

Brane-world stars with solid crust and vacuum exterior

Jorge Ovalle^{1*}, László Á. Gergely^{2,3,4†} and Roberto Casadio^{5,6‡}

¹*Departamento de Física, Universidad Simón Bolívar
Apartado 89000, Caracas 1080A, Venezuela*

²*Department of Theoretical Physics, University of Szeged
Tisza Lajos krt 84-86, Szeged 6720, Hungary*

³*Department of Experimental Physics, University of Szeged,
Dóm Tér 9, Szeged 6720, Hungary*

⁴*Department of Physics, Tokyo University of Science, 1-3
Kagurazaka, Shinjyuku-ku, Tokyo 162-8601, Japan*

⁵*Dipartimento di Fisica e Astronomia, Università di Bologna,
via Irnerio 46, 40126 Bologna, Italy*

⁶*Istituto Nazionale di Fisica Nucleare, Sezione di Bologna,
via B. Pichat 6/2, 40127 Bologna, Italy*

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Abstract

The minimal geometric deformation approach is employed to show the existence of brane-world stellar distributions with vacuum Schwarzschild exterior, thus without energy leaking from the exterior of the brane-world star into the extra dimension. The interior satisfies all elementary criteria of physical acceptability for a stellar solution, namely, it is regular at the origin, the pressure and density are positive and decrease monotonically with increasing radius, finally all energy conditions are fulfilled. A very thin solid crust with negative radial pressure separates the interior from the exterior, having a thickness Δ inversely proportional to both the brane tension σ and the radius R of the star, i.e. $\Delta^{-1} \sim R\sigma$. This brane-world star with Schwarzschild exterior would appear only thermally radiating to a distant observer and be fully compatible with the stringent constraints imposed on stellar parameters by observations of gravitational lensing, orbital evolutions or properties of accretion disks.

1 Introduction

The Randall-Sundrum (RS) second brane-world model [1] is based on perceiving our 4-dimensional space-time as a hypersurface embedded into the 5-dimensional bulk. The observed world is induced on it by virtue of a discontinuity in the extrinsic curvature (expressed by the Lanczos-Sen-Darmois-Israel junction conditions [2]-[5]). Despite efforts spent in recent years for understanding the inner

*jovalle@usb.ve

†laszlo.a.gergely@gmail.com

‡casadio@bo.infn.it

workings of the model, the possible impact of the fifth dimension on the 4-dimensional gravity sector is still not fully assessed.

The original RS model was generalised to allow for a codimension one curved brane embedded into a generic 5-dimensional bulk (for a general review see Ref. [6]). At the level of the formalism the gravitational dynamics was worked out in its full generality, either in the covariant $4 + 1$ (brane plus extra-dimension) decomposition [7]-[9], in the $3 + 1 + 1$ canonical [10]-[11] or in the $3 + 1 + 1$ covariant [12]-[13] approaches.

A necessary ingredient in such theories is the brane tension. Its huge value has been constrained from below by employing tabletop measurements of the gravitational constant [14]-[15], imposing the condition that during a cosmological evolution at the time of Nucleosynthesis brane effects must already be severely dampened [16], or from astrophysical considerations [17].

Further cosmological investigations revealed important modifications in the early universe [18] as compared to standard cosmology. The thermal radiation of an initially very hot brane could even lead to black hole formation in the fifth dimension [19]. Such a black hole modifies the 5-dimensional Weyl curvature, backreacting onto the curvature and the dynamics of the brane. Structure formation was analysed in Refs. [20]-[21]. Unfortunately the lack of closure of the dynamical equations on the brane, despite some progress [22]-[24], hindered the development of a full perturbation theory on the brane, hence observations on the Cosmic Microwave Background and on structure formation could not be explored in full generality within the theory. Confrontation with Nucleosynthesis [25] and type Ia supernova data has been successfully done [26]-[27]. Brane-world effects were also shown to successfully replace dark matter, both in the dynamics of clusters [28] and in galactic dynamics [29]-[31], thus solving the rotation curve problem [32].

Although many fundamental aspects in the RS scenario, in particular the cosmological aspects, were clarified (largely in the sense to push the high-energy modifications so close to the Big Bang that their effects remain unobservable), certain key issues remain unresolved, which are mostly related to self-gravitating systems and black holes, and for which the high-energy regimes could be within observational reach.

The simplest, spherically symmetric brane solution is similar to the general relativistic Reissner-Nordström solution, but the role of the square of the electric charge is taken by a tidal charge originating in the higher dimensional Weyl curvature [33]. The value of the tidal charge was constrained by confronting with observations in the Solar System [34]-[35]. A rotating generalisation is also known [36]. Gravitational collapse on the brane has been investigated in Refs. [37]-[42]. Early work on stellar solutions and astrophysics in brane-world context can be found in Refs. [17], [43]-[45]. The topic is reviewed in Ref. [6].

There is some evidence indicating the existence of RS black hole metrics [46]-[48], but an exact solution of the full set of dynamical equations of the 5-dimensional gravity has not been discovered so far. Solving the full 5-dimensional Einstein field equations has indeed proven an extremely complicated task (see, e.g. [49]-[51], and references therein). Beside, such a black hole solution could exhibit various facets in our 4-dimensional world, as there are many possible ways to embed a 4-dimensional brane into the 5-dimensional bulk to get a 4-dimensional section of it. While it is true that the \mathbb{Z}_2 symmetry across the brane is employed as a common simplifying assumption, it is not mandatory either, and lifting it leads to further freedom in the embedding, explored in Refs. [52]-[56] (different cosmological constants), [57]-[59] (different 5-dimensional black hole masses) or [60]-[62] (both).

The study of exact, physically acceptable solutions to the effective 4-dimensional Einstein field equations on the brane could clarify certain aspects of the 5-dimensional geometry and provide

hints on the ways our observed universe could be embedded into it. When starting from any brane solution, the Campbell-Magaard theorems [63]-[64] can be employed to extend them, at least locally, into the bulk. Consequences of the Campbell-Magaard theorems for General Relativity (GR) were discussed in Refs. [65]-[66]. The rigorous study of the effective Einstein field equations on the 4-dimensional brane therefore qualifies as a first step of this process. It also helps clarifying the role of the 5-dimensional contributions to the sources of the effective 4-dimensional field equations.

Deriving physically acceptable exact solutions in GR is an extremely difficult task, even in vacuum [67], due to the complexity of the Einstein field equations. For inner stellar solutions, the task is even more complicated (a useful algorithm to obtain all static spherically symmetric perfect fluid solutions in GR was presented in [68], and its extension to locally anisotropic fluids in [69]). Only a small number of internal solutions are known [70]. This complexity is further amplified in brane-worlds, where nonlinear terms in the matter fields appear as high-energy corrections.

A useful guide is provided by the requirement that GR should be recovered at low energies, where it has been extensively tested. Fortunately, as we mentioned above, the RS theory contains a free parameter, the brane tension σ , which allows to control this important aspect by precisely setting the scale of high energy physics [71]. This fundamental requirement stands at the basis of the *minimal geometric deformation approach* (MGD) [72], which has made possible, among other things, to derive exact and physically acceptable solutions for spherically symmetric [73] and non-uniform stellar distributions [74], to generate other physically acceptable inner stellar solutions [71]-[75], to express the tidal charge in the metric found in Ref. [33] in terms of the Arnowitt-Deser-Misner mass, to study microscopic black holes [76]-[77], to elucidate the role of exterior Weyl stresses (arising from bulk gravitons) acting on compact stellar distributions [78], as well as extend the concept of variable tension introduced in [9], [79] by analyzing the behaviour [80] of the black strings extending into the extra dimension [81].

For spherically symmetric systems on the brane, the RS scenario provides two quantities of extra-dimensional origin, namely, the dark radiation \mathcal{U} and dark pressure \mathcal{P} , which act as sources of 4-dimensional gravity even in vacuum. How the various choices of these quantities affect stellar structures on the brane is only partially understood so far [82]-[87]. Most remarkably, the Schwarzschild exterior can be generated by a static self-gravitating star only if it consists of a certain exotic fluid. By contrast, if the stellar system contains regular matter, there must be an exchange of energy between its 4-dimensional exterior with the 5-dimensional bulk, leading to non-static configurations. Since the Schwarzschild geometry is not the exterior of brane-world stars consisting of regular matter, the modifications in the exterior geometry due to extra-dimensional effects have been one of the main targets of investigations in the search for ways to find evidence of extra-dimensional gravity.

Nevertheless, the Schwarzschild geometry is strongly supported by weak-field tests of gravity in the Solar System, constraining such possible extra-dimensional modifications (for a recent work on classical test of GR in the brane-world context see Ref. [88]). According to Ref. [89], black holes in the RS brane-world should evaporate by Hawking radiation, thus the existence of long-lived solar mass black holes could constrain the bulk curvature radius. The same conclusion was reached in Ref. [90], so it seems that the price to pay for a static exterior would be Hawking radiation, which has however a temperature smaller than the temperature of the CMB for objects of a few solar masses.

In this paper, we shall employ the MGD approach to show that, despite severe constraints on brane-world stars from available data, there are scenarios in which their existence cannot be ruled out. We shall show that certain brane-world star exteriors might be the same as in GR,

hence automatically fitting all observational constraints. In particular, with the exception of a tiny solid crust, the stellar matter could have the most reasonable physical properties and matching conditions are obeyed between the interior geometry and the exterior Schwarzschild vacuum.

The paper is organised as follows. In Section 2, we briefly review the effective Einstein field equations on the brane for a spherically symmetric and static distribution of matter with density ρ , radial pressure p_r and tangential pressure p_t . We also present the MGD approach. In Section 3, we show that it is possible to have a star made of regular matter with a Schwarzschild exterior and without exchange of energy between the brane and the bulk at the price of including a tiny solid stellar crust. Finally, we summarise our conclusions.

2 Effective Einstein equations and MGD

In the generalised RS brane-world scenario, gravitation acts in five dimensions and modifies gravitational dynamics in the (3+1)-dimensional universe accessible to all other physical fields, the so called brane. The arising modified Einstein equations (with G the 4-dimensional Newton constant, $k^2 = 8\pi G$, and Λ the 4-dimensional cosmological constant)

$$G_{\mu\nu} = -k^2 T_{\mu\nu}^{\text{eff}} - \Lambda g_{\mu\nu} , \quad (2.1)$$

could formally be seen as Einstein equations in which the energy-momentum tensor $T_{\mu\nu}$ is complemented by new source terms, which contribute to an effective source as

$$T_{\mu\nu} \rightarrow T_{\mu\nu}^{\text{eff}} = T_{\mu\nu} + \frac{6}{\sigma} S_{\mu\nu} + \frac{1}{8\pi} \mathcal{E}_{\mu\nu} + \frac{4}{\sigma} \mathcal{F}_{\mu\nu} . \quad (2.2)$$

Here σ is again the brane tension,

$$S_{\mu\nu} = \frac{T T_{\mu\nu}}{12} - \frac{T_{\mu\alpha} T_{\nu}^{\alpha}}{4} + \frac{g_{\mu\nu}}{24} \left(3 T_{\alpha\beta} T^{\alpha\beta} - T^2 \right) \quad (2.3)$$

represents a high-energy correction quadratic in the energy-momentum tensor ($T = T_{\alpha}^{\alpha}$),

$$k^2 \mathcal{E}_{\mu\nu} = \frac{6}{\sigma} \left[\mathcal{U} \left(u_{\mu} u_{\nu} + \frac{1}{3} h_{\mu\nu} \right) + \mathcal{P}_{\mu\nu} + \mathcal{Q}_{(\mu} u_{\nu)} \right] \quad (2.4)$$

is a non-local source, arising from the 5-dimensional Weyl curvature (with \mathcal{U} the bulk Weyl scalar; $\mathcal{P}_{\mu\nu}$ and \mathcal{Q}_{μ} the stress tensor and energy flux, respectively), and $\mathcal{F}_{\mu\nu}$ contains contributions from all non-standard model fields possibly living in the bulk (it does not include the 5-dimensional cosmological constant, which is fine-tuned to σ in order to generate a small 4-dimensional cosmological constant). For simplicity, we shall assume $\mathcal{F}_{\mu\nu} = 0$ and $\Lambda = 0$ throughout the paper.

Let us then restrict to spherical symmetry (such that $\mathcal{P}_{\mu\nu} = \mathcal{P} h_{\mu\nu}$ and $\mathcal{Q}_{\mu} = 0$) and choose as the source term in Eq. (2.2) a perfect fluid,

$$T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} - p g_{\mu\nu} , \quad (2.5)$$

where $u^{\mu} = e^{-\nu/2} \delta_0^{\mu}$ is the fluid 4-velocity field in the Schwarzschild-like coordinates of the metric

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) . \quad (2.6)$$

Here $\nu = \nu(r)$ and $\lambda = \lambda(r)$ are functions of the areal radius r only, ranging from $r = 0$ (the star's centre) to some $r = R$ (the star's surface).

The metric (2.6) must satisfy the effective 4-dimensional Einstein field equations (2.1), which here read [77]

$$k^2 \left[\rho + \frac{1}{\sigma} \left(\frac{\rho^2}{2} + \frac{6}{k^4} \mathcal{U} \right) \right] = \frac{1}{r^2} - e^{-\lambda} \left(\frac{1}{r^2} - \frac{\lambda'}{r} \right) \quad (2.7)$$

$$k^2 \left[p + \frac{1}{\sigma} \left(\frac{\rho^2}{2} + \rho p + \frac{2}{k^4} \mathcal{U} \right) + \frac{4}{k^4} \frac{\mathcal{P}}{\sigma} \right] = -\frac{1}{r^2} + e^{-\lambda} \left(\frac{1}{r^2} + \frac{\nu'}{r} \right) \quad (2.8)$$

$$k^2 \left[p + \frac{1}{\sigma} \left(\frac{\rho^2}{2} + \rho p + \frac{2}{k^4} \mathcal{U} \right) - \frac{2}{k^4} \frac{\mathcal{P}}{\sigma} \right] = \frac{1}{4} e^{-\lambda} \left[2\nu'' + \nu'^2 - \lambda' \nu' + 2 \frac{\nu' - \lambda'}{r} \right], \quad (2.9)$$

with primes denoting derivatives with respect to r . Moreover,

$$p' = -\frac{\nu'}{2}(\rho + p), \quad (2.10)$$

The 4-dimensional GR equations are recovered for $\sigma^{-1} \rightarrow 0$, and the conservation equation (2.10) then becomes a linear combination of Eqs. (2.7)-(2.9).

By simple inspection of the field equations (2.7)-(2.9), we identify the effective density $\tilde{\rho}$, effective radial pressure \tilde{p}_r and effective tangential pressure \tilde{p}_t , which are given by

$$\tilde{\rho} = \rho + \frac{1}{\sigma} \left(\frac{\rho^2}{2} + \frac{6}{k^4} \mathcal{U} \right), \quad (2.11)$$

$$\tilde{p}_r = p + \frac{1}{\sigma} \left(\frac{\rho^2}{2} + \rho p + \frac{2}{k^4} \mathcal{U} \right) + \frac{4}{k^4} \frac{\mathcal{P}}{\sigma}, \quad (2.12)$$

$$\tilde{p}_t = p + \frac{1}{\sigma} \left(\frac{\rho^2}{2} + \rho p + \frac{2}{k^4} \mathcal{U} \right) - \frac{2}{k^4} \frac{\mathcal{P}}{\sigma}, \quad (2.13)$$

clearly illustrating that extra-dimensional effects produce anisotropies in the stellar distribution, that is

$$\Pi \equiv \tilde{p}_r - \tilde{p}_t = \frac{6}{k^4} \frac{\mathcal{P}}{\sigma}. \quad (2.14)$$

A GR isotropic stellar distribution (perfect fluid) therefore becomes an anisotropic stellar system on the brane.

Next, the MGD approach [72] will be introduced in order to generalise GR interior solutions to the brane-world scenario.

2.1 Star interior from MGD

Eqs. (2.7)-(2.10) represent an open system of differential equations on the brane. For a unique solution additional information on the bulk geometry and on the embedding of the 4-dimensional brane into the bulk is required [46]-[49], [51].

In its absence, one can rely on the MGD induced by a GR solution. In order to implement the MGD approach, we first rewrite the field equations (2.7)-(2.9) as

$$e^{-\lambda} = 1 - \frac{k^2}{r} \int_0^r x^2 \left[\rho + \frac{1}{\sigma} \left(\frac{\rho^2}{2} + \frac{6}{k^4} \mathcal{U} \right) \right] dx \quad (2.15)$$

$$\frac{1}{k^2} \frac{\mathcal{P}}{\sigma} = \frac{1}{6} (G^1_1 - G^2_2) \quad (2.16)$$

$$\frac{6}{k^4} \frac{\mathcal{U}}{\sigma} = -\frac{3}{\sigma} \left(\frac{\rho^2}{2} + \rho p \right) + \frac{1}{k^2} (2G^2_2 + G^1_1) - 3p, \quad (2.17)$$

where

$$G^1_1 = -\frac{1}{r^2} + e^{-\lambda} \left(\frac{1}{r^2} + \frac{\nu'}{r} \right) \quad (2.18)$$

and

$$G^2_2 = \frac{1}{4} e^{-\lambda} \left(2\nu'' + \nu'^2 - \lambda' \nu' + 2\frac{\nu' - \lambda'}{r} \right). \quad (2.19)$$

Now, by using Eq. (2.17) in Eq. (2.15), we find an integro-differential equation for the function $\lambda(r) = \lambda(\nu(r), r)$, which is different from the GR case, and is a direct consequence of the non-locality of the brane-world equations. The general solution to this equation is given by [72]

$$\begin{aligned} e^{-\lambda} &= \underbrace{1 - \frac{k^2}{r} \int_0^r x^2 \rho dx}_{\text{GR-solution}} + \underbrace{e^{-I} \int_0^r \frac{e^I}{\frac{\nu'}{2} + \frac{x}{r}} \left[H(p, \rho, \nu) + \frac{k^2}{\sigma} (\rho^2 + 3\rho p) \right] dx}_{\text{Geometric deformation}} + \beta(\sigma) e^{-I}, \\ &\equiv \mu(r) + f(r), \end{aligned} \quad (2.20)$$

where

$$H(p, \rho, \nu) \equiv 3k^2 p - \left[\mu' \left(\frac{\nu'}{2} + \frac{1}{r} \right) + \mu \left(\nu'' + \frac{\nu'^2}{2} + \frac{2\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} \right] \quad (2.21)$$

encodes the effects due to bulk gravity, depending on p , ρ and ν . The exponent

$$I \equiv \int \frac{\left(\nu'' + \frac{\nu'^2}{2} + \frac{2\nu'}{r} + \frac{2}{r^2} \right)}{\left(\frac{\nu'}{2} + \frac{x}{r} \right)} dr, \quad (2.22)$$

and $\beta = \beta(\sigma)$ is a function of the brane tension σ which must vanish in the GR limit. Moreover, in the star interior, the condition $\beta = 0$ must be imposed in order to avoid singular solutions at the center $r = 0$. Finally, note that the function

$$\mu(r) \equiv 1 - \frac{k^2}{r} \int_0^r x^2 \rho dx = 1 - \frac{2m(r)}{r} \quad (2.23)$$

contains the usual GR mass function m .

An important remark is that when a given (spherically symmetric) perfect fluid solution in GR is considered as a candidate solution for the brane-world system of Eqs. (2.7)-(2.10) [or, equivalently, Eq. (2.10) along with Eqs. (2.15)-(2.17)], one exactly obtains

$$H(p, \rho, \nu) = 0. \quad (2.24)$$

Therefore, every (spherically symmetric) perfect fluid solution in GR will produce a *minimal* deformation on the radial metric component (2.20), such that the geometric deformation $f = f(r)$ contains only one contribution, and

$$f^*(r) = \frac{2k^2}{\sigma} e^{-I(r)} \int_0^r \frac{x e^{I(x)}}{x \nu' + 4} (\rho^2 + 3\rho p) dx . \quad (2.25)$$

The geometric deformation $f = f(r)$ of Eq. (2.20) “distorts” the GR solution represented by Eq. (2.23), but the specific form $f^* = f^*(r)$ in Eq. (2.25) represents a “minimal distortion” for any GR solution of choice in the sense that all of the deforming terms in Eq. (2.20) have been removed, except for those produced by the density and pressure, which will always be present in a realistic stellar distribution ¹. Note then that the function f^* can also be found from Eq. (2.16) as

$$\frac{6}{k^2} \frac{\mathcal{P}}{\sigma} = \left(\frac{1}{r^2} + \frac{\nu'}{r} - \frac{\nu''}{2} - \frac{\nu'^2}{4} - \frac{\nu'}{2r} \right) f^* - \frac{1}{4} \left(\nu' + \frac{2}{r} \right) (f^*)' . \quad (2.26)$$

The interior stellar geometry is given by the MGD metric

$$ds^2 = e^{\nu^-(r)} dt^2 - \frac{dr^2}{1 - \frac{2m(r)}{r} + f^*(r)} - r^2 (d\theta^2 + \sin^2\theta d\phi^2) , \quad (2.27)$$

and it is straightforward to introduce the effective interior mass function

$$\tilde{m}(r) = m(r) - \frac{r}{2} f^*(r) . \quad (2.28)$$

Since, from Eq. (2.25), the geometric deformation in Eq. (2.27) is seen to obey the positivity condition

$$f^*(r) \geq 0 , \quad (2.29)$$

the effective interior mass (2.28) is always reduced by the extra-dimensional effects.

2.2 Interior MGD metric and exterior Weyl fluid

The MGD metric in Eq. (2.27), characterising the star interior at $r < R$, should be matched with an exterior geometry associated with the Weyl fluid \mathcal{U}^+ , \mathcal{P}^+ , and $p = \rho = 0$, for $r > R$ [17]. This can be generically written as

$$ds^2 = e^{\nu^+(r)} dt^2 - e^{\lambda^+(r)} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) , \quad (2.30)$$

where the explicit form of the functions ν^+ and λ^+ are obtained by solving the effective 4-dimensional vacuum Einstein equations, namely

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^\alpha_\alpha = \mathcal{E}_{\mu\nu} \quad \Rightarrow \quad R^\alpha_\alpha = 0 , \quad (2.31)$$

where we recall that extra-dimensional effects are contained in the projected Weyl tensor $\mathcal{E}_{\mu\nu}$ and that only a few analytical solutions are known to date [33], [44], [46]-[49]. When both interior and

¹There is a MGD solution in the case of a dust cloud, with $p = 0$, but we will not consider it in the present work.

exterior metrics, respectively given by Eq. (2.27) and (2.30) are considered, continuity of the first fundamental form at the star surface Σ defined by $r = R$ reads [5], [91]

$$[ds^2]_{\Sigma} = 0 , \quad (2.32)$$

where $[F]_{\Sigma} \equiv F(r \rightarrow R^+) - F(r \rightarrow R^-) \equiv F_R^+ - F_R^-$, for any function $F = F(r)$, and yields

$$e^{\nu^-(R)} = e^{\nu^+(R)} , \quad (2.33)$$

$$1 - \frac{2M}{R} + f_R^* = e^{-\lambda^+(R)} , \quad (2.34)$$

where $M = m(R)$. Likewise, continuity of the second fundamental form at the star surface reads [5], [91]

$$[G_{\mu\nu} r^{\nu}]_{\Sigma} = 0 , \quad (2.35)$$

where r_{μ} is a unit radial vector. Using Eq. (2.35) and the general Einstein field equations, we then find

$$[T_{\mu\nu}^{eff} r^{\nu}]_{\Sigma} = 0 , \quad (2.36)$$

which leads to

$$\left[p + \frac{1}{\sigma} \left(\frac{\rho^2}{2} + \rho p + \frac{2}{k^4} \mathcal{U} \right) + \frac{4}{k^4} \frac{\mathcal{P}}{\sigma} \right]_{\Sigma} = 0 . \quad (2.37)$$

Since we assume the distribution is surrounded by a Weyl fluid characterised by \mathcal{U}^+ , \mathcal{P}^+ , and $p = \rho = 0$ for $r > R$, this matching condition takes the final form

$$p_R + \frac{1}{\sigma} \left(\frac{\rho_R^2}{2} + \rho_R p_R + \frac{2}{k^4} \mathcal{U}_R^- \right) + \frac{4}{k^4} \frac{\mathcal{P}_R^-}{\sigma} = \frac{2}{k^4} \frac{\mathcal{U}_R^+}{\sigma} + \frac{4}{k^4} \frac{\mathcal{P}_R^+}{\sigma} , \quad (2.38)$$

where $p_R \equiv p_R^-$ and $\rho_R \equiv \rho_R^-$. Finally, by using Eqs. (2.17) and (2.26) in the condition (2.38), the second fundamental form can be written in terms of the MGD at the star surface, denoted by f_R^* , as

$$p_R + \frac{f_R^*}{8\pi} \left(\frac{\nu'_R}{R} + \frac{1}{R^2} \right) = \frac{2}{k^4} \frac{\mathcal{U}_R^+}{\sigma} + \frac{4}{k^4} \frac{\mathcal{P}_R^+}{\sigma} , \quad (2.39)$$

where $\nu'_R \equiv \partial_r \nu_-|_{r=R}$. The expressions given by Eqs. (2.33) and (2.34), along with Eq. (2.39), are the necessary and sufficient conditions for the matching of the interior MGD metric (2.27) to a spherically symmetric “vacuum” filled by a brane-world Weyl fluid.

The matching condition (2.39) yields an important result: in the Schwarzschild exterior one must have $\mathcal{U}^+ = \mathcal{P}^+ = 0$, which then leads to

$$p_R = -\frac{f_R^*}{8\pi} \left(\frac{\nu'_R}{R} + \frac{1}{R^2} \right) . \quad (2.40)$$

Since we showed above, in Eq. (2.29), that $f^* \geq 0$, an exterior vacuum can only be supported in the brane-world by exotic stellar matter, with negative pressure p_R at the star surface, in agreement with the model discussed in Ref. [42].

3 Stellar solution with a solid crust

In this section we will show that, contrary to what was previously believed, it is possible to have a star predominantly made of regular matter and with a Schwarzschild exterior.

In order to accomplish the above, we consider the exact interior brane-world solution found in Ref. [73],

$$e^\nu = A(1+x)^4, \quad (3.1)$$

$$e^{-\lambda} = 1 - \frac{8}{7}x \frac{(3+x)}{(1+x)^2} + f^*(r), \quad (3.2)$$

with $x = Cr^2$ and matter pressure and density given by

$$p(r) = \frac{2C(2-7x-x^2)}{7\pi(1+x)^3}, \quad (3.3)$$

$$\rho(r) = \frac{C(9+2x+x^2)}{7\pi(1+x)^3}, \quad (3.4)$$

and non-local contributions

$$\begin{aligned} \mathcal{P}(r) = & \frac{32C^2}{441x^2(1+x)^6(1+3x)^2} \left[x(180+2040x+8696x^2+16533x^3+12660x^4 \right. \\ & \left. +146x^5-120x^6+9x^7) - 60\sqrt{C}(1+x)^3(3+26x+63x^2)\arctan(\sqrt{x}) \right], \end{aligned} \quad (3.5)$$

$$\begin{aligned} \mathcal{U}(r) = & \frac{32C^2}{441x^2(1+x)^6(1+3x)^2} \left[x^2(795+4865x+10044x^2+6186x^3 \right. \\ & \left. -373x^4-219x^5-18x^6) - 240x^{3/2}(1+x)^3(5+9x)\arctan(\sqrt{x}) \right]. \end{aligned} \quad (3.6)$$

The corresponding geometric deformation in Eq. (3.2) due to 5-dimensional effects is now given by

$$f^* = \left(\frac{2}{7}\right)^2 \frac{C}{\sigma\pi} \left[\frac{240+589x-25x^2-41x^3-3x^4}{3(1+x)^4(1+3x)} - \frac{80\arctan(\sqrt{x})}{(1+x)^2(1+3x)\sqrt{x}} \right], \quad (3.7)$$

with A and C constants to be determined by the matching conditions. Using this interior brane-world solution, and the Schwarzschild exterior geometry with $\mathcal{M} = \tilde{m}(R)$,

$$e^{\nu^+} = e^{-\lambda^+} = 1 - \frac{2\mathcal{M}}{r}, \quad \mathcal{U}^+ = \mathcal{P}^+ = 0, \quad (3.8)$$

in the matching conditions (2.33) and (2.34), we obtain

$$A = \left(1 - \frac{2\mathcal{M}}{R}\right) (1 + CR^2)^{-4} \quad (3.9)$$

and

$$\begin{aligned} \frac{2\mathcal{M}}{R} = & \frac{2M}{R} - \left(\frac{2}{7}\right)^2 \frac{C}{\pi\sigma} \frac{240+589CR^2-25C^2R^4-41C^3R^6-3C^4R^8}{3(1+CR^2)^4(1+3CR^2)} \\ & + \left(\frac{2}{7}\right)^2 \frac{C}{\pi\sigma} \frac{80\arctan(\sqrt{CR})}{(1+CR^2)^2(1+3CR^2)\sqrt{CR}}. \end{aligned} \quad (3.10)$$

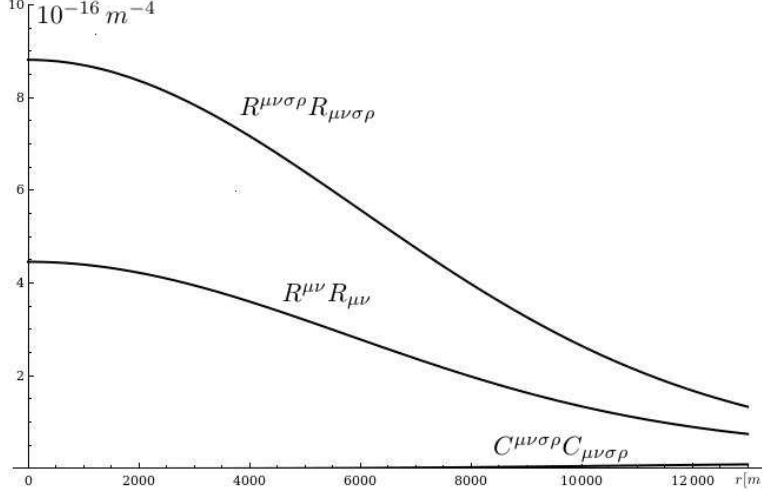


Figure 1: The Kretschmann scalar $R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho}$, Ricci square $R^{\mu\nu}R_{\mu\nu}$ and Weyl square $C^{\mu\nu\sigma\rho}C_{\mu\nu\sigma\rho}$ for a distribution with $R = 13$ km.

Using Eqs. (3.1), (3.3) and (3.7) in the matching condition (2.40), the constant C turns out to be determined by

$$CR^2(2 - 7CR^2 - C^2R^4) + \frac{7}{16}(1 + CR^2)^2(1 + 9CR^2)f_R^*(\sigma) = 0, \quad (3.11)$$

which clearly shows that C is promoted to a function of the brane tension σ due to bulk gravity effects, that is $C = C(\sigma)$. The Kretschmann scalar $R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho}$, Ricci square $R^{\mu\nu}R_{\mu\nu}$ and Weyl square $C^{\mu\nu\sigma\rho}C_{\mu\nu\sigma\rho}$ associated with the interior geometry given by the expressions (3.1) and (3.2) are shown on Fig. 1.

In order to find the extra-dimensional effects on physical variables, i.e., the pressure in Eq. (3.3) and density in Eq. (3.4), we need to fix the function $C = C(\sigma)$ satisfying Eq. (3.11). We first consider

$$C = C_0 + \delta, \quad (3.12)$$

where C_0 is the GR value of C , given by

$$C_0 = \frac{\sqrt{57} - 7}{2R^2}, \quad (3.13)$$

and which is found by using the standard GR condition at the star surface, $p(R) \equiv p_R = 0$, in Eq. (3.3). By using Eq. (3.12) in Eq. (3.11), we then obtain the leading-order brane-world contribution

$$\delta(\sigma) = \frac{7(1 + C_0R^2)^2(1 + 9C_0R^2)}{16C_0R^2(7 + 2C_0R^2)} \frac{f_R^*}{R^2} + \mathcal{O}(\sigma^{-2}). \quad (3.14)$$

The pressure can thus be determined by expanding $p = p(C)$ around C_0 ,

$$p(C_0 + \delta) \simeq p(C_0) + \delta \left. \frac{dp}{dC} \right|_{C=C_0}, \quad (3.15)$$

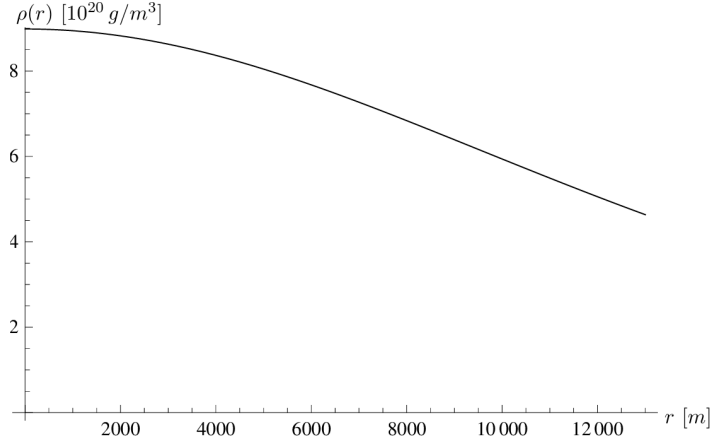


Figure 2: Behaviour of the density $\rho = \rho(r)$ [in units of 10^{20} g/m^3] for $\delta/C_0 = 0.03$, for a typical compact distribution of $R = 13 \text{ km}$.

which leads to

$$p(r) \simeq \frac{2C_0}{7\pi} \frac{(2 - 7C_0r^2 - C_0^2r^4)}{(1 + C_0r^2)^3} + \frac{4}{7\pi} \frac{(1 - 9C_0r^2 + 2C_0^2r^4)}{(1 + C_0r^2)^4} \delta(\sigma). \quad (3.16)$$

Consequently, at the star surface $r = R$, the pressure becomes

$$p_R(\sigma) \simeq \frac{4}{7\pi} \frac{(1 - 9C_0R^2 + 2C_0^2R^4)}{(1 + C_0R^2)^4} \delta(\sigma) < 0, \quad (3.17)$$

that is, negative and proportional to $1/\sigma$, according to Eqs. (3.7) and (3.14).

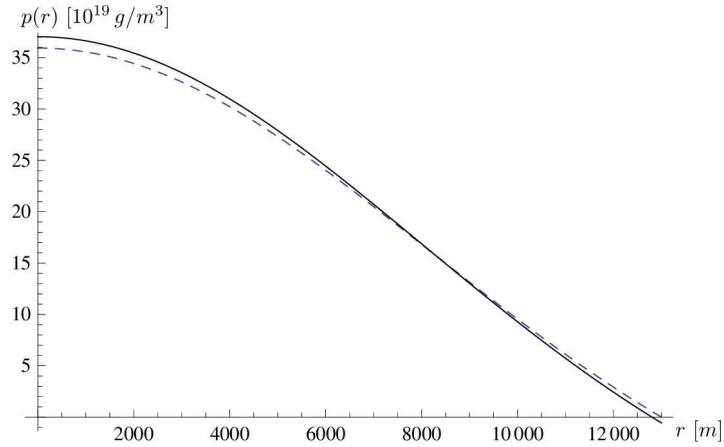


Figure 3: Qualitative comparison of the pressure $p = p(r)$ [in units of 10^{19} g/m^3] for a compact distribution of $R = 13 \text{ km}$ in GR [$p(R) = 0$; dashed curve] and in the brane-world model with $\delta/C_0 = 0.03$ [$p(R) < 0$; solid curve] both with Schwarzschild exterior, showing the thin layer of solid outer crust in the brane-world case.

A typical density profile $\rho = \rho(r)$ is displayed in Fig. 2. In Fig. 3, we likewise display the pressure $p = p(r)$ for different values of the radius, both for the GR case and for its brane-world

generalisation. Remarkably, the pressure is negative only in a thin layer close to the boundary. A negative pressure in this layer acts as a positive tension, a common property for solid materials. Hence we can interpret the structure of the star as an (effectively) imperfect fluid with a solid crust. In this respect, it is now important to look at the energy conditions for our system.

Let us recall the energy conditions are a set of constraints which are usually imposed on the energy-momentum tensor in order to avoid exotic matter sources and, correspondingly, exotic space-time geometries. In particular, we may mention: (a) the Null Energy Condition (NEC), $T_{\mu\nu}K^\mu K^\nu \geq 0$ for any null vector K^μ . For a perfect fluid, this condition implies

$$\rho + p \geq 0 ; \quad (3.18)$$

(b) the Weak Energy Condition (WEC), $T_{\mu\nu}X^\mu X^\nu \geq 0$ for any time-like vector X^μ , which, again for a perfect fluid, yields $\rho \geq 0$ and $\rho + p \geq 0$; (c) the Dominant Energy Condition (DEC), $T^\mu{}_\nu X^\nu = -Y^\mu$, where Y^μ must be a future-pointing causal vector. For a perfect fluid, this means

$$\rho > |p| ; \quad (3.19)$$

and finally (d) the Strong Energy Condition (SEC), $(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})X^\mu X^\nu \geq 0$, or, for a perfect fluid, $\rho + p \geq 0$ and $\rho + 3p \geq 0$. While it is true that these conditions might fail for particular reasonable classical systems, they can be viewed as sensible guides to avoid unphysical solutions. A very well known example is the usual classical fields which obey the WEC, and therefore the energy density seen by any (time-like) observer can never be negative. Hence wormholes, superluminal travel, and construction of time machines can be ruled out. On the other hand, the SEC is violated by Cosmological Inflation (driven by a minimally coupled massive scalar) and by the accelerating universe [92]. Let us also note that $\text{DEC} \Rightarrow \text{WEC} \Rightarrow \text{NEC}$ and $\text{SEC} \Rightarrow \text{NEC}$ (but SEC does not imply WEC). We can now argue how to implement these conditions in our case, where a GR isotropic fluid has been transformed into an anisotropic one due to extra-dimensional effects, as it is clearly seen from Eqs. (2.11)-(2.13). To address this question, we shall consider the weakest condition, namely the NEC, and show with a direct calculation the difference with respect to the perfect fluid case. First of all, we need a null vector field K^μ , which in our case can be written as

$$K^\mu = e^{-\nu/2} \delta_0^\mu + e^{-\lambda/2} \delta_1^\mu , \quad (3.20)$$

for which the NEC reads

$$T_{\mu\nu}K^\mu K^\nu = e^\nu \tilde{\rho} K^0 K^0 + e^\lambda \tilde{p}_r K^1 K^1 = \tilde{\rho} + \tilde{p}_r \geq 0 , \quad (3.21)$$

which looks like the standard condition (3.18) with $\rho \rightarrow \tilde{\rho}$ and $p_r \rightarrow \tilde{p}_r$. In the same way, the DEC leads to $\tilde{\rho} \geq \tilde{p}_r$ and $\tilde{\rho} \geq \tilde{p}_t$, which are precisely the analogue forms of Eq. (3.19).

According to Eqs. (2.11)-(2.13), these inequalities turn into new bounds for the perfect fluid density and pressure,

$$\rho \geq p + \frac{1}{\sigma} \rho p + \frac{4}{k^4 \sigma} (\mathcal{P} - \mathcal{U}) , \quad (3.22)$$

$$\rho \geq p + \frac{1}{\sigma} \rho p - \frac{2}{k^4 \sigma} (\mathcal{P} + 2\mathcal{U}) , \quad (3.23)$$

while the effective strong energy condition becomes

$$\rho + 3p + \frac{1}{\sigma} \left(2\rho^2 + 3\rho p + \frac{12}{k^4} \mathcal{U} \right) > 0 . \quad (3.24)$$

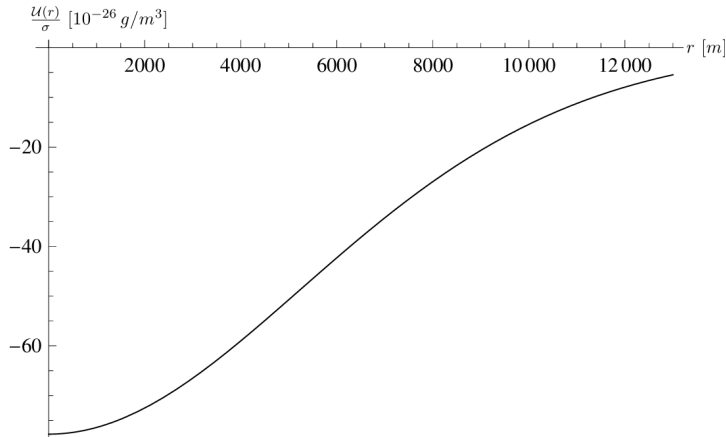


Figure 4: The scalar function $\mathcal{U}(r)/\sigma$ [in units of 10^{-26} g/m^3] for a distribution with $R = 13 \text{ km}$. $\mathcal{U}(r)$ is always negative in the interior, hence it reduces both the effective density and effective pressure.

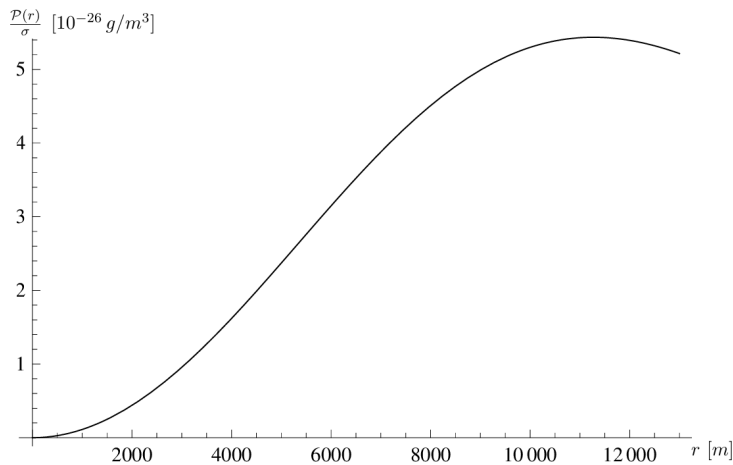


Figure 5: Behaviour of the Weyl function $\mathcal{P}(r)/\sigma$ [in units of 10^{-26} g/m^3] inside the stellar distribution with $R = 13 \text{ km}$.

All of the above effective conditions are satisfied, as well as the WEC, even inside the solid crust. This means that there are no negative (fluid or effective) pressures comparable in magnitude or larger than the density ρ , and therefore the brane-world effects on the GR solution are not strong enough to jeopardise the physical acceptability of the system. In Figs. 4 and 5, we also display the behaviour of $\mathcal{U} = \mathcal{U}(r)$ and $\mathcal{P} = \mathcal{P}(r)$, respectively, for the same star as in Fig. 3. These two plots clearly show the typical energy scale of the Weyl functions and a discontinuity at $r = R$ in the respective quantities. We shall have more to say about this in the last Section.

In order to determine the thickness Δ of the solid crust, we define the critical radius r_c as the areal radius of the sphere on which the pressure vanishes (see Fig. 3 for an example),

$$p(r_c) = 0 . \quad (3.25)$$

Therefore, the crust has a thickness

$$\Delta \equiv R - r_c , \quad (3.26)$$

where r_c can now be found by using the pressure in Eq. (3.3),

$$p(r_c) = \frac{2C(2 - 7Cr_c^2 - C^2r_c^4)}{7\pi(1 + Cr_c^2)^3} = 0 . \quad (3.27)$$

The above immediately yields

$$r_c = \sqrt{\frac{\sqrt{57} - 7}{2C(\sigma)}} , \quad (3.28)$$

where $C(\sigma)$ is given by Eq. (3.12). To leading order in σ^{-1} this gives

$$r_c \simeq R \left(1 + \frac{\delta}{C_0}\right)^{-1/2} \simeq R \left(1 - \frac{\delta(\sigma)}{2C_0}\right) , \quad (3.29)$$

and

$$\Delta \simeq \frac{R\delta(\sigma)}{2C_0} , \quad (3.30)$$

which, according to Eqs. (3.13), (3.14) and (3.7), finally reads

$$\Delta \sim R^3 \delta(\sigma) \sim R f_R^* \sim \frac{1}{R\sigma} . \quad (3.31)$$

We emphasise that the tiny region with $p < 0$ is interpreted a solid crust, consisting of regular matter. A test particle at the star surface $r = R$ would experience a combination of a negative pressure $p(R) < 0$ and gravitational force, both pulling it inwards, and an extra-dimensional effect pushing it out. In this sense, the negative pressure of the crust resembles a fluid tension in a soap bubble. One can consider our solution in analogy with the structure of neutron stars, which have a solid crust surrounding a (superfluid) interior.

The expression in Eq. (3.31) shows that the solid crust becomes thicker as the size of the star becomes smaller, showing that this ‘‘solidification process’’ in the outer layer due to extra-dimensional effects should be particularly important for compact distributions. Note however that for solar size stars $R \gg \sigma^{-1/2}$, and the crust is much thinner than the fundamental length $\sigma^{-1/2}$. This already suggests that the crust is of little physical relevance, if not a pure artefact of the approximations employed, as we shall discuss shortly.

4 Conclusions

Our main result in this paper is that a brane-world compact source, despite previous no-go results, may have a Schwarzschild exterior. The exact solution of the effective Einstein equations on the brane derived here, represents a non-uniform, spherically symmetric, self-gravitating star with regular properties, which is embedded into a vacuum Schwarzschild exterior geometry. This result is obtained at the price of having negative pressure inside a narrow shell at the star surface [$p(r) < 0$ for $R - \Delta < r < R$]. Throughout the star (in the shell and interior), however, all physical properties are perfectly regular.

The negative pressure shell has rather a tension and qualifies as a solid (as illustrated in Fig. 6). The interpretation of a solid material appearing as consequence of the extra dimension in the context

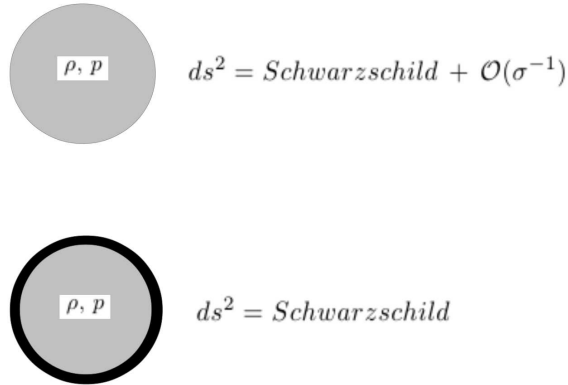


Figure 6: Schematic picture of a brane-world star with and without a solid thin outer crust: the interior is characterised by the density ρ , pressure p and star radius R . The exterior geometry by the ADM mass \mathcal{M} and corrections proportional to σ^{-1} when there is no solid crust, and by the Schwarzschild geometry when there is a solid crust which thickness $\Delta \sim 1/R\sigma$.

of brane-worlds was first advanced in Ref. [93], which presented the homogeneous counterpart of the Einstein brane [94]. Another brane-world star, the perfect fluid material of which in the latest stages of the collapses obeys the dark energy condition, was discussed in Ref. [42]. In the present case however the region with negative pressure is tiny and effectively acts as a solid crust, separating the inner fluid from the vacuum exterior. Moreover, in the crust all energy conditions hold, as they do everywhere inside the star.

The thickness Δ of the solid crust is given by $\Delta^{-1} \sim R\sigma$, showing thus that this “solidification process” in the outer crust due to extra-dimensional effects becomes more important for compact stellar distributions. Moreover, since the dimensionless parameter $\Delta/\sigma^{-1/2} \sim (\sigma^{1/2}R)^{-1} \ll 1$ for astrophysical stars, this crust has negligible thickness, falling below any physically sensible length scale for astrophysical sources. We refrain from trying to develop a detailed mechanism to realise the negative pressure, precisely because the thickness falls below length scales for well-known physics. In fact, the very existence of the crust could be masked by modifications of the fundamental gravitational theory above the brane-world energy scale σ . For testing such modifications, however precision measurements probing physics above the scale set by σ would be necessary.

As the stars with solid crust discussed in this paper are embedded in vacuum, they do not radiate, strictly speaking. In order to include radiation, one should in fact match the star interior with a Vaidya exterior (containing radiation in the geometrical optics limit, or null dust). Nevertheless, the emission of thermal radiation is allowed within our approximations, similarly as our Sun (the exterior of which is also well approximated by a vacuum Schwarzschild solution) can be well described like a black body emitting radiation at approximately 5800 K. One can then argue that a black body radiation outside our brane-world stars with solid crust should not affect the geometry significantly, precisely like this radiation is negligible for the Sun in 4-dimensional GR. Finally, the crust should be transparent to this thermal radiation, otherwise it would accumulate

energy and become quickly unstable.

At a technical level, the Weyl source functions $\mathcal{U} = \mathcal{U}(r)$ and $\mathcal{P} = \mathcal{P}(r)$ exhibit a discontinuity at the star surface $r = R$, where the radial effective pressure $\tilde{p}_r(R) = 0$. Concerning these discontinuities, it is known [17] that $\mathcal{U}(r) = \mathcal{P}(r) = 0$ cannot hold everywhere on the brane if matter is present inside a compact brane-world region (in our case, for $r < R$). The question then arises whether a jump in $\mathcal{U} = \mathcal{U}(r)$ and $\mathcal{P} = \mathcal{P}(r)$ at $r = R$ (see Figs. 4 and 5) might signal some pathological behaviour of the extension into the bulk of the stellar solution. We will argue, that this is not the case. Most brane-world models, including the present one, assume the brane is a (Dirac δ -like) discontinuity along the extra dimension. The material sources on the brane are regular, however, as the brane itself represents a hypersurface, they could only be extended into the fifth dimension as Dirac δ -like distributions. The way to avoid that and have a regular stellar matter distribution in all dimensions would be to consider a thick brane (like a very narrow Gaussian of thickness, say, of order $\sigma^{-1/2}$ along the extra spatial dimension), on top of which the regular distribution of brane-world matter could be placed (for more details, see Ref. [40]). The “brane-world limit” could then be obtained by assuming $\sigma^{-1/2}$ is much shorter than any length scale associated with brane-world matter. The approach followed in this paper instead, is based on the effective 4-dimensional brane-world equations obtained from the δ -like brane energy density in the bulk, and a check that the behaviour of all physical variables is sufficiently well-behaved. Specifically, one can see from the Figs. 4 and 5 that the discontinuities in $\mathcal{U} = \mathcal{U}(r)$ and $\mathcal{P} = \mathcal{P}(r)$ at $r = R$ are negligibly small, most likely generated by the δ -like brane approach.

It is commonly believed that brane-world modifications to a star should be detected through the long range behaviour, manifesting themselves in the weak field regime. Orbital motions or gravitational lensing could then provide information about the parameters of the stars, like its mass, rotation, quadrupole moment and so on. Strong field effects, like those occurring in the inner edge of an accretion disk, supposed to be at the innermost stable circular orbit also depend only on the exterior geometry. Stellar astrophysical processes leading to electromagnetic radiation could also be slightly modified in brane-worlds (although standard model fields remain 4-dimensional, gravity is changed, hence the equilibrium between radiation pressure and gravitational attraction, for example, is shifted). In fact such constraints were already derived for the tidally charged brane black hole [33], and include constraints on the tidal charge from the deflection of light [34, 35, 95], from the radius of the first relativistic Einstein ring due to strong lensing [96] and from the emission properties of the accretion disks, including the energy flux, the emission spectrum and accretion efficiency [97].

The importance of the results presented in this paper relies in explicitly illustrating that no matter how severe constraints from lensing or other tests are derived for brane-world stars with exteriors depending on brane-world parameters, the existence of the brane-world stars cannot be ruled out, as their exterior could be the same as in GR. The brane-world stars presented in this paper composed of a fluid with physically reasonable properties and having a solid crust, immersed into a vacuum Schwarzschild region on the brane, precisely exhibits this property of indistinguishability.

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