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M-MINIMAL COVERS AND SPERNER SYSTEMS WITH APPLICATION TO THE KEY FINDING PROBLEM FOR RELATION SCHEME

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§1. Introduction:

In studying the relation scheme, one important problem is the determination the set of its keys. In [1] C.L. Lucchesi and S.L. Osborn gave a very interesting algorithm to find the set of all keys for any relation scheme $S=\langle \Omega, F \rangle$. Ho Thuan has some new result about keys and superkeys for a relation scheme in [4,5] and J. Demetrovics [2] proved the equivalance of candidate keys with Sperner System.

In [7] we introduced the notion of so-called M-minimal cover for a relation scheme. The necessary and sufficient condition under which a subset X of Ω, $\mathfrak{M}_{\mathcal{G}}(X)$ is a M-minimal cover is established when the set of all keys was known.

Basing upon these results, in this paper we investigate the properties of M-minimal cover when a finite set H and a Sperner System $\mathcal{S}$ on H were given. Specially, we have established a necessary and sufficient condition for which two Sperner Systems are the set of all representative sets of each other. In other words, between the Sperner System $\mathcal{S}$ and the set of representative sets for $\mathcal{S}$ there is a close relationship and they determine each other. This means that from the given set of all keys for relation scheme we can construct the set of all its representative sets and conversely.

The set of keys for the relation scheme is just the set of all representative sets for the set of all representative sets for the set of keys.

§2. Basic definitions:

In this section we give some basic definitions. Let $H=\{a_1,\dots,a_h\}$ be a finite set. The set $\mathcal{S}=\{S_1,\dots,S_s\}$ will be called a Sperner System on the set H if it satisfies the following condition [2] :

a) $S_i \subseteq H$, $S_i \nsubseteq S_j$
for $i\neq j$, $i,j=1,2,\dots,s$.

b) $\bigcup_{i=1}^{s} S_i = H$

2.1: Wihout loss of generality, the ordered set H determines a matrix $\mathcal{M}_{\mathcal{S}}(H)=(\alpha_{ij})$ having h rows and s columns as follows:

$$\alpha_{ij} = \begin{cases} 1 & \text{if } a_i \in S_j \\ 0 & \text{otherwise.} \end{cases}$$

We call r_i , the i-th row of the matrix $\mathcal{M}_{\mathcal{S}}(H)$ for all $i=1,2,\dots,h$ and the following notion is used:

$$r_i \in \mathcal{M}_{\mathcal{S}}(H) \quad , \quad \mathcal{M}_{\mathcal{S}}(H) = \begin{pmatrix} r_1 \\ \vdots \\ r_h \end{pmatrix}$$

Let us define:

$$r_i \leq r_j \iff \alpha_{ik} \leq \alpha_{jk} \quad \begin{matrix} i,j=1,2,\dots,h \\ k=1,2,\dots,s. \end{matrix}$$

Let $X \subseteq H$ be any ordered subset of H. The subset X determines a matrix $\mathcal{M}_{\mathcal{S}}(X)$ which contains all rows r_{i_k} such that $a_{i_k} \in X$. We say that $\mathcal{M}_{\mathcal{S}}(X)$ is the submatrix of $\mathcal{M}_{\mathcal{S}}(H)$ and the meaning of the following notions are obvious:

$$\mathcal{M}_{\mathcal{Y}}(X) \subseteq \mathcal{M}_{\mathcal{Y}}(H) , \quad \mathcal{M}_{\mathcal{Y}}(X) = \begin{pmatrix} r_{i_1} \\ \vdots \\ r_{i_k} \end{pmatrix} .$$

Let $a_j \in X$ be any element, the element a_j determines the row r_j . Let us define :

$$\mathcal{M}_{\mathcal{Y}}(X) - \{r_j\} := \mathcal{M}_{\mathcal{Y}}(X - \{a_j\})$$

$$\mathcal{M}_{\mathcal{Y}}(H) - \mathcal{M}_{\mathcal{Y}}(X) := \mathcal{M}_{\mathcal{Y}}(H - X) .$$

2.2: The row vector $C[\mathcal{M}_{\mathcal{Y}}(X)] = (\eta_1, \ldots, \eta_s)$ is called the characteristic vector of the submatrix

$\mathcal{M}_{\mathcal{Y}}(X) \subseteq \mathcal{M}_{\mathcal{Y}}(H)$ if:

$$\eta_j = \begin{cases} 0 \text{ if} & \sum_{i=i_1}^{i_k} \alpha_{ij} = 0 \quad \text{for } j=1,2,\ldots,s \\ 1 \text{ otherwise} & \end{cases}$$

Where $X = \{a_{i_1}, \ldots, a_{i_k}\} \subseteq H$.

2.3: Let Q be subset of H. The submatrix $\mathcal{M}_{\mathcal{Y}}(Q) \subseteq \mathcal{M}_{\mathcal{Y}}(H)$ is call a M-minimal cover if it satisfies the following conditions:

a) $C[\mathcal{M}_{\mathcal{Y}}(Q)] = (1,1,\ldots,1)$

b) $\bar{\exists}\, Q' \subset Q,\ \mathcal{M}_{\mathcal{Y}}(Q') \subset \mathcal{M}_{\mathcal{Y}}(Q)$

such that $C[\mathcal{M}_{\mathcal{Y}}(Q')] = (1,1,\ldots,1)$.

If $\mathcal{M}_{\mathcal{Y}}(Q)$ only satisfies the condition (a) , we say that $\mathcal{M}_{\mathcal{Y}}(Q)$ is a M-cover.

The set $Q \subseteq H$ is called a representative set of the System $\mathcal{S}$ if the submatrix $\mathcal{M}_{\mathcal{Y}}(Q)$ determined by the subset Q is a M-minimal cover.

Let $\mathcal{Q}^{(\mathcal{S})}$ be the set of all representative sets for the System $\mathcal{S}$.

2.4: Let Y be proper subset of H $(Y \subset H)$. The set Y is called a Sp-antiset for the System $\mathcal{S}$ if it satisfies:

a) $S_i \nsubseteq Y$ for $S_i \in \mathcal{S}$

b) $\forall X: (X \subseteq H \ \& \ Y \subset X) \implies \exists S_i \in \mathcal{S}$ such that $S_i \subseteq X$.

Let $\mathcal{S}^{-1}$ be the set of all Sp-antisets for System $\mathcal{S}$. It is obvious that $\mathcal{S}^{-1}$ is also a Sperner System on H.

2.5: Example

Let $H=\{a_1,a_2,a_3,a_4,a_5,a_6\}$ and

$$\mathcal{S} = \{S_1,S_2,S_3,S_4\}.$$

Where $S_1=\{a_1,a_2\}$ $S_2=\{a_2,a_3,a_4\}$

$S_3=\{a_2,a_4,a_5\}$ $S_4=\{a_4,a_6\}$.

When

$$\mathcal{M}_{\mathcal{S}}(H) = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is M-cover.

And

$$\mathcal{M}_{\mathcal{S}}(Q_1) = \begin{pmatrix} r_1 \\ r_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad Q_1=\{a_1,a_4\}.$$

$$\mathcal{M}_{\mathcal{S}}(Q_2) = \begin{pmatrix} r_2 \\ r_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad Q_2=\{a_2,a_4\}$$

$$\mathcal{M}_{\mathcal{S}}(Q_3) = \begin{pmatrix} r_2 \\ r_6 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_3=\{a_2,a_6\}$$

$$\mathcal{M}_{\mathcal{S}}(Q_4) = \begin{pmatrix} r_1 \\ r_3 \\ r_5 \\ r_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_4=\{a_1,a_3,a_5,a_6\}$$

are M-minimal covers and the representative sets for System $\mathcal{S}$.

$$S_1^{-1}=\{a_2,a_3,a_5,a_6\} , S_2^{-1}=\{a_1,a_4,a_3,a_5\}$$

$$S_3^{-1}=\{a_1,a_3,a_5,a_6\} , S_4^{-1}=\{a_2,a_4\}$$

are Sp-antisets for $\mathcal{S}$.

§3. The properties of the M-minimal cover

In [7] we have proved some properties of the M-minimal cover when the set of all keys for the relation scheme was given. Basing on these results, in this section, the more general properties of M-minimal covers.

First, we recall some results that have been presented in [7] .

Theorem 3.1 [7] :

Let $H=\{a_1,\dots,a_h\}$ be a finite set and $\mathcal{S}=\{S_1,\dots,S_s\}$ be a Sperner System on H. Then for any $a_j \in H$, there exists a set $X \subseteq H$ such that $a_j \in X$ and $\mathcal{M}_{\mathcal{S}}(X) \subseteq \mathcal{M}_{\mathcal{S}}(H)$ is a M-minimal cover.

Corollary 3.1 [7]:

Let X be any subset of H such that $\mathcal{M}_{\mathcal{S}}(X) \subseteq \mathcal{M}_{\mathcal{S}}(H)$ is a M-cover. Then there exists a set $Q \subseteq X$ such that $\mathcal{M}_{\mathcal{S}}(Q) \subseteq \mathcal{M}_{\mathcal{S}}(X)$ is a M-minimal cover.

In other words any M-cover has a M-minimal cover.

Theorem 3.2 [7] :

Let $H=\{a_1,\dots,a_h\}$ be a finite set and $\mathcal{S}=\{S_1,\dots,S_s\}$ be a Sperner System on H. Let any $Q \subseteq H$. Then the matrix $\mathcal{M}_{\mathcal{S}}(Q)$ is a M-minimal cover if and only if the set Q satisfies the following conditions:

a) $\forall S_i \in \mathcal{S} \implies Q \cap S_i \neq \emptyset$

b) $\forall Q' \subset Q \implies \exists S_i \in \mathcal{S}$ such that $Q' \cap S_i = \emptyset$.

The theorem 3.2 can be formulated in an another form as follows:

Let $H = \{a_1,\dots,a_h\}$ be a finite set and $\mathcal{S} = \{S_1,\dots,S_s\}$ be a Sperner System on H. Let any $Q \subseteq H$. Then the set Q is a representative set of the System $\mathcal{S}$ if and only if the set Q satisfies both the conditions (a) and (b) .

Corollary 3.2 [7]

Let Q be any subset of H such that $\mathfrak{M}_{\mathcal{S}}(Q) \subseteq \mathfrak{M}_{\mathcal{S}}(H)$ is a M-minimal cover. Then for any subset $X \subseteq H$, $Q \subset X \implies \mathfrak{M}_{\mathcal{S}}(X) \subseteq \mathfrak{M}_{\mathcal{S}}(H)$ is a M-cover.

Corollary 3.3

Let $H = \{a_1,\dots,a_h\}$ be a finite set and $\mathcal{S} = \{S_1,\dots,S_s\}$, $\mathcal{Q} = \{Q_1,\dots,Q_q\}$ be a Sperner Systems on H.Then the system $\mathcal{Q}$ is the set of all representative sets for $\mathcal{S}$ if and only if the System $\mathcal{Q}$ satisfies the following conditions:

a) $\forall S_i \in \mathcal{S}$, $\forall Q_j \in \mathcal{Q} \implies S_i \cap Q_j \neq \emptyset$

b) $\forall Q_j \in \mathcal{Q}$, $\forall Q' \subset Q_j \implies S_i \in \mathcal{S}$ such that $Q' \cap S_i = \emptyset$.

Theorem 3.3 [7]

Let $H = \{a_1,\dots,a_h\}$ be a finite set and $\mathcal{S} = \{S_1,\dots,S_s\}$ be a Sperner System on H . Let $X \subseteq H$. Then the set X is a representative set for the System $\mathcal{S}$ (i.e. $\mathfrak{M}_{\mathcal{S}}(X) \subseteq \mathfrak{M}_{\mathcal{S}}(H)$ is the M-minimal cover) if and only if the set $H - X$ is a Sp-antiset for $\mathcal{S}$.

Corollary 3.4 [7]

Any Sp-antiset for $\mathcal{S}$ has the following form:

$$S^{-1} = H - \{a_{i_1}, \dots, a_{i_k}\} .$$

Where $Q = \{a_{i_1}, \dots, a_{i_k}\} \subseteq H$ is the representative set for $\mathcal{S}$.

Corollary 3.5 [7]

$$|\mathcal{S}^{-1}| = |\mathcal{Q}^{(\mathcal{S})}|$$

Where $|\mathcal{S}^{-1}|$ is a cardinality of $\mathcal{S}^{-1}$
$|\mathcal{Q}^{(\mathcal{S})}|$ is a cardinality of $\mathcal{Q}^{(\mathcal{S})}$.

Theorem 3.4

Let $H = \{a_1, \dots, a_h\}$ be a finite set and $\mathcal{S} = \{S_1, \dots, S_s\}$ be a Sperner System on H. Let $\mathcal{Q}^{(\mathcal{S})} = \{Q_1, \dots, Q_q\}$ be a set of all representative sets for $\mathcal{S}$. Then the set $\mathcal{Q}^{(\mathcal{S})}$ is a Sperner System on H, i.e. the set $\mathcal{Q}^{(\mathcal{S})}$ satisfies the following relations:

a) $\forall Q_i \subseteq H, Q_i \nsubseteq Q_j$ for all $i \neq j$, $i,j = 1, \dots, q$

b) $\bigcup_{i=1}^{q} Q_i = H$.

Proof: It is obvious that the condition (a) holds. We have to prove that $H \subseteq \bigcup_{i=1}^{q} Q_i$.

For any $a_j \in H$, by Theorem 3.1, there exists a set $Q_i \subseteq H$ such that $a_j \in Q_i$ and $Q_i \in \mathcal{Q}^{(\mathcal{S})}$, i.e. $a_j \in \bigcup_{t=1}^{q} Q_t$. Showing that $H \subseteq \bigcup_{i=1}^{q} Q_i$. It is obvious that $H \supseteq \bigcup_{i=1}^{q} Q_i$. The condition (b) holds. The proof is complete.

Theorem 3.5

Let $H = \{a_1, \dots, a_h\}$ be a finite set and $\mathcal{S} = S_1, \dots, S_s$ be a Sperner System on H.

Let $\mathcal{Q}^{(\mathcal{S})} = \{Q_1,\dots,Q_q\}$ be the set of all representative sets for $\mathcal{S}$. Then $\bigcap_{i=1}^{s} S_i \neq \emptyset$ if and only if there exists a subset $\mathcal{Q}_1^{(\mathcal{S})} \subseteq \mathcal{Q}^{(\mathcal{S})}$ such that $\forall Q_j \in \mathcal{Q}_1^{(\mathcal{S})}: |Q_j| = 1$ and $\bigcap_{i=1}^{s} S_i = \bigcup_{Q_j \in \mathcal{Q}_1^{(\mathcal{S})}} Q_j$.

<u>Proof</u>: Suppose that $\bigcap_{i=1}^{s} S_i \neq \emptyset$. We need prove that there exists a subset $\mathcal{Q}_1^{(\mathcal{S})} \subseteq \mathcal{Q}^{(\mathcal{S})}$ such that

$$\forall Q_j \in \mathcal{Q}_1^{(\mathcal{S})} : |Q_j| = 1 \text{ and } \bigcap_{i=1}^{s} S_i = \bigcup_{Q_j \in \mathcal{Q}_1^{(\mathcal{S})}} Q_j .$$

Let us define $\mathcal{Q}_1^{(\mathcal{S})} = \left\{ \{a\} \,\middle|\, a \in \bigcap_{i=1}^{s} S_i \right\}$. It is obvious that $|\{a\}| = 1$ and

$$\bigcap_{i=1}^{s} S_i = \bigcup_{\{a\} \in \mathcal{Q}_1^{(\mathcal{S})}} \{a\}$$

Since $a \in S_i$ for all $i=1,\dots,s$, it follows that $\{a\}$ is the representative set of $\mathcal{S}$. This means $\{a\} \in \mathcal{Q}^{(\mathcal{S})} \implies \mathcal{Q}_1^{(\mathcal{S})} \subseteq \mathcal{Q}^{(\mathcal{S})}$.

Conversely, let $\mathcal{Q}_1^{(\mathcal{S})} \subseteq \mathcal{Q}^{(\mathcal{S})}$ be a set that $\forall Q_j \in \mathcal{Q}_1^{(\mathcal{S})} : |Q_j| = 1$ and $\bigcup_{Q_j \in \mathcal{Q}_1^{(\mathcal{S})}} Q_j = \bigcap_{i=1}^{s} S_i$. We must prove that $\bigcap_{i=1}^{s} S_i \neq \emptyset$. By the condition (a) of the Corollary 3.3 : $\forall Q_j \in \mathcal{Q}^{(\mathcal{S})}$ and $\forall S_i \in \mathcal{S} \implies Q_j \cap S_i \neq \emptyset$. Since $|Q_j| = 1$, it shows that $Q_j \subseteq S_i$ for every $S_i \in \mathcal{S}$ i.e.

$$Q_j \subseteq \bigcap_{i=1}^{s} S_i \neq \emptyset .$$ The proof is complete.

<u>Corollary 3.6</u>

Let $H = \{a_1,\dots,a_h\}$ be a finite set and $\mathcal{S} = \{S_1,\dots,S_s\}$ be a Sperner System on H. Then $\{a_1\}, \{a_2\}, \dots, \{a_h\}$ are the representative sets of $\mathcal{S}$ if and only if $|\mathcal{S}| = 1$.

§4. The necessary and sufficient condition:

In this section, we will give a necessary and sufficient condition for which two given Sperner Systems on H are the set of all representative sets of each other.

Theorem 4.1:

Let $H = \{a_1, \dots, a_h\}$ be a finite set and $\mathcal{S} = \{S_1, \dots, S_s\}$ be a Sperner System on H. Let $\mathcal{Q}^{(\mathcal{S})} = \{Q_1, \dots, Q_q\}$ be the set of all representative sets for $\mathcal{S}$. Then

$$\mathcal{S} \equiv \mathcal{Q}^{(\mathcal{Q}^{(\mathcal{S})})}$$

In other words the set of all representative sets for the set of all representative sets for $\mathcal{S}$ is just equal to $\mathcal{S}$.

Proof: We prove that

1) $\forall S_i \in \mathcal{S} \implies S_i \in \mathcal{Q}^{(\mathcal{Q}^{(\mathcal{S})})}$ i.e. the set S_i is the representative set for $\mathcal{Q}^{(\mathcal{S})}$. We need show that the set S_i satisfies the following conditions:

   a) $\forall Q_j \in \mathcal{Q}^{(\mathcal{S})} \implies S_i \cap Q_j \neq \emptyset$

   b) $\forall S' \subset S_i \implies \exists Q_j \in \mathcal{Q}^{(\mathcal{S})}$ such that $S' \cap Q_j = \emptyset$.

2) $\forall Q_i^+ \in \mathcal{Q}^{(\mathcal{Q}^{(\mathcal{S})})} \implies \exists S_j \in \mathcal{S}$ such that $Q_i^+ \equiv S_j$.

   This means that the matrix $\mathcal{M}_{\mathcal{S}}(H)$ has exactly one column corresponding to the set Q_i^+ .

Now let us show the statement 1 .

a) Let any $S_i \in \mathcal{S}$. Since $\mathcal{Q}^{(\mathcal{S})}$ is the set of all representative sets for $\mathcal{S}$, by Corollary 3.3:

$$\forall S_i \in \mathcal{S}, \ \forall Q_j \in \mathcal{Q}^{(\mathcal{S})} \implies S_i \cap Q_j \neq \emptyset .$$

b) Let any $S_i \in \mathcal{S}$ and $\forall S' \subset S_i$. We have

$(H - S') \cap S_k \neq \emptyset$ for every $S_k \in \mathcal{S}$. Assume the contrary that there exists $S_k \in \mathcal{S}$ such that $(H - S') \cap S_k = \emptyset$. It is obvious that $S_k \subseteq S' \subset S_i$ i.e. $S_k \subset S_i$. This contradicts to the definition of the Sperner System $\mathcal{S}$ on H. Since $(H - S') \cap S_k \neq \emptyset$ for every $S_k \in \mathcal{S}$, it follows that $\mathcal{M}_{\mathcal{S}}(H - S')$ is a M-cover. By Corollary 3.1, there exists $Q_j \in \mathcal{Q}^{(\mathcal{S})}$ and $Q_j \subseteq (H - S')$. Consequently, $S' \cap Q_j = \emptyset$.

Thus, we have $\mathcal{S} \subseteq \mathcal{Q}^{(\mathcal{Q}^{(\mathcal{S})})}$.

Now let us prove the statement (2) .

Let any $Q_i^+ \in \mathcal{Q}^{(\mathcal{Q}^{(\mathcal{S})})}$. First we show that the matrix $\mathcal{M}_{\mathcal{S}}(H - Q_i^+)$ is not a M-cover. Assume the contrary that the matrix $\mathcal{M}_{\mathcal{S}}(H - Q_i^+)$ is a M-cover. By Corollary 3.1, there exists $Q_k \subseteq (H - Q_i^+)$ such $Q_k \in \mathcal{Q}^{(\mathcal{S})}$, showing that $Q_i^+ \cap Q_k = \emptyset$. We arrive to contradiction to the fact that Q_i^+ is a representative set of $\mathcal{Q}^{(\mathcal{S})}$.

Since the matrix $\mathcal{M}_{\mathcal{S}}(H - Q_i^+)$ is not a M-cover, the characteristic vector $C[\mathcal{M}_{\mathcal{S}}[(H - Q_i^+)]$ has at least one component equal to null. Suppose its j-th component equal to null.

a) We shall prove that for all $a \in Q_i^+$, j-th component of the vector $C[\mathcal{M}_{\mathcal{S}}(\{a\})]$ must equal to 1. The proof is by contradiction. Suppose there exists $a \in Q_i^+$ such that the j-th component of the characteristic vector $C[\mathcal{M}_{\mathcal{S}}(\{a\})]$ equal to null. Consequently, the j-th component of the vector $C[\mathcal{M}_{\mathcal{S}}[(H - Q_i^+) \cup \{a\}]]$ is equal to null. On the other hand, since $Q_i^+ \in \mathcal{Q}^{(\mathcal{Q}^{(\mathcal{S})})}$, it follows that the set $(H - Q_i^+)$ is a Sp-antiset of $\mathcal{Q}^{(\mathcal{S})}$. Since $(H - Q_i^+) \subset (H - Q_i^+) \cup \{a\}$, there exists $Q_k \in \mathcal{Q}^{(\mathcal{S})}$ such that $Q_k \subseteq (H - Q_i^+) \cup \{a\}$. Since Q_k is a representative set of $\mathcal{S}$ then, by Corollary 3.2, the matrix $\mathcal{M}_{\mathcal{S}}((H - Q_i^+) \cup \{a\})$ is a M-cover.

It follows that all components of the vector $C[\mathfrak{M}_{\mathcal{S}}((H - Q_i^+) \cup \{a\})]$ are equal to 1 . We arrive to a contradiction with the assumption that the j-th component of the vector $C[\mathfrak{M}_{\mathcal{S}}((H - Q_i^+) \cup \{a\})]$ equal to null. Thus, we have proved that

(*) All elements in the j-th column of the matrix $\mathfrak{M}_{\mathcal{S}}(Q_i^+)$ are equal to 1 . All elements in the j-th column of the matrix $\mathfrak{M}_{\mathcal{S}}(H - Q_i^+)$ are equal to null.

b) Let us show that the matrix $\mathfrak{M}_{\mathcal{S}}(H)$ has exactly one column which satisfies (*) . In fact, assume the contrary that there exists j-th and t-th columns which satisfy condition (*) . It is clear that $S_j = S_t$, a contradiction (by the definition of the Sperner System $\mathcal{S}$ on H) .

Combined (a) with (b) we concluded that there exists exactly one column in the matrix $\mathfrak{M}_{\mathcal{S}}(H)$ that satisfies (*) and this column just the one which corresponds to set Q_i^+ , i.e. $\mathcal{Q}^{(\mathcal{Q}(\mathcal{S}))} \subseteq \mathcal{S}$.

The proof is complete.

Corollary 4.1:

Let $H = \{a_1,\dots,a_h\}$ be a finite set and $\mathcal{S} = \{S_1,\dots,S_s\}$ be a Sperner System on H. Let $\mathcal{Q}^{(\mathcal{S})} = \{Q_1,\dots,Q_q\}$ be the set of all representative sets for $\mathcal{S}$. Then the following statements are equivalent :

1) a) $\forall S_i \in \mathcal{S},\ \forall Q_j \in \mathcal{Q}^{(\mathcal{S})} \Longrightarrow Q_j \cap S_i \neq \emptyset$

   b) $\forall Q_j \in \mathcal{Q}^{(\mathcal{S})},\ \forall Q' \subset Q_j \Longrightarrow \exists S_i \in \mathcal{S}$ such that $Q' \cap S_i = \emptyset$.

2) a) $\forall S_i \in \mathcal{S},\ \forall Q_j \in \mathcal{Q}^{(\mathcal{S})} \Longrightarrow Q_j \cap S_i \neq \emptyset$

   b) $\forall S_i \in \mathcal{S},\ \forall S' \subset S_i \Longrightarrow \exists Q_j \in \mathcal{Q}^{(\mathcal{S})}$ such that $S' \cap Q_j = \emptyset$.

Corollary 4.2:

Let $H = \{a_1, \ldots, a_h\}$ be a finite set and $\mathcal{S} = \{S_1, \ldots, S_s\}$, $\mathcal{Q}^{(\mathcal{S})} = \{Q_1, \ldots, Q_q\}$ be a Sperner system on H. Then the necessary and sufficient condition for $\mathcal{S}$ and $\mathcal{Q}^{(\mathcal{S})}$ systems are the set of all representative sets of each other is that the systems $\mathcal{S}$ and $\mathcal{Q}^{(\mathcal{S})}$ satisfy either statement (1) or statement (2).

Example 4.1:

Let $H = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$
and $\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5\}$

Where $S_1 = \{a_3, a_6, a_7\}$, $S_2 = \{a_1, a_4, a_5, a_7\}$,

$S_3 = \{a_1, a_3, a_5, a_7\}$, $S_4 = \{a_2, a_5, a_7\}$,

$S_5 = \{a_2, a_6, a_7\}$.

Then

$$\mathfrak{M}_{\mathcal{S}}(H) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathcal{Q}^{(\mathcal{S})} = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\},$$

where $Q_1 = \{a_7\}$, $Q_2 = \{a_2, a_3, a_5\}$,

$Q_3 = \{a_5, a_6\}$, $Q_4 = \{a_1, a_2, a_3\}$,

$Q_5 = \{a_1, a_2, a_6\}$, $Q_6 = \{a_2, a_3, a_4\}$

is the set of all representative sets for $\mathcal{S}$.

And

$$\mathcal{M}_{\mathcal{Q}^{(\mathcal{S})}}(H) = \begin{pmatrix} 0&0&0&1&1&0 \\ 0&1&0&1&1&1 \\ 0&1&0&1&0&1 \\ 0&0&0&0&0&1 \\ 0&1&1&0&0&0 \\ 0&0&1&0&1&0 \\ 1&0&0&0&0&0 \end{pmatrix}$$

$$\mathcal{Q}^{(\mathcal{Q}^{(\mathcal{S})})} = \{ Q_1^*, Q_2^*, Q_3^*, Q_4^*, Q_5^* \} ,$$

where

$$Q_1^* = \{a_3, a_6, a_7\} \equiv S_1 , \ Q_2^* = \{a_1, a_4, a_5, a_7\} \equiv S_2 ,$$

$$Q_3^* = \{a_1, a_3, a_5, a_7\} \equiv S_3 , \ Q_4^* \equiv \{a_2, a_5, a_7\} \equiv S_4 ,$$

$$Q_5^* = \{a_2, a_6, a_7\} \equiv S_5 .$$

It is obvious that $\mathcal{Q}^{(\mathcal{Q}^{(\mathcal{S})})}$ is the set of all representative set for $\mathcal{Q}^{(\mathcal{S})}$.

§5. Algorithms:

In this section, we present the algorithm to find the set of all representative sets of any Sperner system $\mathcal{S}$ on H, and the algorithm to recognize whether a given set $X \subseteq H$ is or is not a representative set of $\mathcal{S}$.

Remark: The algorithm to determine whether $\mathcal{M}_{\mathcal{S}}(X)$ is a M-cover is a simple matrix algorithm, so here we omit its.

5.1 Algorithm 1 :

This is an algorithm for the recognition whether a given set $X \subseteq H$ is a representative set for $\mathcal{S}$.

The block schema of the Algorithm 1 is presented in Fig. 5.1

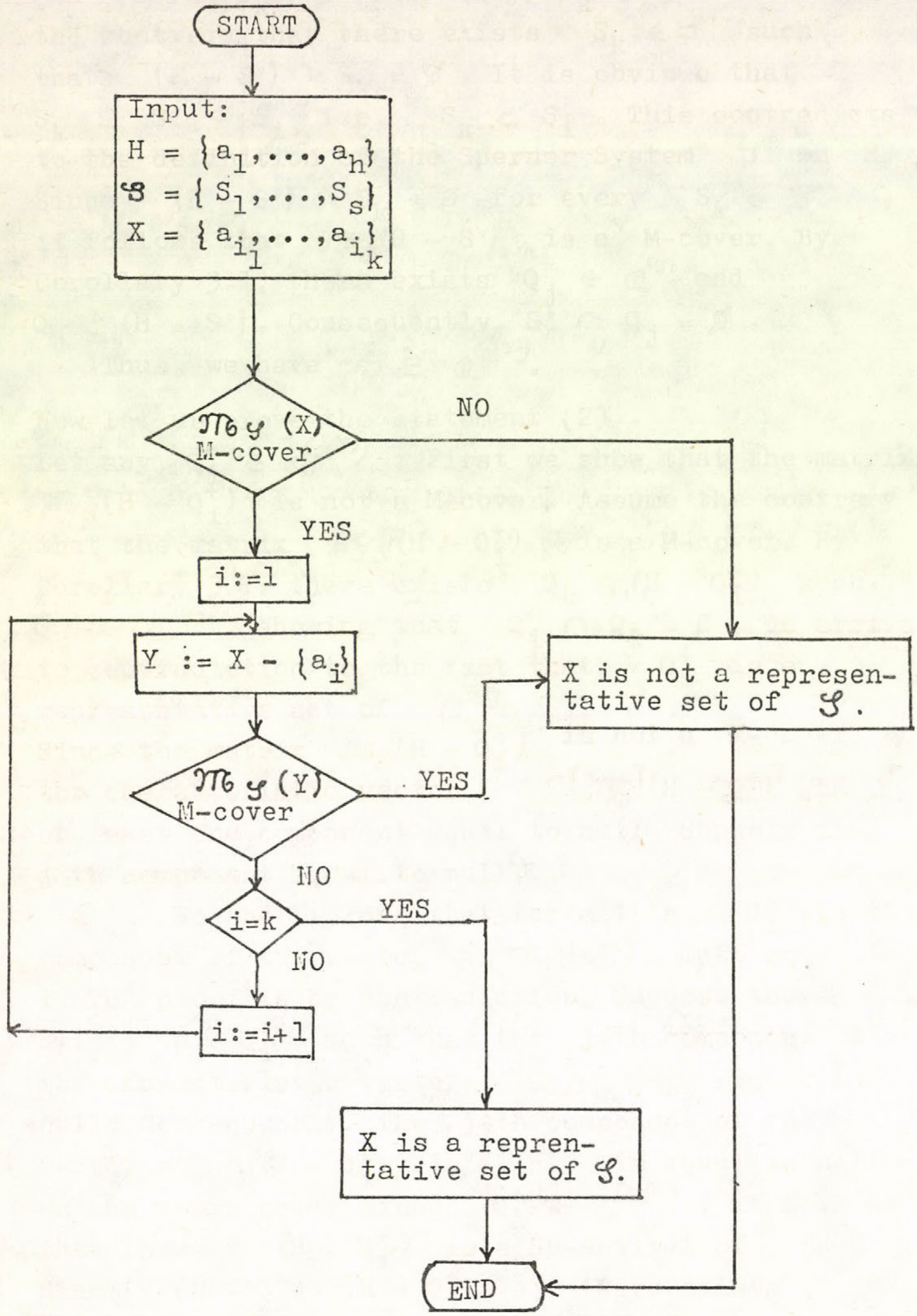

START
Input:
$H = \{a_1, \ldots, a_h\}$
$\mathcal{S} = \{S_1, \ldots, S_s\}$
$X = \{a_{i_1}, \ldots, a_{i_k}\}$
$\mathcal{M}_{\mathcal{S}}(X)$ M-cover
NO
YES
$i := 1$
$Y := X - \{a_i\}$
$\mathcal{M}_{\mathcal{S}}(Y)$ M-cover
YES
NO
$i = k$
YES
NO
$i := i + 1$
X is not a representative set of $\mathcal{S}$.
X is a reprentative set of $\mathcal{S}$.
END

Fig. 5.1

5.2: Algorithm 2:

This is an algorithm to determine the set of all representative sets for $\mathcal{S}$ System.

Input: $H = \{a_1, \ldots, a_h\}$ is the finite set

$\mathcal{S} = \{S_1, \ldots, S_s\}$ is the Sperner System on H.

Output: The set $\mathcal{Q}^{(\mathcal{S})}$ is the set of all representative sets for $\mathcal{S}$.

Method:

I) Let $i := 1$, $H^{(1)} := H = \{a_1, \ldots, a_h\}$,

$$\mathcal{S}^{(1)} = \mathcal{S} = \{S_1, \ldots, S_s\},$$

$$\mathcal{M}^{(1)} := \mathcal{M}_{\mathcal{S}}(H) = \begin{pmatrix} r_1 \\ \vdots \\ r_h \end{pmatrix} .$$

Let $Q := \emptyset$ will be a representative set.

II) Suppose that at i-th step, we have

$$H^{(i)} = \{a_{l_1}, \ldots, a_{l_t}\}$$

$$\mathcal{S}^{(i)} = \{S^{(i)}_{j_1}, \ldots, S^{(i)}_{j_k}\}$$

and

$$\mathcal{M}^{(i)} = \begin{pmatrix} r^{(i)}_{l_1} \\ \vdots \\ r^{(i)}_{l_t} \end{pmatrix}$$

$$|r^{(i)}_{l_1}| \geq |r^{(i)}_{l_2}| \geq \cdots \geq |r^{(i)}_{l_t}| > 0 .$$

The set Q is a representative set of $\mathcal{S} \setminus \mathcal{S}^{(i)}$.

Case 1: If $\mathcal{M}^{(i)}$ is a M-cover.

a) If $|r^{(i)}_{l_1}| = |\mathcal{S}^{(i)}| = k$ then $\{a_{l_1}\}, \ldots, \{a_{l_n}\}$ are the representative sets of $\mathcal{S}^{(i)}$ and

$Q := Q \cup \{a_{l_1}\}$, $Q := Q \cup \{a_{l_2}\}$, ..., $Q := Q \cup \{a_{l_n}\}$

are the representative sets for $\mathcal{S}$, where

$|r^{(i)}_{1_1}| = \ldots = |r^{(i)}_{1_n}| = k$. Hence we only consider

$$H^{(i)} := H^{(i)} - \bigcup_{j=\ell_1}^{\ell_n} \{a_j\} ,$$

$$\mathcal{S}^{(i)} := \mathcal{S}^{(i)} ,$$

$$\mathcal{M}^{(i)} := \mathcal{M}_{\mathcal{S}^{(i)}}(H^{(i)}) ,$$

$$|r^{(i)}_{1_{n+1}}| \geq \ldots \geq |r^{(i)}_{1_t}| > 0 .$$

b) If there exists $\mathcal{S}'' \subseteq \mathcal{S}^{(i)}$ such that $|S^{(i)}_j| = 1$ for all $S^{(i)}_j \in \mathcal{S}''$ then $\bigcup_{S^{(i)}_j \in \mathcal{S}''} S^{(i)}_j = \bigcap_{Q_q \in \mathcal{Q}^{(i)}} Q_q$.

I.e. $\bigcup_{S_j \in \mathcal{S}''} S^{(i)}_j$ is a common part of all the representative sets for $\mathcal{S}^{(i)}$.

Hence, we only consider :

$$H^{(i+1)} := H^{(i)} - \bigcup_{a \in S^{(i)}_j ,\, |S^{(i)}_j| = 1 ,\, S^{(i)}_j \in \mathcal{S}''} \{a\}$$

$$\mathcal{S}^{(i+1)} := \mathcal{S}^{(i)} - \mathcal{S}'' ,$$

and $$\mathcal{M}^{(i+1)} := \mathcal{M}_{\mathcal{S}^{(i+1)}}(H^{(i+1)}) = \begin{pmatrix} r^{(i+1)}_{j_1} \\ \vdots \\ r^{(i+1)}_{j_g} \end{pmatrix} ,$$

$$|r^{(i+1)}_{j_1}| \geq \ldots \geq |r^{(i+1)}_{j_g}| \geq 0 .$$

$Q := Q \cup A$ is the representative set of $\mathcal{S} - \mathcal{S}^{(i+1)}$, where $A := \bigcup_{a \in S^{(i)}_j ,\, S^{(i)}_j \in \mathcal{S}''} \{a\}$

c) If either $|r^{(i)}_{1_1}| < |\mathcal{S}^{(i)}|$ or $\nexists\ \mathcal{S}'' \subseteq \mathcal{S}^{(i)}$ such that $|S^{(i)}_j| = 1$, $S^{(i)}_j \in \mathcal{S}''$ then we construct the matrix $\mathcal{M}^{(i+1)}$ as follows:

$$H^{(i+1)} := H^{(i)} - \{ a_j \mid a_j \in H^{(i)},\ r_j^{(i)} \leq r_{l_1}^{(i)},$$

where a_j determine the row $r_j^{(i)}$,

for all $j = l_2, \ldots, l_t \}$.

$$\mathcal{S}^{(i+1)} := \mathcal{S}^{(i)} - \mathcal{S}'' \text{, where}$$

$$\mathcal{S}'' := \{ S_j^{(i)} \mid S_j^{(i)} \in \mathcal{S}^{(i)},\ a_{l_1} \in S_j^{(i)} \}.$$

$$\mathcal{M}^{(i+1)} := \begin{pmatrix} r_{p_1}^{(i+1)} \\ \vdots \\ r_{p_m}^{(i+1)} \end{pmatrix}$$

$$\left| r_{p_1}^{(i+1)} \right| \geqslant \ldots \geqslant \left| r_{p_m}^{(i+1)} \right| \geqslant 0 .$$

And $Q := Q \cup \{a_{l_1}\}$ is the representative set of $\mathcal{S} \setminus \mathcal{S}^{(i+1)}$.

d) Let us go to the $i:=i+1$ -th step.

Case 2: If $\mathcal{M}^{(i)}$ is not a M-cover, i.e. either

$$C[\mathcal{M}^{(i)}] = (0,\ldots,0) \quad \text{or} \quad C[\mathcal{M}^{(i)}] \neq (1,\ldots,1).$$

a) If $i=1$, the algorithm stop.

b) If $i > 1$, let $i := i - 1$, and

$Q := Q - \{a_{l_1}\}$. We consider a matrix

$$\mathcal{M}^{(i)} := \mathcal{M}^{(i)} - \{ r_{l_1}^{(i)} \} := \mathcal{M}_{\mathcal{S}^{(i)}}(H^{(i)} - \{a_{l_1}\})$$

$$\left| r_{l_2}^{(i)} \right| \geqslant \left| r_{l_3}^{(i)} \right| \geqslant \ldots \geqslant \left| r_{l_t}^{(i)} \right| > 0 .$$

Go to the case 1.

Theorem 5.1:

Let $H = \{a_1, \dots, a_h\}$ be finite set and

$\mathcal{S} = \{S_1, \dots, S_s\}$ be a Sperner System on H.

Then the algorithm 2 make a clearn sweep all the representative sets for $\mathcal{S}$.

Proof: Let $\mathcal{Q}^*$ be a set of all sets determined by the algorithm 2. We must prove that

1) $\forall Q \in \mathcal{Q}^* \implies Q \in \mathcal{Q}^{(\mathcal{S})}$

2) $\forall Q \in \mathcal{Q}^{(\mathcal{S})} \implies Q \in \mathcal{Q}^*$.

Now, we prove (1):

Let any $Q \in \mathcal{Q}^*$, $Q := \{a_{i_1}, \dots, a_{i_k}\}$.

a) It is obvious that $\forall S_j \in \mathcal{S} : S_j \cap Q \neq \emptyset$

b) $\forall a_{i_j} \in Q, \exists S_n \in \mathcal{S}'' := \{S_t \mid a_{i_j} \in S_t\}$

such that $(Q - \{a_{i_j}\}) \cap S_n = \emptyset$.

2) We need prove that $Q \in \mathcal{Q}^{(\mathcal{S})} \implies Q \in \mathcal{Q}^*$

a) If the set Q satisfies either $|\mathcal{M}_{\mathcal{S}}(Q)| = s$ or $|r_{i_1}| = \dots = |r_{i_k}| = 1$, it is obvious that $Q \in \mathcal{Q}^*$.

b) Since $Q \in \mathcal{Q}^{(\mathcal{S})}$, there exists $X \subseteq H$ such that $Q \subseteq X$ and a maximal row[1] of a matrix

$\mathcal{M}_{\mathcal{S}}(Q)$ is just a maximal row of a matrix

$\mathcal{M}_{\mathcal{S}}(X)$. Suppose, it is r_{t_1} .

Let $X^{(1)} = X$, $Q^{(1)} = Q$, $\mathcal{S}^{(1)} = \mathcal{S}$,

$\mathcal{S}'' = \{S_j \mid S_j \in \mathcal{S}^{(1)}, a_{t_1} \in S_j , a_{t_1}$ detemmine the row $r_{t_1}\}$.

Let $\mathcal{S}^{(2)} = \mathcal{S}^{(1)} \setminus \mathcal{S}''$.

Since Q is the representative set of $\mathcal{S}$, there exists $X^{(2)} \subset X^{(1)}$ such that

$Q^{(2)} := (Q^{(1)} - \{a_{t_1}\}) \subseteq X^{(2)}$ and the maximal row of a matrix $\mathcal{M}_{\mathcal{G}^{(2)}}(Q^{(2)})$ is just the maximal row of a matrix $\mathcal{M}_{\mathcal{G}^{(2)}}(X^{(2)})$.

Since the set Q has k elements, it follows that there exists $p > 0$ such that:

$$Q^{(1)} \supset Q^{(2)} \supset \ldots \supset Q^{(p-1)} \supset Q^{(p)} = \emptyset$$

$$X^{(1)} \supset X^{(2)} \supset \ldots \supset X^{(p-1)} \supset X^{(p)} \supseteq \emptyset$$

and $Q^{(m)} \subseteq X^{(m)}$ $m=1,\ldots p$.

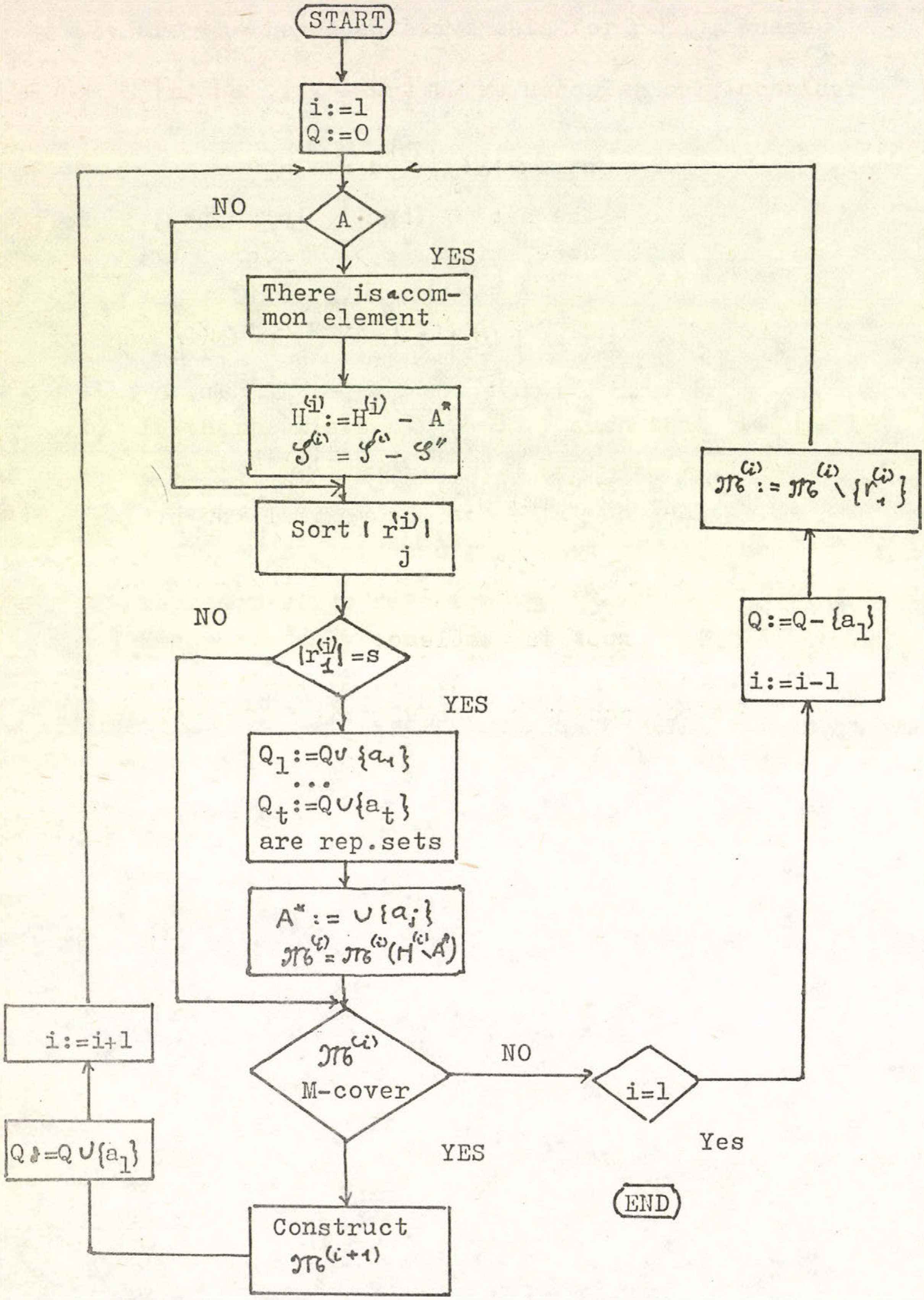
The row $r_{t_j}^{(m)}$ is the maximal row of $\mathcal{M}_{\mathcal{G}^{(m)}}(Q^{(m)})$ and $\mathcal{M}_{\mathcal{G}^{(m)}}(X^{(m)})$. It follows that $Q \in \mathcal{Q}^*$

The proof is complete.

The block schema of the algorithm 2 is presented in Fig. 5.2 .

---

1) The row r_i is called the maximal row of the matrix $\mathcal{M}_{\mathcal{G}}(X)$ if $|r_i| = \max\{|r_1|, \ldots, |r_t|\}$.



$A^{\circ} := \{ \exists\, \mathcal{S}'' \subset \mathcal{S}^{(i)} : \forall S_j \in \mathcal{S}''\ |S_j| = 1 \}$

Fig. 5 2

We close our paper with an example.

5.3 <u>Example</u>:

Let $H = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ and

$\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5, S_6\}$

where

$S_1 = \{a_7\}$, $S_2 = \{a_2, a_3, a_5\}$, $S_3 = \{a_5, a_6\}$,

$S_4 = \{a_1, a_2, a_6\}$, $S_5 = \{a_1, a_2, a_3\}$, $S_6 = \{a_2, a_3, a_4\}$.

$$\mathfrak{M}_{\mathcal{S}}(H) = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \end{array} \begin{array}{c} \begin{array}{cccccc} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

a_7 is a common element of all the representative sets for $\mathcal{S}$. Hence , we consider only $H^{(1)} = H \setminus \{a_7\}$ and $\mathcal{S}^{(1)} = \mathcal{S} - \{S_1\}$.

$|r_2^{(1)}| \geq |r_3^{(1)}| \geq |r_1^{(1)}| \geq |r_5^{(1)}| \geq |r_6^{(1)}| \geq |r_4^{(1)}| > 0$.

$r_2^{(1)}$:

$$\mathfrak{M}_{\mathcal{S}^{(2)}}(H^{(2)}) = \begin{array}{c} \\ r_5^{(2)} \\ r_6^{(2)} \end{array} \begin{array}{c} S_3 \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array}$$

$Q_1 := \{a_7, a_2, a_5\}$, $Q_2 := \{a_7, a_2, a_6\}$.

We consider

$$\mathfrak{M}_{\mathcal{S}^{(1)}}(H^{(1)} \setminus \{a_2\}) = \begin{array}{c} \\ r_1^{(3)} \\ r_3^{(3)} \\ r_4^{(3)} \\ r_5^{(3)} \\ r_6^{(3)} \end{array} \begin{array}{c} \begin{array}{ccccc} S_2 & S_3 & S_4 & S_5 & S_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$r_3^{(1)}$:

$$\mathfrak{M}_{\mathcal{S}^{(2)}}(H^{(2)}) = \begin{array}{cc} & \begin{array}{cc} S_3 & S_4 \end{array} \\ \begin{array}{c} r_1^{(2)} \\ r_5^{(2)} \\ r_6^{(2)} \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \end{array}$$

$Q_3 := \{a_7, a_3, a_1, a_5\}$, $Q_4 := \{a_7, a_3, a_6\}$.

$$\mathfrak{M}_{\mathcal{S}^{(1)}}(H^{(1)} - \{a_2, a_3\}) := \begin{array}{cc} & \begin{array}{ccccc} S_2 & S_3 & S_4 & S_5 & S_6 \end{array} \\ \begin{array}{c} r_1^{(1)} \\ r_4^{(1)} \\ r_5^{(1)} \\ r_6^{(1)} \end{array} & \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$r_1^{(1)}$:

$$\mathfrak{M}_{\mathcal{S}^{(2)}}(H^{(2)}) = \begin{array}{cc} & \begin{array}{ccc} S_2 & S_3 & S_6 \end{array} \\ \begin{array}{c} r_4^{(2)} \\ r_5^{(2)} \\ r_6^{(2)} \end{array} & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

$Q_5 := \{a_7, a_1, a_4, a_5\}$.

$\mathfrak{M}_{\mathcal{S}^{(1)}}(H^{(1)} - \{a_1, a_2, a_3\})$ is not a M-cover .

Thus, we have the set of all representative sets for $\mathcal{S}$:

$Q_1 = \{a_7, a_2, a_5\}$, $Q_2 = \{a_7, a_2, a_6\}$, $Q_3 = \{a_1, a_7, a_3, a_5\}$,

$Q_4 = \{a_7, a_3, a_6\}$, $Q_5 = \{a_7, a_1, a_4, a_5\}$, .

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M-minimal covers and Spredner systems with application to the key finding problem for relation scheme

Phan The Que

Summary

Based on results in [7], in this paper the properties of M-minimal covers when a finite set H and Sperner system $\mathcal{S}$ on H are given are investigated. Specially, the necessary and sufficient condition for which two Sperner systems are sets of all representative sets of each other are established.

This means that from the given set of all keys for a relation scheme , its set of all representative sets can be constructed and conversely, from the set all representative sets for the set of keys, the set of keys for the relation scheme can be determined.

The set of keys the relation scheme is just the set of all representative sets for the set of keys.

M-minimális lefedések, Sperner-rendszerek és alkalmazásuk a relációs sémák kulcs-keresési problémájára

Phan The Que

Összefoglaló

A [7] eredményeire alapozva, a szerző az M-minimális lefedések tulajdonságait vizsgálja adott véges H halmaz és rajta egy Sperner-rendszer esetén. Annak szükséges és elégséges feltételét is megadja, hogy két Sperner--rendszer egymásnak teljes reprezentáló rendszerét alkotják. Ennek segitségével, ha adva van egy reláció séma kulcsainak halmaza, meg lehet konstruálni a séma teljes reprezentáló halmazát és forditva, ha adva van a teljes reprezentáló halmaz, akkor a kulcsok halmazát lehet meghatározni.

A reláció séma kulcsainak halmaza tehát semmi más, mint a kulcsok halmazát reprezentáló teljes halmaz.