t|^p$.

Next, let $\Omega = \mathbb{Z}_4$. By symmetry arguments we can assume that $x = (1, 1, 1, -1)$ or $x = (1, 1, -1, -1)$. Set $x = (1, 1, 1, -1)$. A simple calculation implies that

$$f - \mathbb{E}f = \frac{1}{4}(a_1, a_2, a_3, a_4),$$

where

$$a_j = 3f_j - \sum_{i \neq j} f_i.$$

Moreover,

$$f\mathbb{E}x - \mathbb{E}(fx) = \frac{1}{4} \begin{pmatrix} +f_1 - f_2 - f_3 + f_4 \\ -f_1 + f_2 - f_3 + f_4 \\ -f_1 - f_2 + f_3 + f_4 \\ -f_1 - f_2 - f_3 + 3f_4 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} -a_2 - a_3 \\ -a_1 - a_3 \\ -a_1 - a_2 \\ 2a_4 \end{pmatrix}.$$

Therefore, it is enough to check that

$$\left| \frac{a_2 + a_3}{2} \right|^p + \left| \frac{a_1 + a_3}{2} \right|^p + \left| \frac{a_1 + a_2}{2} \right|^p \leq |a_1|^p + |a_2|^p + |a_3|^p,$$

which follows again by the convexity of the function $t \mapsto |t|^p$.

Lastly, consider the remaining case $x = (1, 1, -1, -1)$. Then

$$f\mathbb{E}x - \mathbb{E}(fx) = \frac{1}{4} \begin{pmatrix} -f_1 - f_2 + f_3 + f_4 \\ -f_1 - f_2 + f_3 + f_4 \\ -f_1 - f_2 + f_3 + f_4 \\ -f_1 - f_2 + f_3 + f_4 \end{pmatrix} = \frac{1}{16} \begin{pmatrix} -a_1 - a_2 + a_3 + a_4 \\ -a_1 - a_2 + a_3 + a_4 \\ -a_1 - a_2 + a_3 + a_4 \\ -a_1 - a_2 + a_3 + a_4 \end{pmatrix}.$$

Since

$$4 \left| \frac{-a_1 - a_2 + a_3 + a_4}{4} \right|^p \leq |a_1|^p + |a_2|^p + |a_3|^p + |a_4|^p,$$

by a convexity argument as seen before, we get the statement of the proposition. \square

Example 2.8. The statement of Proposition 2.7 does not hold in general. Let $n \geq 5$ and $p = 1$, for instance. Let $x = (1, \dots, 1, -1)$ and $f = (1, 0, \dots, 0, -1)$ in $\ell_n^\infty(\lambda_n)$. Obviously, $\mathbb{E}f = 0$, $\mathbb{E}x = 1 - \frac{2}{n}$, $\mathbb{E}(fx) = \frac{2}{n}$, $\|x\|_\infty = 1$, furthermore,

$$\|f - \mathbb{E}f\|_1 = \frac{2}{n},$$

and

$$\|f\mathbb{E}x - \mathbb{E}(fx)\|_1 = \left\| \left(1 - \frac{4}{n}, -\frac{2}{n}, \dots, -\frac{2}{n}, -1 \right) \right\|_1 = \frac{4n - 8}{n^2} \quad (n \geq 5).$$

Thus

$$\|f\mathbb{E}x - \mathbb{E}(fx)\|_1 = \left(2 - \frac{4}{n} \right) \|f - \mathbb{E}f\|_1 > \|f - \mathbb{E}f\|_1.$$

Example 2.9. In the case of non-uniform distributions, the inequality of Proposition 2.7 is not true even on $\Omega = \{1, 2, 3\}$. To see this, define the measure $\mu(1) = \frac{1}{8}$, $\mu(2) = \frac{3}{4}$, $\mu(3) = \frac{1}{8}$ and consider $f = (1, 0, -1)$ and $x = (1, 1, -1)$. Then $\mathbb{E}f = 0$, $\mathbb{E}x = \frac{3}{4}$, $\mathbb{E}(fx) = \frac{1}{4}$, and

$$\|f - \mathbb{E}f\|_1 = \frac{1}{4},$$

while

$$\|f\mathbb{E}x - \mathbb{E}(fx)\|_1 = \left\| \left(\frac{1}{2}, -\frac{1}{4}, -1 \right) \right\|_1 = \frac{3}{8}.$$

As we have seen before in the proof of Theorem 2.6, we can infer the next statement on discrete measure spaces.

Corollary 2.10. *For $1 \leq n \leq 4$ and $1 \leq p < \infty$, the seminorm σ_p is strongly Leibniz on the real ℓ_n^∞ endowed with uniform distribution.*

Surprisingly, we cannot prove or disprove the last statement on measure spaces which contain more than 4 atoms. Computer simulations suggest us that Corollary 2.10 might be true for any n which would imply that σ_p is a strongly Leibniz seminorm for every $1 \leq p < \infty$ (see Proposition 2.1). Now we have only a very few particular results on general measure spaces. Denote λ_n the uniform distribution on the set \mathbb{Z}_n , as usual.

Proposition 2.11. *Let $1 \leq p < \infty$ and $f, g \in \ell_n^\infty(\lambda_n)$ be such that the coordinates of f, g and fg have the same order. Then*

$$\|fg - \mathbb{E}(fg)\|_p \leq \|g\|_\infty \|f - \mathbb{E}f\|_p + \|f\|_\infty \|g - \mathbb{E}g\|_p$$

holds.

Proof. We use the fact that the ℓ^p norm with uniform distribution and $1 \leq p \leq \infty$ is a Schur-convex function [5, Ch. 3 Example I.1]. Therefore, it suffices to prove that the vector $fg - \mathbb{E}(fg)$ is majorized by $\|f\|_\infty(g - \mathbb{E}g) + \|g\|_\infty(f - \mathbb{E}f)$. To see this, we may assume without loss of generality that $f_1 \geq f_2 \geq \dots \geq f_n$, thus we also have $g_1 \geq g_2 \geq \dots \geq g_n$ and $f_1g_1 \geq f_2g_2 \geq \dots \geq f_n g_n$. Then we have to verify that

$$\sum_{j=1}^k (f_j g_j - \mathbb{E}(fg)) \leq \sum_{j=1}^k (\|g\|_\infty (f_j - \mathbb{E}f) + \|f\|_\infty (g_j - \mathbb{E}g)),$$

for all $1 \leq k \leq n-1$, and equality holds when $k = n$. The latter equality is obvious because both sides are zero if $k = n$. In the remainder of the proof, a simple calculation gives that

$$n \left(\sum_{j=1}^k (f_j - \mathbb{E}f) \right) = (n-k) \sum_{j=1}^k f_j - k \sum_{j=k+1}^n f_j = \sum_{j=1}^k \sum_{i=k+1}^n (f_j - f_i)$$

and analogously

$$\begin{aligned} n \left(\sum_{j=1}^k (f_j g_j - \mathbb{E}(fg)) \right) &= \sum_{j=1}^k \sum_{i=k+1}^n (f_j g_j - f_i g_i) \\ &= \sum_{j=1}^k \sum_{i=k+1}^n (f_j (g_j - g_i) + g_i (f_j - f_i)). \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} &\sum_{j=1}^k (\|f\|_\infty (g_j - \mathbb{E}g) + \|g\|_\infty (f_j - \mathbb{E}f) - (f_j g_j - \mathbb{E}(fg))) \\ &= \frac{1}{n} \left(\sum_{j=1}^k \sum_{i=k+1}^n (g_j - g_i) (\|f\|_\infty - f_j) + (f_j - f_i) (\|g\|_\infty - g_i) \right) \geq 0. \end{aligned}$$

□

Analogously to the proof of Proposition 2.1, we readily obtain the following corollaries.

Corollary 2.12. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $1 \leq p < \infty$. For any non-negative $f \in L^\infty(\Omega, \mu)$,*

$$\|f^2 - \mathbb{E}f^2\|_p \leq 2\|f\|_\infty \|f - \mathbb{E}f\|_p.$$

Corollary 2.13. *Let $1 \leq p < \infty$ and μ be a probability measure on the interval $[0, 1]$. For any non-negative, bounded and monotone increasing (or decreasing) functions f and g , we have*

$$\|fg - \mathbb{E}fg\|_p \leq \|g\|_\infty \|f - \mathbb{E}f\|_p + \|f\|_\infty \|g - \mathbb{E}g\|_p.$$

3. STANDARD DEVIATION IN C^* -ALGEBRAS

In this section we shall complete Rieffel's argument on the standard deviation in non-commutative probability spaces. Let \mathcal{A} be a unital C^* -algebra and denote ω any faithful state of it. Denote $L^2(\mathcal{A}, \omega)$ the GNS Hilbert space obtained by completing \mathcal{A} for the inner product $\langle a, b \rangle = \omega(b^*a)$, as usual. Obviously, every $a \in \mathcal{A}$ has a natural representation; i.e. the left-regular representation L_a , in the operator algebra of $L^2(\mathcal{A}, \omega)$. Consider now the projection (or Dirac operator) $E: a \mapsto \omega(a)\mathbf{1}_{\mathcal{A}}$ on $L^2(\mathcal{A}, \omega)$. Direct calculations for the norm of the commutator $\delta(L_a) = [E, L_a] = EL_a - L_aE$ give that

$$\|\delta(L_a)\| = \omega(|a - \omega(a)|^2)^{1/2} \vee \omega(|a^* - \omega(a^*)|^2)^{1/2}.$$

Thus it immediately follows that Rieffel's non-commutative standard deviation is a strongly Leibniz $*$ -seminorm, see [8, Theorem 3.7]. Moreover, an application of the 'independent copies trick' in C^* -algebras gives that

$$\sigma_2^\omega(a) := \omega(|a - \omega(a)|^2)$$

is strongly Leibniz as well if one assumes that ω is tracial [8, Proposition 3.6]. Actually, the 'strong' part of the statement requires only the tracial assumption. Computer simulations for matrices indicate that σ_2^ω might be strongly Leibniz for any state ω but the question remained open in [8]. Now we shall provide the affirmative answer by means of an elementary argument.

Pick a faithful state ω of \mathcal{A} . Let $\|a\|_2 = \omega(|a|^2)^{1/2}$ denote the norm on $L^2(\mathcal{A}, \omega)$. We begin with

Lemma 3.1. *For any a and $x \in \mathcal{A}$,*

$$\|\omega(x)a - \omega(xa)\|_2 \leq \|x\| \|a - \omega(a)\|_2.$$

Proof. There is no loss of generality in assuming that $\omega(a) = 0$. Denote E the orthogonal projection from $L^2(\mathcal{A}, \omega)$ onto its subspace $\mathbb{C}\mathbf{1}_{\mathcal{A}}$. Then

$$\|\omega(x)a - \omega(xa)\|_2 = \|\omega(x)(I - E)a - E\omega(xa)\|_2 = \|\omega(x)a\|_2 + |\omega(xa)|.$$

Notice that the Cauchy-Schwarz inequality readily gives that

$$|\omega(xa)| = |\omega((x - \omega(x))a)| \leq \|a\|_2 \|x^* - \omega(x^*)\|_2.$$

Hence

$$\begin{aligned} \|\omega(x)a - \omega(xa)\|_2 &= \|\omega(x)a\|_2 + |\omega(xa)| \\ &\leq |\omega(x^*)| \|a\|_2 + \|a\|_2 \|x^* - \omega(x^*)\|_2 \\ &= \|x^*\|_2 \|a\|_2 \\ &\leq \|x^*\| \|a\|_2 \\ &= \|x\| \|a\|_2, \end{aligned}$$

and the proof is finished. \square

Now the main theorem of the section reads as follows.

Theorem 3.2. *For any invertible $a \in \mathcal{A}$, the inequality*

$$\|a^{-1} - \omega(a^{-1})\|_2 \leq \|a^{-1}\|^2 \|a - \omega(a)\|_2$$

holds.

Proof. We clearly have that

$$\|xa\|_2 \leq \|x\| \|a\|_2$$

for any $x \in \mathcal{A}$. In fact,

$$\omega(|xa|^2) = \omega(a^*|x|^2a) \leq \omega(a^*\|x\|^2a) = \|x\|^2\omega(|a|^2).$$

Combining the previous inequality with Lemma 3.1, it follows that

$$\begin{aligned} \|a^{-1} - \omega(a^{-1})\|_2 &= \|a^{-1}(\omega(a^{-1}a) - \omega(a^{-1})a)\|_2 \\ &= \|a^{-1}(\omega(a^{-1}a) - \omega(a^{-1})a)\|_2 \\ &\leq \|a^{-1}\| \|\omega(a^{-1}a) - \omega(a^{-1})a\|_2 \\ &\leq \|a^{-1}\|^2 \|a - \omega(a)\|_2, \end{aligned}$$

and the proof is complete. \square

With [8, Proposition 3.4] at hand, we immediately obtain the following

Theorem 3.3. *Let \mathcal{A} be a unital C^* -algebra. For any faithful state ω of \mathcal{A} , $\sigma_2^\omega(a)$ is a strongly Leibniz seminorm.*

Alternatively, for any faithful tracial state ω , we can define a derivation on a Banach algebra to infer the above corollary. In fact, let us consider the Banach space

$$\mathcal{A} \oplus L^2(\mathcal{A}, \omega)$$

endowed with the norm $\|(x, y)\| = \max(\|x\|, \|y\|_2)$. The linear operators

$$E: (x, y) \mapsto (0, \omega(x)\mathbf{1}_{\mathcal{A}})$$

and

$$T_a: (x, y) \mapsto (xa, ya)$$

on $\mathcal{A} \oplus L^2(\mathcal{A}, \omega)$ define a strongly Leibniz seminorm L on \mathcal{A} via the norm of the derivation $L(a) = \|\delta(T_a)\| = \|[E, T_a]\|$. From Lemma 3.1, we have that

$$\|\delta(T_a)(x, y)\| = \|(0, \omega(x)a - \omega(xa))\| \leq \|a - \omega(a)\|_2 \|x\| \leq \|a - \omega(a)\|_2 \|(x, y)\|.$$

With the choice of $(\mathbf{1}_{\mathcal{A}}, 0)$, we get

$$\|\delta(T_a)\| = \|a - \omega(a)\|_2.$$

Since ω is tracial, $\|xa\|_2 \leq \|a\| \|x\|_2$. Hence it clearly follows that $\|T_a\| = \|a\|$. Notice that $T_{ab} = T_a T_b$. Now a direct application of the derivation rules gives that $\|\delta(T_a)\| = L(a) = \sigma_2^\omega(a)$ is a strongly Leibniz seminorm.

REFERENCES

- [1] K.M.R. Audenaert, Variance bounds, with an application to norm bounds for commutators, *Linear Algebra Appl.*, **432** (2010), 1126–1143.
- [2] T. Bhattacharyya and P. Grover, Characterization of Birkhoff–James orthogonality, *J. Math. Anal. Appl.*, **407** (2013), 350–358.
- [3] C. Franchetti, The norm of the minimal projection onto hyperplanes in $L_p[0, 1]$ and the radial constant, *Boll. Un. Mat. Ital. B*, (7) 4, (1990), 803–821.
- [4] Z. Léka, A note on central moments in C^* -algebras, <http://arxiv.org/abs/1402.6528>, to appear in *J. Math. Inequal.*
- [5] A. W. Marshall, I. Olkin, B. C. Arnold, *Inequalities: Theory of Majorization and Its Applications*, second edition, Springer, 2011.
- [6] S. Rolewicz, On minimal projections of the space $L_p[0, 1]$ on 1-codimensional subspace, *Bull. Polish Acad. Sci. Math.*, **34** (1986), 151–153.
- [7] M.A. Rieffel, Leibniz seminorms and best approximation from C^* -subalgebras, *Sci. China Math.*, **54** (2011), 2259–2274.
- [8] M.A. Rieffel, Standard deviation is a strongly Leibniz seminorm, *New York J. Math.*, **20** (2014), 35–56.

DEPARTMENT OF APPLIED ANALYSIS, EÖTVÖS LORÁND UNIVERSITY, H-1117 BUDAPEST, PÁZMÁNY
P. SÉTÁNYI 1/C, HUNGARY
E-mail address: badam@cs.elte.hu

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, 1053 BUDAPEST, REÁLTANODA U. 13-15
E-mail address: leka.zoltan@renyi.mta.hu