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ON THE CARDINALITY OF SELF-DUAL CLOSED CLASSES IN k-VALUED LOGICS

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Introduction

Let $E_k = \{0,1,\dots,k-1\}$. By a k-valued function we shall mean a function $f: E_k^n \to E_k$, and by P_k we denote the set of all those functions. If A is a subset of P_k, $[A]$ will denote the set of all superpositions over A. The definition of a superposition over A is the following:

1. $f \in A$ is a superposition over A

2. If $g_o(x_1 \dots x_n)$, $g_1(x_{11},\dots,x_{1m_1}),\dots,$ $g_n(x_{n1},\dots,x_{nm_n})$ are either superpositions over A, or $g_i(x_{i1},\dots,x_{im_i}) = x_{ij}$ then $g_o(g_1(x_{11},\dots,x_{1m_1}) \dots g_n(x_{n1},\dots,x_{nm_n}))$ is a superposition over A.

The set A is closed if $A = [A]$. Let s be a permutation of $0,1,\dots,k-1$. We say, that $f \in P_k$ preserves s, if

$$f(x_1,\dots,x_n) = s^{-1}[f(s(x_1) \dots s(x_n))] .$$

We shall denote by $\mathfrak{c}$ the cardinality of the continuum.

Ju. I. Janov and A.A. Mučnik [5] have proved, that if $k \geq 3$, then the cardinality of the set of all closed sets in P_k is continuum. E.Post's general result implies that there are countably many closed sets in P_k for k = 2.

It is well known, [see [4], [8]], that there exist 6 types of maximal closed sets in P_k. The characterisation of these sets can be found in [8]. J.Demetrovics and J.Bagyinszki have proved in [2] that the linear classes in P_k (k prime) contain a finite number of closed classes. J.Bagyinszki and A.Szendrei [1], [9] have proved that if k is square-free, then there are also finitely many closed linear classes in P_k. D.Lau in [6] have shown, that the cardinality of the so-called quasi-linear closed classes is countable. In [3] the authors have proved, that the so-called central, k-regular, monotonous and equivalence-preserving maximal classes in P_k, for $k \geq 3$ contain as many as $\mathfrak{c}$ closed classes. In this paper it is also shown that the maximal classes, which preserve a permutation s, conatain $\mathfrak{c}$ closed classes provided k is not prime. Marcenkov in [7] has proved that for all $k \in \{13,14,16,17,\}$ and for all permutation $s : E_k \to E_k$ there exist a set of closed classes preserving s with cardinality $\mathfrak{c}$. In the case k=2. E.Post's result ([10]) implies that there are finitely many closed classes pereserving a permutation of E_2.

The purpose of this paper is to show that for all $k \geq 3$ and for all permutation $s : E_k \to E_k$ /except for two cases, namely k=3 and s = (012) or k=4, s = (0123)/ there exist $\mathfrak{c}$ closed sets in P_k preserving s. We shall also prove that for all $k \geq 3$ there is at least a countable number of closed sets preserving s, for all permutation $s: E_k \to E_k$.

§.1.

A permutation s of E_k can be written as a product of disjoint cycles. Such a cycle will be denoted by C_i. If

$$s = C_1 . C_2 \dots C_m \quad \text{and} \quad \begin{array}{l} C_1 = (a_{11}, \dots, a_{n_1 1}) \\ \vdots \\ C_m = (a_{1m}, \dots, a_{n_m m}), \text{ then} \end{array}$$

$|C_i|$ will denote the number of the elements of the set

$$\{a_{1i}, \dots, a_{n_i i}\}$$

<u>Lemma 1.</u> Let $k \geq 3$, s a permutation in the form $s = C_1 . C_2 . \dots . C_m$. If $m > 1$ and there are $i,j \leq m$ such that $i \neq j$, $|C_i| = k_1$, $|C_j| = k_2$ and k_1 / k_2 then it can be constructed $\mathfrak{c}$ closed classes preserving s.

<u>Proof.</u> We can assume that $s = C_1 . C_2 . \dots . C_m$, where

$C_1 = (0,\ldots,a_{m_1})$ $\quad C_2 = (1,2,\ldots,a_{m_2})$ and $|C_1| / |C_2|$

We shall prove, that there is a set $\{f_i\} = F$ of functions such that for all $f_i \in F$, $f_i \notin [F \setminus f_i]$ and all f_i preserve s. This is sufficient since in this case all subsets of F generate a closed class, and $H_1 \subset F$, $H_2 \subset F$ $H_1 \neq H_2$ implies $[H_1] \neq [H_2]$.

Let $f_m(x_1,x_2,\ldots,x_m)$, $m \geq 3$ be defined as follows:

$$f_m(a_1,\ldots,a_m) = \begin{cases} b \in C_2, \text{ if } (a_1,\ldots,a_m) \subset C_2 \ \ |\{i / a_i = b\}| = 1 \\ \quad \text{and all } a_i \neq b \text{ are equal to } s(b); \\ d \in C_1, \text{ if } \{a_1,\ldots,a_m\} \subset C_1 \cup C_2 \text{ and the previous} \\ \quad \text{condition does not hold;} \\ a_1 \quad , \text{ in all other cases.} \end{cases}$$

One can easily see that since $/C_1/ / /C_2/$, $f_m(x_1,\ldots,x_n)$ preserves s.

Let us suppose, that $f_k(x_1,\ldots x_k) \in [F \setminus f_k]$. This means that

$$f_k(x_1,\ldots,x_k) = \mathfrak{A}(x_1,\ldots,x_k)$$

where $\mathfrak{A}$ is a superposition over $F \setminus f_k$.
Let $f_s(x_{i_1},\ldots,x_{i_s})$ be a function in $\mathfrak{A}$.
If $s < k$, then we can find an x_ℓ such that $x_\ell \notin \{x_{i_1},\ldots,x_{i_s}\}$
If $x_\ell = 1$, and all $x_i = 2$ $(i \neq \ell)$, then – by the definition – $f_k(x_1,\ldots,x_k) = 1$.
If we choose $(x_1,\ldots,x_k)$ as above, then $f_s(x_{i_1},\ldots,x_{i_s}) \in C_1$ that is $\mathfrak{A}$ cannot be equal to 1. (f_m preserves the set $C_1 \cup C_2$ and if $\{a_1,\ldots,a_m\} \cap C_1 \neq \emptyset$ then $f_m(a_1,\ldots,a_m) \in C_1$.) If $s > k$, then we have at least one pair x_{i_k}, x_{i_ℓ} such that $i_k = i_\ell$.

Let $x_{i_k} = x_{i_\ell} = 1$, and all $x_j = 2$ $(j \neq i_k)$. In this case $f_s(x_{i_1},\ldots,x_{i_s}) \in C_1$ and $f_k(x_1,\ldots,x_k) = 1$. This is a contradiction, thus Lemma 1 is proved.

<u>Corollary:</u>

1. if k is not prime, then in the maximal closed class S_k of P_k there exists $\mathfrak{c}$ closed classes. (S_k denotes the class of all functions pre-

serving a permutation π; π is the product of cycles C_i of length p, where p is prime.)

2. if π is a permutation of the form $\pi = (1)\ C_1 \dots\dots C_m$ then there is a continuum cardinality set of closed classes pereserving π.

Lemma 2. Let $k \geq 5$, let s be a permutation consisting of one cycle of length k. Then we can construct a set of closed classes in P_k of cardinality $\mathfrak{C}$ which preserves s.

Proof. We can assume, that

$$s = (01234\ \dots).$$

Analogously to the proof of Lemma 1 we shall give a set $\{g_i\} = G$ of functions so that $g_i \notin [G \setminus g_i]$ and g_i preserves s.

We define g_i, $i>3$ on the set $\{0,1,2\}^i$. It can be easily verified that the definition does not contradict the assumption that g_i preserves s.

Let:

$$g_k(a,\dots,a) = a$$

$$g_k(\{\{0,1\}^k \setminus (1,\dots,1)\}) = 0$$

$$g_k(\{\{0,2\}^k \setminus (2,\dots,2)\}) = 0$$

$$g_k(\{\{1,2\}^k \setminus (2,\dots,2)\}) = 1$$

and for $\{0,1,2\}^k \setminus \{0,1\}^k \setminus \{0,2\}^k \setminus \{1,2\}^k$:

$$g_k(a_1,\dots,a_k) = \begin{cases} 1, & \text{if } /\{a_i/a_i = 0\}/ = 1 \\ & \quad /\{a_i/a_i = 2\}/ = 1 \\ & \quad /\{a_i/a_i = 1\}/ = k-2; \\ 0, & \text{in all other cases.} \end{cases}$$

A vector $(a_1,\dots,a_k) = \underline{a} \in \{0,1,2\}^k$ is called characteristic if

$/\{a_i / a_i = 0\}/ = 1,$
$/\{a_i / a_i = 2\}/ = 1,$ and
$/\{a_i / a_i = 1\}/ = k-2.$

Let us suppose $g_k \in [G \setminus g_k]$, that is $g_k(x_1,\ldots,x_k) = \mathfrak{A}(x_1,\ldots,x_k)$.

If $g_k(x_1,\ldots,x_k) = \mathfrak{A}$ then there exists at least one superposition over $G \setminus g_k$ such that $g_k(x_1,\ldots,x_k)$ is equal to this superposition on the characteristic vectors. Hence we can choose a minimal formula $\mathfrak{A}^*$ which equals $g_k(x_1,\ldots,x_k)$ on the characteristic vectors. The minimality of $\mathfrak{A}^*$ means that if $\mathfrak{A}^* = g_m(\mathcal{L}_1,\ldots,\mathcal{L}_m)$ then $\mathcal{L}_1,\ldots,\mathcal{L}_m$ cannot be equal to $g_k(x_1,\ldots,x_k)$ on the characteristic vectors.

We shall prove that such an $\mathfrak{A}^*$ cannot exist. $\mathfrak{A}^*$ can be written in the form $g_m(\mathcal{L}_1,\ldots,\mathcal{L}_m)$ where $\mathcal{L}_i = x_{ij}$ or $\mathcal{L}_i$ is a superposition over $G \setminus g_k$.

a./ if all $\mathcal{L}_i$ are superpositions over $G \setminus g_k$ then all $\mathcal{L}_i$ equal 1 or 0 on the characteristic vectors.

$g_\ell(\{\{0,1,2\}^\ell \ (2,\ldots,2)\}) \subseteq \{0,1\}$

Since $\mathfrak{A}^*$ is minimal /in the above sence/, there is exists a characteristic vector $\underline{c}$ such that $\mathcal{L}_1(\underline{c}) = 0$ that is $\mathfrak{A}^*(\underline{c}) = 0$. On the other hand $g_\ell(\underline{c}) = 1$ holds. This is a contradiction;

b./ We have seen, that there is a $\mathcal{L}_\ell = x_q$ in the superposition $\mathfrak{A}^* = g_m(\mathcal{L}_1,\ldots,\mathcal{L}_m)$.

Let $\underline{x} = x_1,\ldots,x_k$ be a characteristic vector so that $x_q = 0$, and $x_n = 2$. If $x_n \neq \mathcal{L}_1$, $x_n \neq \mathcal{L}_2, \ldots$ $x_n \neq \mathcal{L}_m$ then all $\mathcal{L}_i$ are equal to 1 or 0 on this characteristic vector, and hence $\mathfrak{A}^*(\underline{x}) = 0$. $[(\mathcal{L}_1(x),\ldots,\mathcal{L}_m(x))] \neq (1,1,\ldots,1)$ and by the definition $g_m(\{\{0,1\}^m \setminus (1,\ldots,1)\}) = 0.)$ This is also a contradiction.

c./ By a/ and b/ $\mathfrak{A}^*$ can be written in the form

$$g_m(\mathcal{L}_1,\ldots,\mathcal{L}_q, \ x_1,\ldots x_k) .$$

The assumption that $\mathfrak{A}^*$ is minimal implies that $\mathcal{L}_1$ cannot be equal to 1 on all characteristic vectors. Let $\underline{x}$ be a characteristic vector so that $\mathcal{L}_1 = 0$.

In this case, $\mathscr{L}_2, \ldots \mathscr{L}_q = 0$ or 1, and there is one $x_j = 0$. Since $(\mathscr{L}_1, \ldots \mathscr{L}_q, x_1, \ldots, x_k) \notin \{1,2\}^n$ and it cannot be characteristic, $\mathfrak{A}^*(\underline{x}) = \emptyset$. This implies that $\mathfrak{A}^* = g_m(x_{i_1}, \ldots, x_{i_m})$. If $m < k$, then there is a $x_q \notin \{x_{i_1}, \ldots, x_{i_m}\}$. On the characteristic vector $x_q = 2$, $x_{i_1} = 0$, $x_j = 1$ $(j \neq q,\ j \neq i_1)$, the statements $g_k = 1$ and $\mathfrak{A}^* = 0$ hold. If $m > k$ then there exists at least one pair i_ℓ, i_j such that $i_\ell = i_j$. In this case let $x_{i_\ell} = 0$, $x_j = 2$ $(j \neq i_\ell)$ and $x_t = 1$ $(t \neq j,\ t \neq i_\ell)$. On this characteristic vector $g_k(x_1 \ldots x_k) = 1$ and $\mathfrak{A}^* = 0$ hold. This is also a contradiction, thus lemma 2 is completely proved.

<u>Lemma 3.</u> Let $k = 5$ and π a permutation of the form $C_1 . C_2$ where $/C_1/ = 2$, $/C_2/ = 3$ or let $k = 7$ and π be a permutation of the form $C_1 . C_2$ where $/C_1/ = 3$, $/C_2/ = 4$. Then there is a set of closed sets in P_5 or in P_7 preserving π which has cardinality $\mathfrak{c}$.

It is easy to see that it is sufficient to consider the cases when

$\pi = (03)(124)$ and

$\pi = (034)(1256)$

The definition of g_m in Lemma 2 does not contradict the property g_m preserves π.

If we define h_m so that $h_m(a_1, \ldots, a_n) = g_m(a_1, \ldots, a_n)$ on the set $\{o,1,2\}^m$ and h_m preserves π, then $H = \{h_m / m \geq 3\}$ is a set with the property $h_m \notin [H \setminus h_m]$. Thus analogously to Lemma 1 $H^* = \{[S] / S \subset H\}$ is a set consisting of closed classes preserving π, and the cardinality of H^* is $\mathfrak{c}$.

<u>Theorem 1:</u> Let $k \geq 2$ and π be a permutation of E_k. If

$\pi \neq (a_1 a_2 a_3)$ for $k=3$ and

$\pi \neq (a_1 a_2 a_3 a_4)$ for $k=4$

then there are as many as $\mathfrak{c}$ closed classes in P_k preserving π.

Proof.: If π contains a cycle C such that $|C| \geq 5$, the statement is implied by Lemma 2.

If π contains a cycle C such that $|C| = 1$ or two cycles with equal lengths, then the statement follows from Lemma 1. If π contains at least 4 cycles with lengths 2,3,4 then two of them have equal lengths.

Thus we have the following cases:

$$\pi = C_1 . C_2, \quad /C_1/ = 2, \quad /C_2/ = 3 \quad \text{or}$$
$$/C_1/ = 3, \quad /C_2/ = 4$$

$$\pi = C_1 . C_2 . C_3 \quad /C_1/ = 2, \quad /C_2/ = 3, \quad /C_3/ = 4$$

The first case is treated in Lemma 3.
In the second case $/C_1/ \mid /C_3/$, therefore the assumptions of Lemma 1 hold. Thus the proof of Theorem 1 is complete.

§. 2.

In §.1. we have seen, that for all but three permutations π $\mathfrak{c}$ closed sets in P_k ($k \geq 2$) preserving π can be constructed.

In the case k=2 there is only a finite number of closed sets in P_2 which preserve (01) ([10]). In the cases k=3, π = (012) and k=4, π = (0123) we cannot give an "independent" set of functions with cardinality $\aleph_0$. However we can prove.

Theorem 2: For all k>2 and all permutations π there is at least a countably many closed sets in P_k that preserve π.

Proof: It is sufficient to consider the following two cases: k=3 and π = (012); k=4 and π = (0123). We will construct a set $\{t_i\} = T$ of functions such that $t_i \notin [\bigcup_{j>i} \{t_j\}] = T_i$, and t_i preserves π.

If we have such a family of functions, then the set $\{T_i \mid i \in \omega\}$ contains countably many closed classes, and it can be ordered as

$$T_1 \supset T_2 \supset T_3 \supset \dots$$

We define t_i as follows:

$$t_m(a_1,\dots,a_m) = \begin{cases} b, \text{ if } (a_1,\dots,a_m) = b \text{ or} \\ \quad a_1,\dots,a_{j-1},a_{j+1}\dots a_m = b \text{ and} \\ \quad a_j = \pi^{-1}(b); \\ \pi^{-1}(b), \text{ if} \\ \quad a_1,\dots,a_m \in \{\pi^{-1}(b),b\}^m \quad \text{and} \\ \quad \{a_i/a_i = b\} < m-1; \\ a_1 \quad \text{otherwise.} \end{cases}$$

A vector $\underline{a} = (a_1,\dots,a_m)$ is called characteristic, if $|\{i/a_i = 0\}| = 1$ and $|\{i/a_i| = 1\}| = m-1$. The definition implies that t_m preserves π. Let us suppose, that

$$t_m(x_1,\dots,x_m) = \mathfrak{A},$$

where $\mathfrak{A}$ is a superposition over T_i.

We can choose - analogously to Lemma 2 - a minimal formula $\mathfrak{A}^*$ which equals 1 on all characteristic vectors. This $\mathfrak{A}^*$ cannot be equal to x_i, that is $\mathfrak{A}^*$ can be written in the form

$$t_s(\mathcal{L}_1,\dots,\mathcal{L}_s) \quad \text{where} \quad s > m$$

Denote by y_j the characteristic vector with $x_j = 0$. Let us consider the matrix

$$\begin{vmatrix} \mathcal{L}_1(y_1) & \dots & \mathcal{L}_s(y_1) \\ \mathcal{L}_1(y_2) & \dots & \mathcal{L}_s(y_2) \\ \vdots & & \\ \mathcal{L}_1(y_m) & \dots & \mathcal{L}_s(y_m) \end{vmatrix}$$

By the minimality of $\mathfrak{A}^*$ every column of the matrix contains at least one 0. $s > m$ implies, that at least one row in the matrix contains two or more 0's. If the e'th row in the matrix contains at least two 0 - elements then $\mathfrak{A}^*(y_\ell) = 0$. This is a contradiction, since $t_m(y_i) = 1$ for all $i \in \{1,2,\ldots,m\}$.

Thus Theorem 2 is proved.

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Ö s s z e f o g l a l ó

A k-értékü logika önduális osztályairól

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A jelen dolgozatban a szerzök bebizonyitják, hogy $\forall s(x) \in P_k$, $k \geq 3$ - kivéve, ha s(x) = (0 1 2) ill. s(x) = (0 1 2 3), - (s(x)-permutáció) az önduális zárt osztályok száma kontinuum.

Ha s(x) = (0 1 2) ill. s(x) = (0 1 2 3), akkor is legalább megszámlálható sok önduális osztály van.

Резюме

О мощностях самодейственных заминутных классов в P_k

Я. Деметрович, Л. Ханнак

В настоящей работе авторы изучают самодвойственные замкнутые классы в P_k $(k \geq 3)$. Они доказывают, что

а/ для любого $S(x) \in P_k$ /где S(x) -перестановка; S(x) ≠ (0 1); S(x) ≠ (0 1 2) и S(x) ≠ (0 1 2 3)/, существует континуум самодвойственных замкнутых классов относительно S(x);

б/ если S(x) = (0 1 2) из P_3 или S(x) = (0 1 2 3) из P_4, то существует по крайней мере счетное число самодвойственных замкнутых классов относительно S(x).