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ON THE GENERATION OF BINARY VECTORS BY BOOLEAN FUNCTIONS

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Dedicated to Prof. Dr. W. Engel on the occasion of his 50th birthday.

1. Introduction and notation

This is the last paper in a series of four. In (2) the author began studies in the following direction.

Let k be an integer, $k \geq 2$.

Let $V_k = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix}, \ a_1, \ldots, a_k \in \{0,1\} \right\}$ and $M = \{X_1, \ldots, X_n\} \subseteq V_k$.

Then we define $f(M)$, where f is a Boolean function and

$$X_i = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{ik} \end{pmatrix} \quad \text{for} \quad i = 1, \ldots, n, \quad \text{by}$$

$$f(M) = f(X_1, \ldots, X_n) = \begin{pmatrix} f(a_{11}, \ldots, a_{n1}) \\ \vdots \\ f(a_{1k}, \ldots, a_{nk}) \end{pmatrix}$$

For a set K of Boolean functions we define the closure $[M]_K$ of M with respect to K.

Definition. *Let a sequence* $M_K^i \subseteq V_k$ *defined by*

1^0 $M_K^0 = M$ and

2^0 $M_K^{i+1} = M_K^i \cup \{ X : \exists f \in K, \ X_1, \ldots, X_n \in M_K^i : X = f(X_1, \ldots, X_n) \}$

for $i = 0,1,2, \ldots$

Then let $[M]_K = \lim_{i \to \infty} M_K^i$.

We notice that the successor of M_K^i is a superset of M_K^i and all members of this sequence are subsets of V_k. Hence, starting by some M_K^a this sequence has to be constant. This M_K^a is denoted by $\lim_{i \to \infty} M_K^i$ or by $[M]_K$, accordingly.

We will investigate the following problems:

1. Find K-conditions for M such that M is *K-complete*, i.e. $[M]_K = V_k$.

2. Find the cardinality of a *K-base*, i.e. M is K-complete, but any proper subset of

M is not K-complete. If there are K-bases of different cardinalities, find the minimal and the maximal cardinality of K-bases.

In (2),(3) and (4) we solved these problems for some closed sets of Boolean functions, namely for all closed sets of nonmonotonic functions. In (2), (3) and (4) we used M_K^1 for the closure of M with respect to K. Without loss of generality these restrictions are possible, because $M_K^1 = [M]_K$ was proved for closed sets K in (4). Moreover, in this paper (section2) we will prove $[M]_K = [M]_{[K]}$ for arbitrary sets K of Boolean functions, where $[K]$ is the usual closure of functions. Hence, in order to solve our problems, we only have to solve the problems for closed sets K. All closed sets of Boolean functions are known. For a survey and notations of these closed sets see (1).

In section 3 we give a survey of the results for all closed sets. In section 4 we prove these results.

2. A theorem

In this section we will prove the following

Theorem 1. *Let M be an arbitrary subset of V_k and let K be an arbitrary set of Boolean functions.*
Then $[M]_K = [M]_{[K]}$.

We give the following version of the definition of the closure $[K]$ which we will use in the proof of Theorem 1.

If $id \epsilon K$, $[K]$ is defined by (1), p.4, 1.,2.,3., and 4!:

Definition. *Let $id \epsilon K$. Let $K^i (i = 0,1, \ldots)$ be a sequence of sets of Boolean functions as follows.*

1^0 *If a function f belongs to K^i $(i = 0,1, \ldots)$, all functions which can be generated by f by adding fictive variables to f, identification of variables belong to K^i too.*

2^0 – $K^0 = K$

– $K^{i+1} = K^i \cup \{ f: \exists g, g_1, \ldots, g_m \epsilon K^i : f = g(g_1, \ldots, g_m) \}$
if $i = 0,1, \ldots$ *and*

– $[K] = \bigcup_{i=0}^{\infty} K^i$.

We notice that we only need a finite set of Boolean functions for the generation of $[M]_{[K]}$, i.e. there is an integer a with

(1) $[M]_{[K]} = [M]_{K^a}$.

Moreover, we only need functions of K^a with a finite number of variables. Finally it is worthy of remark that only K^0 has to contain all functions obtained by 1^0. This property we will use.

Proof of Theorem 1.

a) Let $id \epsilon K$. First we prove $[M]_{[K]} \subseteq [M]_K$. Let b be an integer, $b \geqslant 1$. Then there is an integer q satisfying

(2) $M^q_{K^b} = [M]_{K^b}$

by our remarks at the definition of the closure of M with respect to K.

We prove

(3) $[M]_{K^{b-1}} \supseteq M^i_{K^b}$

by induction on $i (i \geqslant 1)$, for all integers $b \geqslant 1$.

1. $i = 1$. Then $M^1_{K^b} = M^0_{K^b} \cup \{X : \ \exists f \epsilon K^b, \exists X_1, \ldots, X_n \epsilon M: X = f(X_1, \ldots, X_n)\}$.

Let us assume there is a vector $X \epsilon M^1_{K^b} \setminus [M]_{K^{b-1}}$.

Then there is a function $f \epsilon K^b$ and there are vectors $X_1, \ldots, X_n \epsilon M = M^0_{K^b}$ with $X = f(X_1, \ldots, X_n)$. If $f \epsilon K^{b-1}$ then $X \epsilon [M]_{K^{b-1}}$, which is a contradiction to our assumption. Hence, $f \epsilon K^b \setminus K^{b-1}$. Then there are functions $g, g_1, \ldots, g_m \epsilon K^{b-1}$ with $f = g(g_1, \ldots, g_m)$, i.e. $X = g(g_1(X_1, \ldots, X_n), \ldots, g_m(X_1, \ldots, X_n))$. By $g_j \epsilon K^{b-1}$ and $X_l \epsilon M$ $(j = 1, \ldots, m;\ l = 1, \ldots, n)$ it follows $X_j = g_j(X_1, \ldots, X_n) \epsilon [M]_{K^{b-1}}$

Hence using $g \epsilon K^{b-1}$, we obtain $X = g(X_1, \ldots, X_m) \epsilon [M]_{K^{b-1}}$, which is also a contradiction to our assumption.

2. We have $M^{i+1}_{K^b} = M^i_{K^b} \cup \{X : \exists f \epsilon K^b, \exists X_1, \ldots, X_n \epsilon M^i_{K^b} : X = f(X_1, \ldots, X_n)\}$.
Let us consider an arbitrary vector $x \epsilon M^{i+1}_{K^b}$. If $X \epsilon M^i_{K_b}$ then $X \epsilon [M]_{K^{b-1}}$ follows by induction assumption.

Let $X \epsilon M^{i+1}_{K^b} \setminus M^i_{K^b}$. Then there is a function $f \epsilon K^b$ and there are vectors $X_1, \ldots, X_n \epsilon M^i_{K^b}$ with $X = f(X_1, \ldots, X_n)$. By the induction assumption we have $X_1, \ldots, X_n \epsilon [M]_{K^{b-1}}$. $f \epsilon K^b$ implies by the definition of K^b that there are functions $g, g_1, \ldots, g_m \epsilon K^{b-1}$ with $f = g(g_1, \ldots, g_m)$. Hence $X'_j = g_j(X_1, \ldots, X_n) \epsilon [M]_{K^{b-1}}$ $(j = 1, \ldots, n)$ and finally $X = g(g_1(X_1, \ldots, X_n), \ldots, g_m(X_1, \ldots, X_n)) =$
$= g(X'_1, \ldots, X'_m) \epsilon [M]_{K^{b-1}}$. Therefore (3) is proved for arbitrary $i \geqslant 1$ and arbitrary fixed $b \geqslant 1$. Inparticular (3) is proved for $i = q$, where q is defined as in (2). Using (2) we have

$$[M]_{K^{b-1}} \supseteq [M]_{K^b}$$

for arbitrary integer $b \geq 1$.

We observe $[M]_K = [M]_{K^0}$ (The vectors, which can be generated by functions of $K^0 \setminus K$, we also obtain by functions of K, what follows by the definition of the closure $[M]_K$.).

By induction we get for arbitrary integer $b \geq 1$:

$$(4) \qquad [M]_{K^b} \subseteq [M]_K .$$

In particular, (4) holds for $b = a$, where a is defined in (1), i.e.

$$[M]_{[K]} \subseteq [M]_K .$$

Clearly, $K \subseteq [K]$ implies the converse direction

$$[M]_K \subseteq [M]_{[K]} .$$

If $id \in K$, the theorem is proved.

b) Let $id \notin K$. By the definition of $[M]_K$ we have $[M]_{K'} = [M]_{K' \cup \{id\}}$ for all sets of functions K'.

If c_0 and c_1 are the constant functions,
$K \subseteq \{c_0, c_1\}$ implies $[K] = K$ and $[K \cup \{id\}] = [K] \cup \{id\}$ and $K \not\subseteq \{c_0, c_1\}$ implies $id \in [K]$ and $[K \cup \{id\}] = [K] \cup \{id\}$ too.
Hence, using part a), we obtain

$$[M]_K = [M]_{K \cup \{id\}} = [M]_{[K \cup \{id\}]} = [M]_{[K] \cup \{id\}} = [M]_{[K]} .$$

q.e.d.

3. A survey on results

In this section we will give the answers to our problems for each closed set of Boolean functions.

Let the closed sets of Boolean functions denote by the notation by Post, see (1).

Further we use the following notations:

- $\underline{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in V_k$, $\underline{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in V_k$.

- $e_i (i = 1, \ldots, k)$ denotes the vector of V_k containing a 1 exactly in the i-th component.

If $X \in V_k$, $\overline{X}$ denotes the vector of V_k which does not coincide with X in any component.

– If $M \subseteq V_k$, we consisder M also as a matrix. We say M has the property $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$, if and only if for each pair (i,j), $1 \leqslant i < j \leqslant k$, the 2-rows-matrix M_{ij}, whose first row is the i-th row of M and the second row is the j-th row of M, has a column $\binom{0}{1}, \binom{1}{0}, \binom{0}{0}, \binom{1}{1}$, respectively. Further M has the property $\boldsymbol{C}(\mu), \boldsymbol{D}(\mu)$, $\mu \geqslant 2$, if and only if every matrix consisting of μ rows of M has a column

$$\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}, \quad \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix},$$

respectively. Let $\boldsymbol{P}$ be a Boolean function of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$. Then M has the property $\boldsymbol{P}^i$, $i \in \{0,1\}$, if and only if M has the property $\boldsymbol{P}$ and does not contain, in addition, rows consisting only of i's. Accordingly, let $\boldsymbol{P}^{01}$ defined as $\boldsymbol{P}^0 \wedge \boldsymbol{P}^1$.

Now we are able to formulate the main results.

2. **A theorem** *Let* $K \in \{0_i, S_i, P_i, L_i\}$. $M \subseteq V_k$ *is*

1^0 *K-complete, if and only if* M *satisfies the condition of table* 1,

2^0 *a K-base, if and only if* M *is K-complete and has the cardinality given in table* 1.

set	criterion of completeness	m
0_1	$M = V_k$	2^k
$0_2, 0_5$	$M \supseteq V_k \setminus \{1\}$	$2^k - 1$
$0_3, 0_6$	$M \supseteq V_k \setminus \{0\}$	$2^k - 1$
0_4	$\forall X \in V_k$ we have $X \in M$ or $\overline{X} \in M$	2^{k-1}
$0_7, 0_8$	$M = V_k \setminus\ 0,1$	$2^k - 2$
0_9	$\forall X \in V_k \setminus\ 0,1$ we have $X \in M$ or $\overline{X} \in M$	$2^{k-1} - 1$
S_1, S_3	$M \supseteq \{\underline{0}, e_1, \dots, e_k\}$	$k + 1$
S_5, S_6	$M \supseteq \{e_1, \dots, e_k\}$	k
P_1, P_3	$M \supseteq \{\underline{1}, \overline{e}_1, \dots, \overline{e}_k\}$	$k + 1$
P_5, P_6	$M \supseteq \{\overline{e}_1, \dots, \overline{e}_k\}$	k
L_1	$\exists X_1, \dots, X_{k-1} \in M$: $rg(X_1, \dots, X_{k-1}, \underline{1}) = k$	$k - 1$
L_2	$\exists X_1, \dots, X_k \in M$: $rg(\overline{X}_1, \dots, \overline{X}_k) = k$	k
L_3	$\exists X_1, \dots, X_k \in M$: $rg(X_1, \dots, X_k) = k$	k
L_4	$\exists X_1, \dots, X_k \in M$: $rg(X_1, \dots, X_k) = k$ and $X \in M \setminus [\{X_1, \dots, X_k\}]_{L_4}$	$k + 1$
L_5	$\exists X_1, \dots, X_{k-1} \in M$: $rg(X_1, \dots, X_{k-1}) = k-1$ $\exists X \in M \setminus [\{X_1, \dots, X_{k-1}\}]_{L_4}$ and $\nexists$ even number of vectors of $X_1, \dots, X_{k-1}, X$ with sum $\underline{1}$	k

Table 1.

These results are proved in (2), (3) and (4).

Theorem 3. *Let* $K \in \{ C_i, D_i, A_i, F_i^{\mu}, F_i^{\infty} \}$. $M \subseteq V_k$ *is K-complete, if and only if* M *satisfies the condition* $\boldsymbol{P}$ *of table* 2. *The K-bases have the minimal cardinality* m *and the maximal cardinality* p, *given in table* 2.

In table 2 let

1. 1. $a \in \{ 0,1 \}$,
   2. $]x[\; = \min (y \colon y \in N, \; y \geqslant x)$,
   3. $\varphi_1(k) = x \in N \leftrightarrow \binom{x}{[\frac{x}{2}]} \geqslant k > \binom{x-1}{[\frac{x-1}{2}]}$,
   4. $\varphi_2(k) = x \in N \leftrightarrow \binom{x}{[\frac{x-1}{2}]} \geqslant k > \binom{x-1}{[\frac{x-2}{2}]}$.

   5. *) For $\mu = 3$ we do not give an explicit formula; see the remark at the end of this section.

K		$\boldsymbol{P}$	m	p
C_1		$\boldsymbol{A} \vee \boldsymbol{B}$	$]\log_2 k[$	$k - 1$
C_2		$(\boldsymbol{A} \vee \boldsymbol{B})^1$	$]\log_2 (k + 1)[$	k
C_3		$(\boldsymbol{A} \vee \boldsymbol{B})^0$	$]\log_2 (k + 1)[$	k
C_4		$(\boldsymbol{A} \vee \boldsymbol{B})^{01}$	$]\log_2 (k + 2)[$	$k + 1$
D_1		$((\boldsymbol{A} \vee \boldsymbol{B})(\boldsymbol{C} \vee \boldsymbol{D}))^{01}$	$]\log_2 (k + 1)[+ 1$	$k + 1$
D_2		$\boldsymbol{ABCD}$	$\varphi_2(k) + 1$	$\begin{cases} 2k & 2 \leqslant k \leqslant 4 \\ (\frac{k}{2}) & k \geqslant 5 \end{cases}$
D_3		$(\boldsymbol{A} \vee \boldsymbol{B})(\boldsymbol{C} \vee \boldsymbol{D})$	$]\log_2 k[+ 1$	k
A_1, A_2, A_3, A_4		$\boldsymbol{AB}$	$\varphi_1(k)$	$\begin{cases} 2k - 2 & 2 \leqslant k \leqslant 6 \\ [\frac{k^2}{4}] & k \geqslant 7 \end{cases}$
F_1^{μ}	$(\mu < k)$	$((\boldsymbol{A} \vee \boldsymbol{B})\boldsymbol{C}(\mu))^0$	$]\log_2 (k + 1)[+ 1$	$\begin{cases} (\frac{k}{\mu}) & 2 \leqslant \mu \leqslant k - 2 \\ k + 1 & \mu = k - 1 \end{cases}$
F_5^{μ}	$(\mu < k)$	$((\boldsymbol{A} \vee \boldsymbol{B})\boldsymbol{D}(\mu))^1$		
F_{2+a}^{μ}	$(\mu < k)$	$\boldsymbol{ABC}(\mu)$	$\begin{cases} \varphi_2(k) & \mu = 2 \\ \varphi_1(k) + 1, & \mu \geqslant 4 \end{cases}$	$\begin{cases} 2k - 1 \begin{cases} 2 \leqslant \mu = k - 1 \leqslant 5 \\ 2 = \mu = k - 2 \end{cases} \\ [\frac{k^2}{4}] + 1 & 6 \leqslant \mu \leqslant k - 1 \\ (\frac{k}{\mu}) & 2 \leqslant \mu \leqslant k - 2 \geqslant 3 \end{cases}$
F_{6+a}	$(\mu < k)$	$\boldsymbol{ABD}(\mu)$		
F_4^{μ}	$(\mu < 2)$	$(\boldsymbol{A} \vee \boldsymbol{B})\boldsymbol{C}(\mu)$	$]\log_2 k[+ 1$	$(\frac{k}{\mu})$
F_8^{μ}	$(\mu < k)$	$(\boldsymbol{A} \vee \boldsymbol{B})\boldsymbol{D}(\mu)$		
F_1^{∞}, F_1^{μ}	$(\mu \geqslant k)$	$((\boldsymbol{A} \vee \boldsymbol{B})\boldsymbol{C}(k))^0$	$]\log_2 (k + 1)[+ 1$	$k + 1$
$F_5^{\infty}, F_5^{\infty}$	$(\mu \geqslant k)$	$((\boldsymbol{A} \vee \boldsymbol{B})\boldsymbol{D}(k))^1$		
$F_{2+a}^{\infty}, F_{2+a}^{\mu}$	$(\mu \geqslant k)$	$\boldsymbol{ABC}(k)$	$\varphi_1(k) + 1$	$\begin{cases} 2k - 1 & 2 \leqslant k \leqslant 6 \\ [\frac{k^2}{4}] + 1 & k \geqslant 7 \end{cases}$
$F_{6+a}^{\infty}, F_{6+a}^{\mu}$	$(\mu \geqslant k)$	$\boldsymbol{ABC}(k)$		
F_4^{∞}, F_4^{μ}	$(\mu \geqslant k)$	$(\boldsymbol{A} \vee \boldsymbol{B})\boldsymbol{C}(k)$	$]\log_2 k[+ 1$	k
F_8^{∞}, F_8^{μ}	$(\mu \geqslant k)$	$(\boldsymbol{A} \vee \boldsymbol{B})\boldsymbol{D}(k)$		

4. Proof of Theorem 3

1. Completeness

By Theorem 2 of (3) we have only to consider the following closed sets of Boolean functions: $C_1, C_3, C_4, D_1, D_2, D_3, A_1, A_3, A_4, F_i^\mu, F_i^\infty (i = 1,2,3,4)$. The problems were solved for the sets $C_1, C_3, C_4, F_i^\mu, F_i^\infty$ $(i = 1,4)$ in (3) and for the sets D_1 and D_3 in (2). So we have to prove the statement of Theorem 3 for the closed sets

$$K \epsilon \{ D_2, A_1, A_3, A_4, F_2^\mu, F_2^\infty, F_3^\mu, F_3^\infty \} .$$

1. First we show that the conditions of table 2 are necessary. Let M be K-complete.

1.1 The monotony of the functions of K implies that M satisfies $\boldsymbol{AB}$ (i.e. $\boldsymbol{A} \wedge \boldsymbol{B}$). To show this, let (i,j) be a pair with $i,j \epsilon \{ 1, \ldots, k\}$ and $i \neq j$ such that $M_{i,j}$ does not contain a column $\binom{1}{0}$. Then it is impossible to generate vectors, having the i-th component 1 and the j-th componenet o, by monotonic functions, i.e. $[M]_K \neq V_k$.
Hence M has to satisfy $\boldsymbol{B}$ and, in analogy, $\boldsymbol{A}$ too.

1.2. If $K = D_2$, M has to satisfy $\boldsymbol{CD}$ too. To show this, let (i,j) be a pair with $i,j \epsilon \{1, \ldots, k\}$ and $i \neq j$ such that M_{ij} does not contain a column $\binom{1}{1}$. Let M'_{ij} be a matrix of the same type as M_{ij}, whose elements of the first row coincide with the correspondent elements of the first row of M_{ij}, while this does not hold for any element of the second row of M_{ij}. Then M'_{ij} does not contain a column $\binom{1}{0}$, it is impossible to generate the vector $\binom{1}{0}$ by M'_{ij} and by a monotonic function. Thus, it is impossible to generate vectors, having 1 as the i-th and j-th component, by M_{ij} and by functions of $K = D_2$.
Hence, M has to satisfy $\boldsymbol{D}$ and, in analogy, $\boldsymbol{C}$ too.

1.3. Let $K \epsilon \{F_i^\mu, F_i^\infty\}$ $(i = 2,3)$. Then M has the property $\boldsymbol{C}(\mu)$ for $\mu < k$ and $\boldsymbol{C}(k)$ for $\mu \geqslant k$ and $\mu = \infty$. Either $\underline{0} \epsilon M$ or $\underline{0} \notin M$.
In the first case M satisfies $\boldsymbol{C}(\mu)$ and in the second case there is a function $f \epsilon K$ with $f(M) = \underline{0}$. Now the statement follows by the definition of the functions of F_i^μ or F_i^∞.

1.4. If $K \epsilon \{ A_3, A_4, F_2^\mu, F_2^\infty \}$, $f(\text{o}, \text{o}, \ldots, \text{o}) = \text{o}$ holds for each function $f \epsilon K$. Thus, M does not contain rows consisting of o's only.

1.5. If $K \epsilon \{A_4, F_2^\mu, F_2^\infty\}$ we obtain, in analogy to 1.4., that M has no rows consisting of 1's only.
We notice that M satisfying $\boldsymbol{AB}$ implies M has no rows consisting of o's or 1's only.

2. In order to show that the conditions of table 2 are sufficient, let M be a matrix having the property $\boldsymbol{P}(K)$ of table 2. Denote the rows of M by α_i and let

$$\underline{a} = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ a_k \end{pmatrix}$$

be an arbitrary chosen vector of V_k. Then we give a function $f \epsilon K$ satisfying $f(M) = \underline{a}$. If $\beta = (b_1, \ldots, b_t)$ and $\gamma = (c_1, \ldots, c_t)$, $\beta < \gamma$ means that $b_i \leqslant c_i$ for $i = 1, \ldots, t$, and at least for one i we have the inequality.

2.1 Let $K \epsilon \{A_1, A_3, A_4, F_2^{\mu}, F_2^{\infty}, F_3^{\mu}, F_3^{\infty}\}$.

Then

$$f(\alpha) = \begin{cases} a_i & \text{if } \alpha = \alpha_i, \\ 0 & \text{if there is a } \alpha_i \text{ with } \alpha < \alpha_i, \\ 1 & \text{otherwise.} \end{cases}$$

2.2 Let $K = D_2$.
Then

$$f(\alpha) = \begin{cases} a_i & \text{if } \alpha = \alpha_i, \\ \overline{a_i} & \text{if } \alpha = \alpha_i, \\ 0 & \text{if there is a } \alpha_i \text{ with } \alpha < \alpha_i \text{ or } \alpha < \overline{\alpha_i}, \\ 1 & \text{if there is a } \alpha_i \text{ with } \alpha > \alpha_i \text{ or } \alpha > \overline{\alpha_i}, \\ 0 & \text{for all other } \alpha \text{ with } \alpha = (0, \sim), \\ 1 & \text{for all other } \alpha \text{ with } \alpha = (1, \sim). \end{cases}$$

Thus this part is proved.

2. Cardinality of bases

If we consider the matrices M as an incidence matrix of a family F of k subsets of an r-element set R, the determination of m is equivalent to the determination of the maximal cardinality $n(r)$ of families of a finite set satisfying a certain K-condition, according to $m = \min\{x : x \epsilon N, \ n(x) \geqslant k\}$.

The following conditions for M and F are equivalent:

- **AB** $\leftrightarrow$ $X \not\subset Y$ for all different $X, Y \epsilon F$,
- **CD** $\leftrightarrow$ $X \cap Y \neq 0, X \cup Y \neq R$ for all $X, Y \epsilon F$,
- $\boldsymbol{C}(\mu)$ $\leftrightarrow$ $\bigcup_{i=1}^{\mu} X_i \neq R$ for all $X_1, X_2, \ldots, X_\mu \epsilon F$.

The maximal cardinality of families satisfying the conditions related to ***AB***, ***ABCD***, ***ABC***(2), ***ABC***(μ) $\mu \geqslant 4$ was determined by Sperner [12], Katona [9] and Schönheim [11] and Brace and Daykin [7], Milner [10], the author [5], respectively.

Fraknl [8] and the author [5] solved this problem in the ***ABC***(3) case for sufficiently large r. These maximal cardinalities have different structures for even and odd r. So we did not give an explicit formula in table 2 in this case.

The values of p were determined by the author in [6].

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Összefoglaló

Bináris vektoroknak Boole függvényekkel való generálásáról

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Legyen $M \subset \{0,1\}^k$, ahol k természetes szám. Jelölje K a Boole függvények egy zárt halmazát. Az összes zárt Boole függvényhalmazra megadja a szerző annak szükséges és elégséges feltételét, hogy M K-teljes legyen, azaz hogy M K-lezárása megegyezzen a $\{0,1\}^k$ halmazzal. Továbbá meghatározza $\{0,1\}^k$ K-bázisainak lehetséges minimális és maximális számosságát, ahol M K-bázis ha minimális a K-tejességre nézve.

Резюме

О порождении бинарных векторов булевым функциям

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Пусть $M \leq \{0,1\}^k$, где k натуральное число, и K замкнутое множество Булевых функций. Автор дает необходимые и достаточные условия K-полности множества M. Под K-полностью понимается, что замыкание по K множества M равно множеству $\{0,1\}^k$. В дальнейшем будут определены возможные минимальные и максимальные мощности K-базисов множества $\{0,1\}^k$, где M является K-базисом, если оно минимально относительно K-полноты.