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ON THE DENSITY OF TRANSLATES OF A DOMAIN

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Let w be a domain in the Euclidean plane. We shall denote the density of the densest packing of translates of w by d(w) and the density of the densest lattice packing of translates of w by d'(w).

It has been shown ([1], [2], [3]) that if w is convex then

$$d(w) = d'(w). \qquad (1)$$

In a recent paper L. Fejes Tóth [4] started an interesting field of research trying to extend the validity of (1) to more general domains. He has proved that if w is the union of two intersecting equal circles than equation (1) is still valid. The range of validity of this property has been curbed by a construction of A. Bezdek and G. Kertész [5]. They have constructed a domain consisting of 5 convex domains that can be arranged to have higher density if you do not require the packing to be latticelike. Fejes Tóth's conjecture [1] is that this can not be done with a domain that is the union of two convex domains with a point in commmon. The analoguous question has been raised for starlike domains as well.

In the present paper we are going to give a construction for a domain u with the following properties:

(i) u is the union of three convex domains
(ii) u is starlike,
(iii) $d(u) > d'(u)$, i.e. the densest packing of u is not latticelike.

To describe the domain u and to show its properties we shall use the 2-dimensional coordinate system. In what follows A(x) will denote the area of x, and the sum of a domain and a vector denotes the translate of the domain by that vector.The three conponents that we use to construct our domain u are two rhombs R1 and R2 and a hexagon H. We define them by listing their corners as follows (Fig.1) :

R1 $(a, -a)$, $(a, 1-a)$, $(-1+a, 2-a)$, $(-1+a, 1-a)$
R2 (a, a), $(a, -1+a)$, $(-1+a, -2+a)$, $(-1+a, -1+a)$
H $(0, 0)$, $(1, 1)$, $(1+L, 1)$, $(2+L, 0)$, $(1+L, -1)$, $(1, -1)$

Here L denotes a sufficiently large and $a > 0$ denotes a sufficiently small number.

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The union u of R1, R2 and H is clearly starlike with respect to any point of the triangle (0,0), (a,a), (-a,-a) thus u shares properties (i) and (ii).

Consider now the translate u1 = u + (1,2) and the union v of u and u1. On the one hand u and u1 have no interior point in common, on the other hand the vectors (0, 4) and (L+3, 2) define a latticelike arrangement of v that is a packing (Fig.2). Thus we have a packing of translates of u of density

A(u)/(2*L+6).

Although we are convinced that the best lattice packing is generated by the vectors (1, 2) and (L+3-a,-1+a) (Fig.3) we need not prove that to reach our goal.All we are left to show is that any lattice-packing of u has a smaller density than A(u)/(2*L+6).

Let us consider a lattice-packing of u. First we define the 'side strip' and the 'neck' of u. The side strip of u is a rhomb of area b*(L+1) given by its corners: (0, 1), (L+1, 1), (L+1-b, 1+b), (-b, 1+b), and the neck is a triangle given again by its corners (a, a), (a+b,a+b), (a, a+b); where b is a sufficiently small but positive number.

We distinguish two cases. First we assume that in the lattice-packing the hexagonal parts of the neighbouring domains are not close to each other, more precisely, we assume that the side strip of u does not contain a point of the hexagonal part of a translate. Then - considering that no more than a single rhombic part of the whole packing can have a point in common with the side strip of u, and that the area of that common part is certainly smaller than b, to each translate there belongs an uncovered part of area > b*L. Since A(u) = 2*L+4-2*a*a, to any prefixed a and b L can be chosen so that A(u)+b*L > 2*L+6.

In the other case the sides of certain pairs of hexagons are closer than b. Of the logically symmetric two subcases we assume that the rhombic part of a translate u2 uf u enters the neck triangle of u Then u2 = u + (1+t1, 2+t2), where 0 <= t1 <= t2 < b. The domains u and u2 define a stripe of the lattice-packing, and the whole lattice is defined by two neighbouring stripes. Since the closest position of two such stripes is defined by the translation (L+3-a-t2, -1+t2), the area of the fundamental domain of the lattice of the densest such lattice packing is 2*L+7 +(L+3)*t2 -2*(a+t2) -(t2-t1) -t1*t2. For sutably chosen a and b this area is > 2*L+6.5, and this is what we wanted to show.

References:

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[5] A. Bezdek and G. Kertész, Counter-examples to a packing problem of L. Fejes Tóth, Colloquia Mathematica Soc. János Bolyai 48. Intuitive Geometry, Siófok, 1985. 29-36

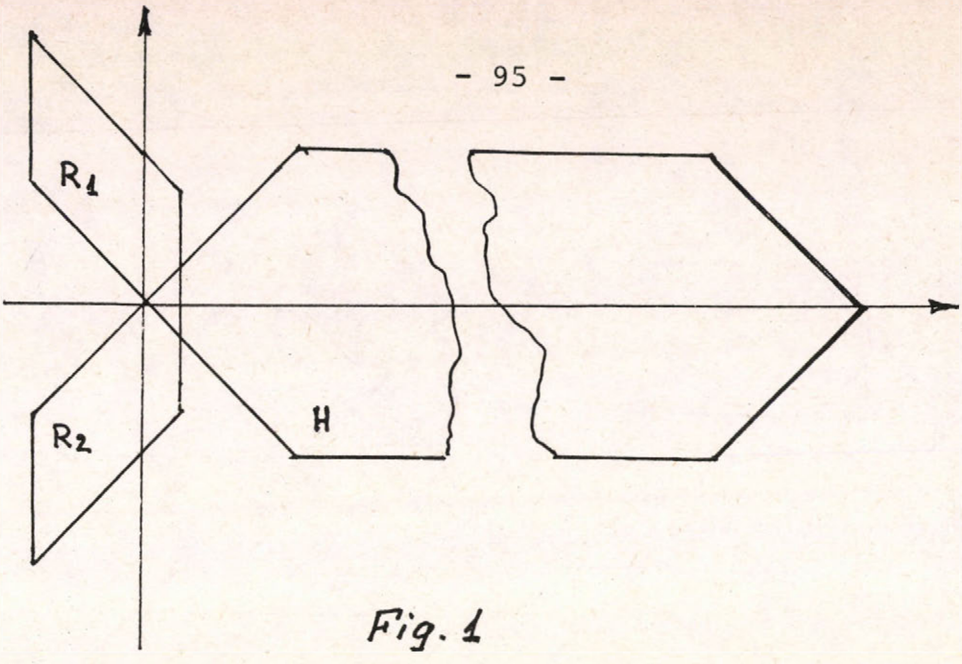

Fig. 1

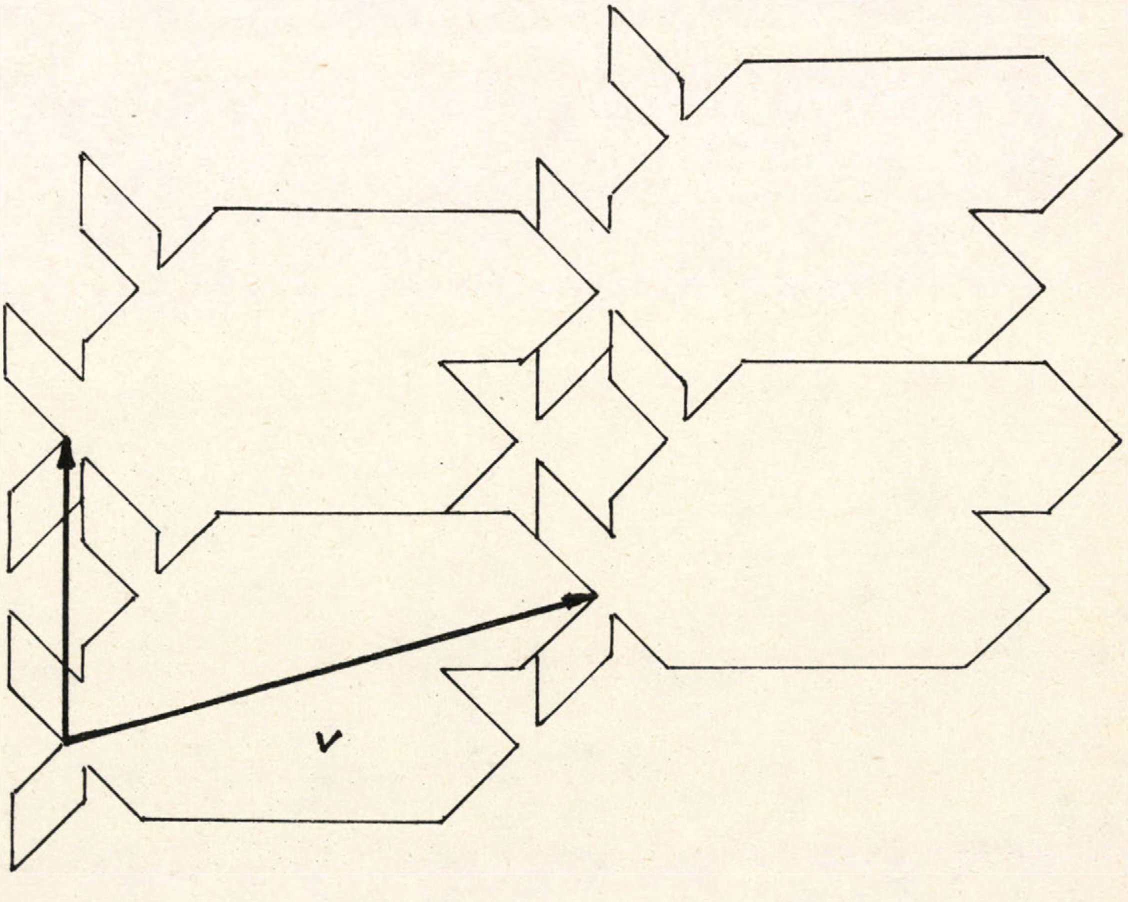

Fig. 2

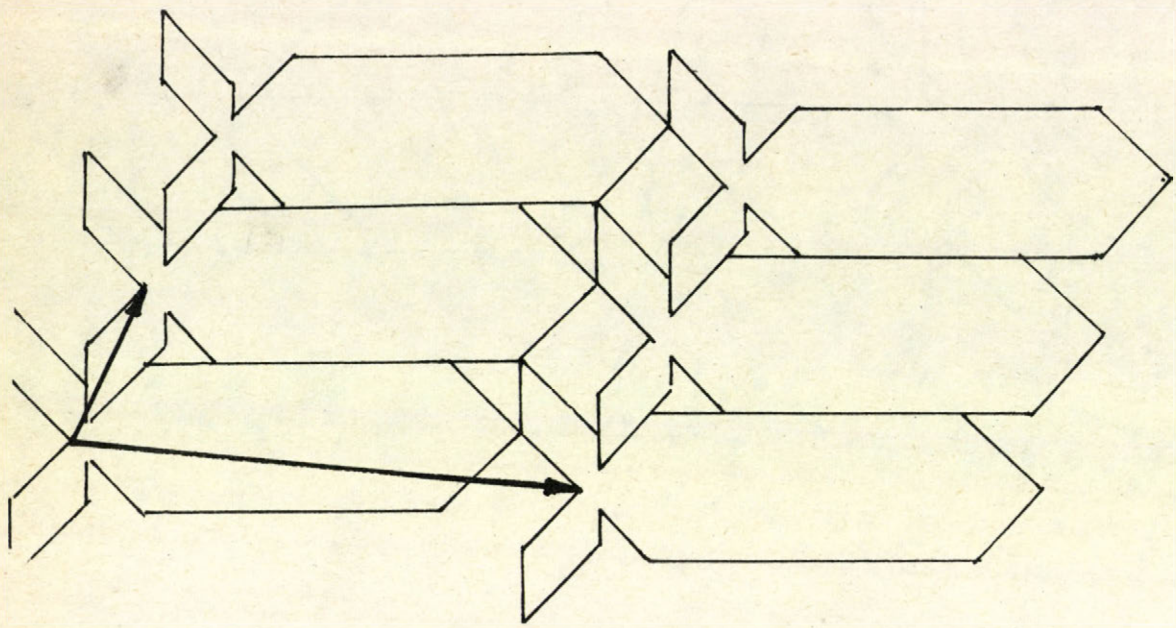

Fig. 3

Плотность трансляций одной области

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Резюме

Статья изучает выполнения плоскости трансляциями данной облисти, так что две области не имеют совместные внутренние точки. Исследуя одну проблему Л. Фейеш Тот-а, автор конструктивным образом доказывает, что: а/ существует область /связное объединение трех выпуклых областей/ такая, что трансляция дающая максимальную плотность не решеточная, и б/ существует звездочная область с такими же свойствами.

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Összefoglaló

A szerző Fejes Tóth László által felvetett problémákat vizsgálva konstrukció utján bizonyitja, hogy a/ létezik olyan tartomány a sikon, amelyen 3 konvex tartomány /összefüggő/ egyesitése és amelynek legsürübb átfedés nélküli elrendezése nem rácsszerü, b/ létezik olyan csillagszerü tartomány a sikon, amelynek legsürübb átfedés nélküli elrendezése nem rácsszerü.