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ON THE SOLIDITY OF PACKINGS OF INCONGRUENT CIRCLES I.

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1. Introduction.

A packing of convex discs is said to be *solid* if no finite subset of the discs can be rearranged so as to obtain a packing *not* congruent to the original one [1].

In the present paper we shall prove a general theorem that contains sufficient conditions for the solidity of circle packings in the Euclidean plane.

2. Definitions.

Let $\mathcal{R} = \{r_1, r_2, \dots, r_k\}$ be a set of positive numbers. Consider three disjoint open circles of a radius $\rho_i \in \mathcal{R}$ $(i=1,2,3)$. This triple as well as the triangle determined by the centers of the circles will be called *normal* if none of the segments connecting two centers intersects the third circle.

We say that a set of normal triangles *generates a packing* if the triangles cover the plane without gaps and without overlapping and the circle sectors of the individual normal triangles fit together to from complete circles.

A positive *weight* $w(r_i)$ will also be assigned to all circles of radius r_i $(i=1,2,\dots,k)$.

Let O_j and ρ_j denote the centers and the radii of a normal triple, respectively, and α_j the angle of the trialgle at vertex O_j $(j=1,2,3)$. The *weighted density* of the triple (in the triangle) is defined by

$$\sigma = \frac{\frac{1}{2}.\pi.\sum_{j=1}^{3}\rho_j^2.\alpha_j.\omega(\rho_j)}{A},$$

where A denotes the area of the triangle $O_1O_2O_3$.

For the sake of simplicity the term density will be used instead of weigthed density throughout this paper.

A normal triangle will be called *tight* (*spanned*) if the circles are mutually tangent (if one circle is tangent to the other two and to the opposite side) (Fig. 1).

3. Preparations.

First we show the validity of the following

LEMMA 1. Let the radii r_i and weights $\omega(r_i)$ (i=1,2,3) be given. We consider all normal triples consisting of circles the radii of which belong to the set $\mathcal{R} = \{r_1,\dots,r_k\}$ and we claim that each triple of maximal density is either tight or spanned.

The proof of LEMMA 1 is based on the following result of Hárs [2]:

LEMMA 2. Let **a**, **b**, **c**, α, β, γ and **A** denote the sides, the opposite angles and the area of a triangle. For given positive weights **u**, **v**, and **w** we consider the weighted *angle-density*

$$\vartheta = \frac{\mathbf{u}.\alpha + \mathbf{v}.\beta + \mathbf{w}.\gamma}{\mathbf{A}}.$$

For fixed **a**, **b**, **u**, **v** and **w** the function $\vartheta(\gamma)$ is strictly quasiconvex in $(0, \pi)$, i.e. for any given interval $0 < \gamma_1 < \gamma < \gamma_2 < \pi$ $\vartheta(\gamma)$ attains its maximum only at one or both ends of the interval.

Proof of LEMMA 1.

As the density in a large triangle is small, when looking for the densest arrangement it is enough to consider normal triangles of restricted size. However, the set of normal triangles of sidelength not greater than K is compact, thus the existence of a triangle of maximal density follows easily. Therefore, it is sufficient to show that a normal triangle that is neither tight nor spanned is not one of greatest density.

We consider a normal triangle that is neither tight nor spanned and distinguish two cases.

Case 1. No circle is tangent to the opposite side of the triangle (consequently it is nor spamnned) and there are two circles, say the first and the second that are not tangent (thus it is not tight either). Let us apply LEMMA 2 using the weights

$$u = \rho_1^2.\omega(\rho_1)/2,$$
$$v = \rho_2^2.\omega(\rho_2)/2,$$
$$w = \rho_3^2.\omega(\rho_3)/2,$$

where ρ_j denotes the actual values of the radii (j=1,2,3). (By this choice the weighted angle-density and the weighted density of the circles coincide). The role of γ_1 and γ_2 in Lemma 2 will be plaied by those values of angle γ - the angle opposite to side c - for which the triangle stops being normal, or, with other words, where a further touching occurs (Fig. 2). According to LEMMA 2 the density can not attain its maximum for an angle γ lying strictly between γ_1 and γ_2 thus the triangle in question is nor extremal.

Case 2. One circle, say the third one, is tangent to the opposite side O_1O_2 of the triangle (thus it is not tight), however it does not touch both of the other circles, say the first and the third are not tangent, (therefore it is not spanned). Let us reflect the triangle in straight line O_1O_2 and denote the mirror image of O_3 by O'_3 (Fig 3). Clearly, both isosceles triangles $O_1O_3O'_3$ and $O_2O_3O'_3$ are normal, and, for the densities σ_0, σ_1, σ_2 of the triangles $O_1O_2O_3$, $O_1O_3O'_3$, $O_2O_3O'_3$ it holds

$$A_1\sigma_1 + A_2\sigma_2 = 2.A.\sigma_0 = (A_1 + A_2).\sigma_0,$$

where A_i denotes the area of $O_iO_3O'_3$ (i=1,2). Consequently, σ_0 cannot be the maximum of σ except for $\sigma_1 = \sigma_2 = \sigma_0$. But, since neither triangle is spanned and - according to our assumption in Case 2 - $O_1O_3O'_3$ is not tight either it belongs to Case 1. Hence neither this triangle nor $O_1O_2O_3$ can be of maximal density.

This completes the proof of LEMMA 1.

Remark. Applying the same reflection we used in the discussion of Case 2 it is easy to see that whenever the maximal density is attained by a spanned triangle there is at least one tight triangle of the same density. Consequently, to find the maximal density for a given set of radii $\rho_i \in \mathcal{R}$ and weights $\omega(\rho_i)$ it is enough to compare the densities for the $k+2\binom{k}{2}+\binom{k}{3}$ tight triangles.

4. The THEOREM.

The proofs of the solidity of certain packings can be based on the following general

THEOREM. A packing of circles of radius $r_1, r_2, \ldots, r_k$ is solid if

(i) The packing can be decomposed into tight triangles. The actual types of triangles used in this decomposition will be called *tile* triangles.

(ii) Positive weights $\omega(r_i)$ can be assigned to the circles of radius r_i $(i=1,2,\ldots,k)$ in such a way that all tile triangles have equal weighted density while the density in any other tight triangle is smaller.

(iii) The union **U** of an arbitrary finite set of triangles of the decomposition can be filled (without gaps and without overlapping) by tile triangles generating a packing only in one way - according to the original pattern.

To prepare the proof of the THEOREM we refrase a result of Fejes Tóth and Molnár [3]:

LEMMA 3. Any saturated packing[1] of circles of radius $\geq p > 0$ can be decomposed into normal triangles - even so that each segment connecting the centers of tangent circles is a side of a triangle of the decomposition[2].

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[1] A packing of circles of radius p is called *saturated* if there is no room left for a further circle of radius p without overlapping.

[2] This formulation of the result is rather a corollary of their method than the exact citing of a statement in the paper.

Proof of the THEOREM. Let P be a packing for which the assumptions are valid, S an arbitrary finite set of circles of P and S' a rearrangement of these circles that together with the rest P - S of the packing forms a new packing P'. We shall show that P and P' are congruent, i.e. the packing is solid.

Let **U** be the union of a finite set of triangles of the decomposition that covers S and S' as well. Now we define a weighted packing problem for **U**. We consider all sets of circles of radius $r_1, r_2, \ldots, r_k$ that completely lie in **U** and, together with P - S, form a packing and maximize the density of these packings within **U**, when all circles of radius r_i are taken with weight $w(r_i)$ defined in assumption (ii).

Clearly, the original set S provides an extremal solution since **U** can be decomposed into tile triangles each maximizing the density. The contribution of S and S' is the same to the density in **U**, thus the extremality of the corresponding packing implies that P' is saturated. Then - by LEMMA 3 - P' can be decomposed into normal triangles in such a way that the boundary of **U** (consisting of segments each connecting the centers of a pair of touching circles) is not "crossed" by triangles, e.g. **U** is the union of a finite set of these triangles.

From the equality of the contributions mentioned above it also follows that each triangle of this second decomposition of **U** also maximize the density. By LEMMA 1 each of these triangles is either tight or spanned. In fact none of them is spanned, because spanned triangles could occure in P' only in symmetrical pairs implying the existence of touching pairs of circles the centers of which are not connected by a side of triangle. But, this would contradict the basic property of the second decomposition guaranteed by LEMMA 3.

Consequently, all triangles of the second decomposition must be tile triangles. These triangles fill **U** and generate a packing, thus - according to assumption (iii) - the second decomposition coincides with the first one.

This completes the proof of the THEOREM.

References

[1] L.Fejes Tóth, Solid Circle-packings and Circle-coverings, Studia Scientiarum Mathematicarum Hungarica 3 (1968) 401-409.

[2] A.Florian, L.Hárs and J.Molnár, On the ρ-system of circles, Acta Math. Acad, Sci. Hung. **34** (1979), 205-221.

[3] L.Fejes Tóth and J.Molnár, Unterdeckung und Überdeckung der Ebene durch Kreise, Math. Nachrichten, 18 (1958), 236-243.

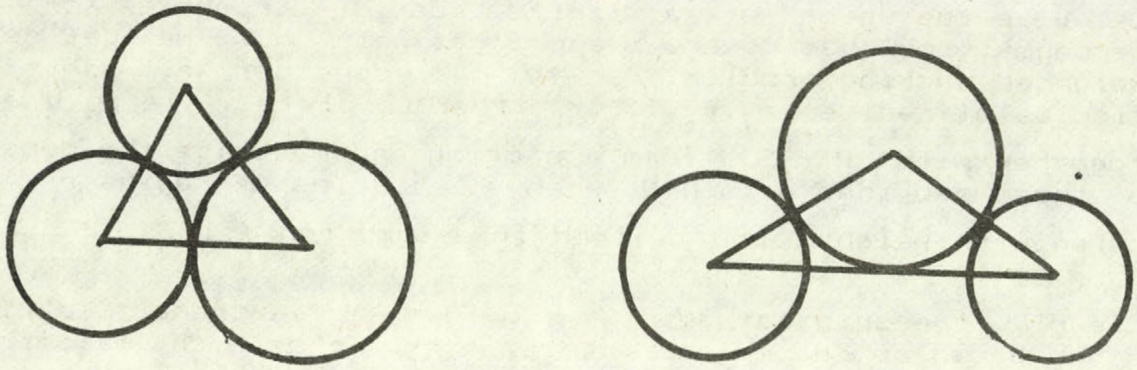

Fig.1

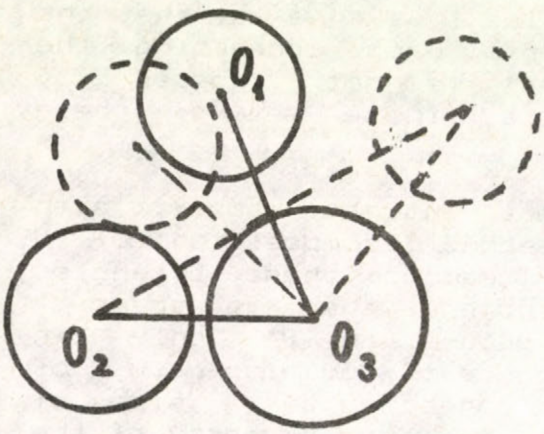

Fig.2

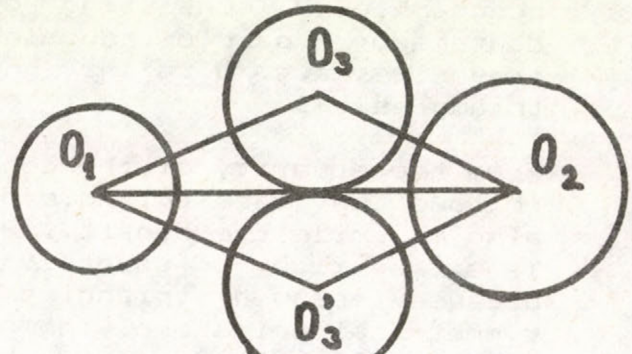

Fig.3

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Summary

A packing of convex discs is said to be solid if no finite subset of the discs can be rearranged so as to obtain a packing not congruent to the original one [1].
In the paper a general theorem is proved that contains sufficient conditions for the solidity of circle packings in the Euclidean plane.

INKONGRUENS KITÖLTÉSEK SZOLIDITÁSÁRÓL I.

Heppes Aladár

Összefoglaló

A sik konvex lemezekkel való kitöltését szolidnak nevezzük, ha a lemezek bármely véges részhalmaza csak ugy rendezhető át, hogy az uj kitöltés az eredetivel egybevágó lesz [1]. A cikkben a szerző a soliditásnak egy elégséges feltételét adja meg.