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A REDUCTION THEOREM FOR THE MEASURES OF SUM-SETS IN R^n

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1. INTRODUCTION

If we have two compact sets $A,B \subset R^1$ such that $A \cap B = \{0\}$ (the zero) and A is to the left of 0 and B is to the right of 0, then the algebraic (Minkowski) sum A+B contains $A \cup B$, hence

$$(1.1) \qquad \mu_{1*}(A+B) \geq \mu_1(A)+\mu_1(B),$$

where μ_1 and μ_{1*} are Lebesque-measure and inner L-measure in R^1, respectively (A+B is in general not L-measureable). This implies that (1.1) holds for any L-measureable sets $A,B \subset R^1$, because the measures are translation invariant and A,B can be approximated from inside by compact sets.

The inequality (1.1) is the 1-dimensional Brunn-Minkowski -Lusternik (B-M-L)-inequality. We see that the proof of (1.1) is quite simple. We know also that equality occurs in (1.1) if and only if A and B are homothetic intervals. While the "if" part of this statement is trivial, the "only if" part has been rigorously proved first in early fifties by Henstock and Macbeath [1]. They solved the problem via the following integral inequality: Given $\alpha>0$ and L-integrable functions $f,g\colon R^1 \to R^1_+$ such that $0<\gamma := \sup_x f(x) < +\infty$, $0 < \delta := \sup_x g(x) < +\infty$, we have

$$(1.2)\qquad \int_{R^1}^{*} h^{\alpha}(t)\,dt \geq (\gamma+\delta)^{\alpha}\left(\gamma^{-\alpha}\int_{R^1} f^{\alpha}(x)\,dx + \delta^{-\alpha}\int_{R^1} g^{\alpha}(x)\,dx\right),$$

where $\int^*$ is the inner L-integral, $h(t) := \sup\{h(x,y): x+y=t\}, t\in R^1$, and $h(x,y)$ equals to $f(x)+g(y)$ when $f(x)g(y) > 0$ and zero otherwise [1].

Henstock and Macbeath used for the proof of (1.2) the following nice idea which goes back to the Bonnesen's proof of (1.1), [2] (see also [3]).

Denote

$$(1.3)\qquad A(\xi) := \{x\in R^1 : f^{\alpha}(x) \geq \gamma^{\alpha}\xi\},\ B(\xi) := \{x\in R^1 : g^{\alpha}(x) \geq \delta^{\alpha}\xi\},\quad 0 \leq \xi \leq 1,$$

$$(1.4)\qquad C(\xi) := \{t\in R^1 : h^{\alpha}(t) \geq (\gamma+\delta)^{\alpha}\xi\},\qquad 0\leq \xi \leq 1.$$

Then

$$(1.5)\qquad C(\xi) \supseteq A(\xi) + B(\xi),\qquad 0\leq \xi\leq 1,$$

hence, using (1.1) we get

$$(1.6)\qquad \mu_{1*}(C(\xi)) \geq \mu_1(A(\xi)) + \mu_1(B(\xi)),\qquad 0\leq \xi \leq 1$$

and integrating (1.6) over $0\leq \xi \leq 1$, we get (1.2).

In the last step we used the obvious identity

$$(1.7)\qquad \int_{R^1} \varphi(x)\,dx = \int_0^{+\infty} \mu_1(\{x:\varphi(x)\geq \xi\})\,d\xi.$$

It can be seen easily that the inequality (1.2) holds for any $-\infty \leq \alpha \leq +\infty$, $\alpha \neq 0$, as well as (1.2) can be given a more "integral-theoretic" form taking "ess-sup" instead of "sup" in the definitions of $h(t)$, γ and δ. Namely, using the steps (1.3) $\div$ (1.7) we can easily prove

$$(1.8) \qquad \int_{R^1} k(t)\,dt \geq \frac{\lambda}{\gamma} \int_{R^1} f(x)\,dx + \frac{1-\lambda}{\delta} \int_{R^1} g(x)\,dx,$$

where $0 < \gamma := \operatorname{ess-sup}_x f(x) < +\infty$, $0 < \delta := \operatorname{ess-sup}_x g(x) < +\infty$,

and

$$(1.9) \qquad k(t) := \operatorname{ess-sup}_x \min \{\gamma^{-1} f(x/\lambda), \delta^{-1} \cdot g((t-x)/(1-\lambda))\}, \qquad t \in R^1.$$

Define for $a,b \geq 0$, $0 \leq \lambda \leq 1$ and $-\infty < \alpha < +\infty$, $\alpha \neq 0$ the "extended" means as follows

$$(1.10) \qquad M_\alpha^{(\lambda)}(a,b) := \begin{cases} 0 & \text{if } a \cdot b = 0 \\ (\lambda a^\alpha + (1-\lambda) b^\alpha)^{1/\alpha} & \text{if } a \cdot b > 0, \end{cases}$$

$$(1.11) \qquad M_0^{(\lambda)}(a,b) := \lim_{\alpha \to 0} M_\alpha^{(\lambda)}(a,b) = a^\lambda b^{(1-\lambda)}$$

$$(1.12) \qquad M_{-\infty}^{(\lambda)}(a,b) := \lim_{\alpha \to -\infty} M_\alpha^{(\lambda)}(a,b) = \min\{a,b\},$$

$$(1.13) \qquad M(a,b) := M_{+\infty}^{(\lambda)}(a,b) := \lim_{\alpha \to +\infty} M_\alpha^{(\lambda)}(a,b) = \begin{cases} 0 & \text{if } a\,b = 0 \\ \max\{a,b\} & \text{if } a \cdot b > 0. \end{cases}$$

It is clear that for any $a,b,c,d \geqslant 0$ and $-\infty \leq \alpha \leq +\infty$ we have

(1.14) $$M_{\alpha}^{(\lambda)}(a,b)\cdot M_{-\alpha}^{(\lambda)}(c,d) \geqslant \min\{ac,bd\}.$$

Hence we get from (1.8) immediately

(1.15) $$\int_{R^1} h_{\alpha}^{(\lambda)}(t)\,dt \geqslant M_{\alpha}^{(\lambda)}(\gamma,\delta)\left(\frac{\lambda}{\gamma}\int_{R^1} f(x)\,dx + \frac{1-\lambda}{\delta}\int_{R^1} g(x)\,dx\right),$$

where

(1.16) $$h_{\alpha}^{(\lambda)}(t) := \operatorname*{ess-sup}_{x\in R^1} M_{\alpha}^{(\lambda)}\left(f(x/\lambda),\ g\left(\frac{t-x}{1-\lambda}\right)\right),\quad t\in R^1.$$

(The function $h_{\alpha}^{(\lambda)}(t)$ is already measurable [4], while $h(t)$ is in general not.)

The "ess-sup" definition of $h_{\alpha}^{(\lambda)}(t)$ has an interesting auxiliary effect: taking characteristic functions of two sets χ_A and χ_B instead of f and g, the function $h_{\alpha}^{(\lambda)}(t)$ does not depend on α and it is the characteristic function of the set

(1.17) $$\lambda A \boxplus (1-\lambda)B := \{x\in R^1 : \mu_1(\lambda A\cap(x-(1-\lambda)B)) > 0\}.$$

The set (1.17) is empty if one of the sets has the measure zero. This set has been later called in [5] the "essential sum" of the sets λA and $(1-\lambda)B$. This sum is already measureable ([4]) and

$$(1.18)\quad \lambda A ⊞ (1-\lambda)B \subseteq \lambda A+(1-\lambda)B := \{ x\in R^1 : \lambda A \cap (x-(1-\lambda)B) \neq \emptyset \}.$$

One can see easily that for compact sets A,B

$$(1.19)\quad \lambda A ⊞ (1-\lambda)B \supseteq \lambda A^* + (1-\lambda)B^*,$$

where A^*, B^* are the sets of density points of the sets A,B. We recall that $x\in A^*$ iff (see, e.g. [1])

$$\lim_{\delta \to 0+} \frac{\mu_1(A \cap [x-\delta, x+\delta])}{\delta} = 2.$$

After that using the inequality (1.1) and the facts $\mu_1(A) = \mu_1(A^*)$, $\mu_1(B) = \mu_1(B^*)$, we see that

$$(1.20)\qquad \mu_1(\lambda A ⊞ (1-\lambda)B) \geqslant \lambda\mu_1(A) + (1-\lambda)\mu_1(B)$$

holds for any measureable $A,B \subset R^1$.
The later inequality is a slight sharpening of the B-M-L inequality (1.1) (see [6] for details).

The first multidimensional extension of (1.2) is due to Dancs and Uhrin [7] (see the case $k = n-1$ of Theorem 3.1 below and remarks in Section 4). The main problem in extending (1.2) (or (1.15)) to many dimensions is the presence of γ and δ. Applying to the right hand side of (1.2) the Hölder inequality, we get a weakening of (1.2) where γ and δ are already not involved. After that taking an induction on dimension, one can easily prove:

If $\alpha > 0$ then, denoting $\omega := \frac{\alpha}{1+n\alpha}$ we have

$$(1.21) \quad \int^*_{R^n} h^{\alpha}(t)dt \geqslant ((\int_{R^n} f^{\alpha}(x)dx)^{\omega} + (\int_{R^n} g^{\alpha}(x)dx)^{\omega})^{1/\omega} .$$

This inequality is due to Dinghas [8].

As to a weakening of (1.15), one first prove that for $a,b,c,d \geqslant 0$ and $\alpha+\beta \geqslant 0$ (see, [9],[10] for details):

$$(1.22) \quad M^{(\lambda)}_{\alpha}(a,b) \cdot M^{(\lambda)}_{\beta}(c,d) \geqslant M^{(\lambda)}_{\frac{\alpha\beta}{\alpha+\beta}}(ac,bd).$$

Now applying this inequality (with $\alpha \geqslant -1$, $\beta = 1$) to the right hand side of (1.15) and performing an induction on the dimension n, we get for $\alpha \geqslant -1/n$ the inequality

$$(1.23) \quad \int_{R^n} h^{(\lambda)}_{\alpha}(t)dt \geqslant M^{(\lambda)}_{\frac{\alpha}{1+n\alpha}}(\int_{R^n} f(x)dx, \int_{R^n} g(x)dx) .$$

(1.23) or a weaker form of it (when "sup" is used instead of "ess-sup" in the definition of $h^{(\lambda)}_{\alpha}(t)$) has been proved and studied by many authors ([5],[7],[11],[12]). Taking in (1.23) $\alpha = +\infty$ and $f := \chi_A$, $g := \chi_B$ we get a following strengthened form of B-M-L inequality ([5])

$$(1.24) \quad \mu_n(\lambda A \boxplus (1-\lambda)B) \geqslant (\lambda \mu_n(A)^{1/n} + (1-\lambda)\mu_n(B)^{1/n})^n ,$$

where μ_n is the L-measure in R^n and the essential sum is defined analogously to (1.17). In what follows we shall refer to the weaker form of (1.23) ("sup " instead of "ess-sup" and $\int^*_{R^n}$ instead of $\int_{R^n}$) as (1.23').

In this paper we prove some multidimensional extensions of (1.15) that sharpen and extend the result in [7]. Similar but weaker results have been proved also in [10]. However, our main aim is to prove an extension of the reduction theorem in [6] proved for the Lebesque measure μ_n. The measures involved will be generated by so called essentially α-concave functions. We note that many density functions of mathematical statistics belong to this class (see Section 4 for details).

2. PRELIMINARY REMARKS

Before turning to multidimensional extensions of (1.15) we shall write an elementary lower estimation for $\nu(\lambda A \boxplus (1-\lambda)B)$, where ν is a measure defined on L-measureable sets $\mathcal{L}_n \subset R^n$. In general, one can expect only that

$$(2.1)\qquad \nu(\lambda A \boxplus (1-\lambda)B) \geqslant \nu(\lambda A)+\nu((1-\lambda)B).$$

(The inequalities of type $m(A+B) \geqslant m(A)+m(B)$ in the more general setting in locally compact Abelian groups are studied in [13]).
We shall see that for well defined classes of measures estimations much better than (2.1) can be proved.
In what follows, we shall frequently use the following identity (an extension of (1.7) to higher dimensions):

$$(2.2)\qquad \int_{R^n} \varphi(x)\,dx = \int_0^{+\infty} \mu_n(\{x \in R^n : \varphi(x) \geqslant \xi\})\,d\xi,$$

where φ is any non-negative L-integrable function (here and everywhere below $\int \cdot\,dx$ means L-integral).

If $f: R^n \to R^1_+$ is an L-integrable function, then it generates a measure

$$(2.3) \qquad \nu(A) := \int_A f(t)\,dt, \quad A \in \mathcal{L}_n.$$

The function $f \equiv 1$ is the generator of the L-measure μ_n (the integrable here means that $\int_{R^n} f(x)\,dx$ is also meaningful but it can have the value $+\infty$). Let $A \in \mathcal{L}_n$ be an essentially bounded set (i.e. $\mathrm{ess}_x\text{-}\sup \chi_A(x) < +\infty$) such that

$$(2.4) \qquad 0 < m_f(A) := \operatorname*{ess-sup}_{x \in R^n} \chi_A(x) f(x) < +\infty.$$

Then denoting

$$(2.5) \qquad A_f(\xi) := \{x \in A : f(x) \geq m_f(A)\,\xi\},$$

we have

$$(2.6) \qquad \nu(A) = m_f(A) \int_0^1 \mu_n(A_f(\xi))\,d\xi.$$

Now, assume that for some $0 \leq \lambda \leq 1$ and $-\infty \leq \alpha \leq +\infty$ the function f is such that

$$(2.7) \qquad f(t) \geq \operatorname*{ess-sup}_{x \in R^n} M_\alpha^{(\lambda)}(f(x/\lambda), f((t-x)/(1-\lambda)))$$

for a.e. $t \in R^n$.

Denote the function on the right side of (2.7) by $p_{\alpha}^{(\lambda)}(t)$, $t \in R^n$ (this is defined for all $t \in R^n$).

Let $A, B \in \mathcal{L}_n$ be two essentially bounded sets and let $t \in R^n$ be such that

$$(2.8) \qquad \mu_n(\lambda A_f(\xi) \cap (t-(1-\lambda)B_f(\xi))) > 0.$$

For any $x \in R^n$ belonging to the intersection in (2.8) we have

$$(2.9) \qquad \min\left\{ \frac{f(x/\lambda)}{m_f(A)}, \frac{f((t-x)/(1-\lambda))}{m_f(B)} \right\} \geq \xi ,$$

i.e. (2.9) holds for all x from a subset of positive μ_n-measure.

This implies, denoting

$$(2.10) \qquad p(t) := \operatorname*{ess-sup}_{x \in R^n} \min\left\{ \frac{f(x/\lambda)}{m_f(A)}, \frac{f((t-x)/(1-\lambda))}{m_f(B)} \right\} ,$$

that

$$(2.11) \qquad (\lambda A \boxplus (1-\lambda)B)_p(\xi) \supseteq \lambda A_f(\xi) \boxplus (1-\lambda)B_f(\xi),$$

hence using the trivial inequality (1.14) we see that

$$(2.12) \qquad \nu(\lambda A \boxplus (1-\lambda)B) \geq M_{\alpha}^{(\lambda)}(m_f(A), m_f(B)) \cdot \int_0^1 \mu_n(\lambda A_f(\xi) \boxplus (1-\lambda)B_f(\xi))\, d\xi .$$

If (2.7) is satisfied for all $0 \leq \lambda \leq 1$, then we call the function f essentially α-concave.
Using (1.24) and the inequality

$$(2.13) \qquad M_\alpha^{(\lambda)}(a,b) \cdot M_\beta^{(\lambda)}(c,d) \geq \min\{\lambda^{\frac{\alpha+\beta}{\alpha\beta}} ac, (1-\lambda)^{\frac{\alpha+\beta}{\alpha\beta}} bd\},$$

which holds for $a,b,c,d \geq 0$ and $\alpha+\beta \leq 0$, $\alpha \cdot \beta < 0$ (see [10]), we get immediately from (2.12)

$$(2.14) \qquad \nu(\lambda A \boxplus (1-\lambda)B) \geq \min\{\lambda^{n+(1/\alpha)}\nu(A), (1-\lambda)^{n+(1/\alpha)}\nu(B)\},$$

where $-\infty \leq \alpha \leq -1$.

This inequality has been proved for $-\infty \leq \alpha \leq -1/n$ in [7] (see Section 4 for details). We see that already the trivial reduction inequality (2.12) is sharper than a known result.

3. AN INTEGRAL INEQUALITY AND REDUCTION THEOREM

Here we use the definitions and notations of the previous sections. First, some new notations.
Let $S \subset R^n$ be a k-dimensional linear subspace, $0 \leq k \leq n$, and $T \subset R^n$ be an (n-k)-dimensional linear subspace such that $S \oplus T = R^n$ (the direct sum). The L-measures in S and T will be denoted by μ_k and μ_{n-k}, respectively.
We shall denote the L-integrals both in S and T by $\int \cdot \, dx$ (the meaning will be clear from the context). By definition $\mu_0(\theta) = 1$, $\mu_0(\emptyset) = 0$.

Given an L-integrable non-negative function $f:R^n \to R^1_+$, we shall denote

$$(3.1) \qquad i(f,u) := \int_S f(x+u)\,dx, \quad u \in T,$$

$$(3.2) \qquad m_k(f) := \operatorname*{ess\text{-}sup}_{u \in T} i(f,u),$$

in particular

$$(3.3) \qquad m_o(f) := \operatorname*{ess\text{-}sup}_{x \in R} f(x), \; m_n(f) := \int_{R^n} f(x)\,dx,$$

and

$$(3.4) \qquad H_f(\xi) := \{ x \in R^n : f(x) \geq m_o(f)\xi \}, \quad 0 \leq \xi \leq 1.$$

Given two L-integrable functions $f,g:R^n \to R^1_+$, denote for $-\infty \leq \alpha \leq +\infty$ and $0 \leq \lambda \leq 1$

$$(3.5) \qquad h^{(\lambda)}_{\alpha}(t) := \operatorname*{ess\text{-}sup}_{x \in R^n} M^{(\lambda)}_{\alpha}(f(x/\lambda), g((t-x)/(1-\lambda))), \; t \in R^n,$$

$$(3.6) \qquad k^{(\lambda)}_{\alpha}(\tau) := \operatorname*{ess\text{-}sup}_{u \in T} M^{(\lambda)}_{\alpha}\left(\frac{i(f,\frac{u}{\lambda})}{m_k(f)}, \frac{i(g,\frac{\tau-u}{1-\lambda})}{m_k(g)}\right), \; \tau \in T,$$

(in particular if $k = n$ then $k^{(\lambda)}_{\alpha}(\theta) = 1$).

Now, we have

<u>Theorem 3.1.</u> The following two inequalities hold

$$(3.7)\qquad \int_{R^n} \operatorname*{ess-sup}_{x\in R^n} \min\left\{\frac{f(x/\lambda)}{m_o(f)}, \frac{g((t-x)/(1-\lambda))}{m_o(g)}\right\} dt \geq$$

$$\geq \int_o^1 \mu_n(\lambda H_f(\xi) \boxplus (1-\lambda) H_g(\xi))\, d\xi ;$$

if $0 < k \leq n$, $\alpha+\beta \geq 0$, $(\alpha\beta/(\alpha+\beta)) \geq -1/k$, then

$$(3.8)\qquad \int_{R^n} h_\alpha^{(\lambda)}(t)\,dt \geq M_{-\beta}^{(\lambda)}(m_k(f), m_k(g)) . \int_T k_\omega^{(\lambda)}(\tau)\, d\tau ,$$

where $\omega := ((1/\alpha) + (1/\beta) + k)^{-1}$. □

Proof. The proof of (3.7) is quite simple.
Denoting by $q(t)$ the integrand in the left hand side of (3.7), we can easily see (analogously to (2.11)) that

$$(3.9)\qquad H_q(\xi) \geq \lambda H_f(\xi) \boxplus (1-\lambda) H_g(\xi), \qquad 0 \leq \xi \leq 1,$$

which gives (3.7).

The case $k = n = 1$ of the inequality (3.8) is a simple consequence of (3.7): apply first (1.20) to the integrand in the right hand side of (3.7), integrate over $0 \leq \xi \leq 1$, take into account (1.7) and finally use (1.14) to get (1.15) (in fact, we have proved (1.15) along these lines also in Section 1). Now, applying (1.22), $\alpha \geq -1$, $\beta = 1$, we get from (1.15) the case $k = n = 1$ of (3.8).

Assume for the moment that (3.8) is already proved for $k = n-1$ $(n > 1)$. Then the case $k = n$ can be derived in the following way.
Let α and β be such that

$$\alpha+\beta \geq 0, \quad \alpha\beta/(\alpha+\beta) \geq \frac{-1}{n-1}. \tag{3.10}$$

Then, we know that

$$\int_{R^n} h_{\alpha}^{(\lambda)}(t)\,dt \geq M_{-\beta}^{(\lambda)}(m_{n-1}(f), m_{n-1}(g)) \cdot \int_T k_{\omega}^{(\lambda)}(\tau)\,d\tau, \tag{3.11}$$

where $\omega = (1/\alpha + 1/\beta + n-1)^{-1}$.

T is now 1-dimensional, hence applying (3.7) for $n = 1$ and after that using (1.20) and (1.7) we get

$$\int_{R^n} h_{\alpha}^{(\lambda)}(t)\,dt \geq M_{-\beta}^{(\lambda)}(m_{n-1}(f), m_{n-1}(g)) \cdot \left(\lambda \frac{m_n(f)}{m_{n-1}(f)} + (1-\lambda)\frac{m_n(g)}{m_{n-1}(g)}\right). \tag{3.12}$$

The conditions (3.10) are equivalent to the conditions

$$\alpha \geq -1/(n-1), \quad \beta \geq \frac{-\alpha}{1+(n-1)\alpha}. \tag{3.13}$$

Let α, β be such that $\alpha+\beta \geq 0$ and $\alpha\beta/(\alpha+\beta) \geq -\frac{1}{n}$, or equivalently

$$\alpha \geq -\frac{1}{n}, \quad \beta \geq \frac{-\alpha}{1+n\alpha}. \tag{3.14}$$

It is clear that for such α and β the conditions (3.13) are also fulfilled, and we can write (3.12) for $-\beta = \frac{\alpha}{1+(n-1)\alpha}$. For this β we have $-\beta \geq -1$ (because $\alpha \geq -\frac{1}{n}$), hence using (1.22) to the right hand side of (3.12) we get

$$(3.15) \qquad \int_{R^n} h_{\alpha}^{(\lambda)}(t)\,dt \geq M^{(\lambda)}_{\frac{\alpha}{1+n\alpha}}(m_n(f),m_n(g)) \geq M^{(\lambda)}_{-\beta}(m_n(f),m_n(g))$$

for any α and β fulfilling (3.14). This proves (3.8) for the pair $(n, k = n)$ (assuming that it holds for $(n, k = n-1)$. Now, take the pairs (n,k), $1 \leq k \leq n$, into lexicographic order, i.e. $(n_1,k_1) \prec (n_2,k_2)$ iff either $n_1 < n_2$ or $\{ n_1 = n_2$ and $k_1 < k_2 \}$.

We get a sequence

$$(3.16) \qquad (1,1) \prec (2,1) \prec (2,2) \prec (3,1) \prec (3,2) \prec (3,3) \prec \ldots$$

We proceed with the proof of (3.8) by induction on this sequence. For $n = k = 1$ (3.8) is true. Assume we have proved (3.8) for all first $N-1$ members of (3.16). Let (n,k) be the N-th member of the sequence. If $k = n$ we are ready by the above reasoning because the case $(n, k = n-1)$ is assumed to be proved by the induction. So assume $1 \leq k \leq n-1$, $n > 1$ and

$$(3.17) \qquad \alpha \geq -\frac{1}{k}, \quad \beta \geq \frac{-\alpha}{1+k\alpha}$$

(these conditions are equivalent to $\beta+\alpha \geq 0$, $\alpha\beta/(\alpha+\beta) \geq -\frac{1}{k}$).

We can write using (1.22)

$$(3.18)\qquad M_{\beta}^{(\lambda)}(1/m_k(f),\ 1/m_k(g))\cdot\int_{R^n} h_{\alpha}^{(\lambda)}(t)\,dt \geqslant$$

$$\geqslant \int_T \operatorname*{ess-sup}_{\in T}\Big(\int_S \operatorname*{ess-sup}_{x\in S} M^{(\lambda)}_{\frac{\alpha\beta}{\alpha+\beta}}\Big(\frac{1}{m_k(f)} f\Big(\frac{x}{\lambda}+\frac{u}{\lambda}\Big), \frac{1}{m_k(g)}\, g\Big(\frac{\tau-u}{1-\lambda}+\frac{z-x}{1-\lambda}\Big)\Big)\,dz\Big)d\tau .$$

Applying (3.8) case (k,k) to the inner integral $\int_S \ldots dz$, where now $\alpha\beta/(\alpha+\beta)$ plays the role of α, we get that the right hand side of (3.18) is not less then

$$(3.19)\qquad \int_T \operatorname*{ess-sup}_{u\in T} M_{\omega}^{(\lambda)}\Big(\frac{i(f,\frac{u}{\lambda})}{m_k(f)}, \frac{i(g,\frac{\tau-u}{1-\lambda})}{m_k(g)}\Big)\,d\tau.$$

By this (3.8) and the whole theorem is proved. ■

Let $f:R^n \to R_+^1$ be L-integrable function and $A,B \subset R^n$ be two essentially bounded L-measurable sets.
Denote

$$(3.20)\qquad m_k(A) := m_k(\chi_A\cdot f),\quad m_k(B) := m_k(\chi_B\cdot f)$$

and for $0 \leqslant \xi \leqslant 1$

$$(3.21)\qquad \begin{aligned} A(\xi) &:= \{u\in T:\ i(\chi_A\cdot f,u) \geqslant m_k(A)\,\xi\} \\ B(\xi) &:= \{u\in T:\ i(\chi_B\cdot f,u) \geqslant m_k(B)\,\xi\}. \end{aligned}$$

Now we have

<u>Theorem 3.2.</u> If $-\frac{1}{k} \leq \alpha \leq +\infty$, $0 \leq \lambda \leq 1$, $0 \leq k \leq n$ and A,B and f are such that

(3.22) $$0 < m_k(A),\ m_k(B) < +\infty$$

and

(3.23) $$f(t) \geq \underset{x\in R^n}{\text{ess-sup}}\ M_\alpha^{(\lambda)}(f(x/\lambda), f(\tfrac{t-x}{1-\lambda})) \quad \text{for a.e. } t\in R^n,$$

then for the measure ν generated by the f we have

(3.24) $$\nu(\lambda A \boxplus (1-\lambda)B) \geq M_\beta^{(\lambda)}(m_k(A), m_k(B)),$$

$$\int_0^1 \mu_{n-k}(\lambda A(\xi) \boxplus (1-\lambda) B(\xi))\, d\xi,$$

where

$$\beta = \frac{\alpha}{1+k\alpha}. \quad \square$$

<u>Proof</u>. After some technical observations, (3.24) will follow from the previous theorem.
Denote the right hand side of (3.23) by $r_\alpha^{(\lambda)}(t)$ and

(3.25) $$s_\alpha^{(\lambda)}(t) := \underset{x\in R^n}{\text{ess sup}}\ M_\alpha^{(\lambda)}(\chi_A(\tfrac{x}{\lambda})\cdot f(\tfrac{x}{\lambda}), \chi_B(\tfrac{t-x}{1-\lambda})\cdot f(\tfrac{t-x}{1-\lambda})).$$

First we prove that

$$(3.26)\qquad \int_{\lambda A \boxplus (1-\lambda) B} r_{\alpha}^{(\lambda)}(t)\,dt = \int_{R^n} s_{\alpha}^{(\lambda)}(t)\,dt.$$

It is clear that

$$(3.27)\qquad \chi_{\lambda A \boxplus (1-\lambda) B}(t) = \underset{x \in R^n}{\text{ess-sup}}\, M\left(\chi_A\left(\tfrac{x}{\lambda}\right), \chi_B\left(\tfrac{t-x}{1-\lambda}\right)\right)$$

where M is defined by (1.13).

Now, using the trivial inequalities

$$(3.28)\qquad \underset{x}{\text{ess-sup}}\, \varphi(x) \cdot \underset{x}{\text{ess-sup}}\, \psi(x) \geqslant \underset{x}{\text{ess-sup}}(\varphi(x) \cdot \psi(x)),$$

$$(3.29)\qquad M(a,b) M_{\alpha}^{(\lambda)}(c,d) \geqslant M_{\alpha}^{(\lambda)}(ac,bd),$$

we see that the left hand side of (3.26) is not less than the right hand side.
On the other hand, if for given t and x

$$(3.30)\qquad M_{\alpha}^{(\lambda)}\left(\chi_A\left(\tfrac{x}{\lambda}\right) \cdot f\left(\tfrac{x}{\lambda}\right), \chi_B\left(\tfrac{t-x}{1-\lambda}\right) \cdot f\left(\tfrac{t-x}{1-\lambda}\right)\right) > 0,$$

then clearly $\chi_A\left(\frac{x}{\lambda}\right) = \chi_B\left(\frac{t-x}{1-\lambda}\right) = 1$ ($M_{\alpha}^{(\lambda)}$ is the "extended" mean), i.e.

(3.31) $$x \in \lambda A \cap (t-(1-\lambda)B).$$

The definition of ess-sup shows that if $s_{\alpha}^{(\lambda)}(t) > 0$ then there is a set E such that $\mu_n(E) > 0$ and for all $x \in E$ (3.30) holds, i.e. $t \in \lambda A \boxplus (1-\lambda)B$. This shows that the right hand side of (3.26) is not less than the left hand side. Apply now Theorem 3.1 to functions $\chi_A \cdot f$ and $\chi_B \cdot f$.
Similarly to (2.11), we can easily check that

(3.32) $$C(\xi) \geq \lambda A(\xi) \boxplus (1-\lambda)B(\xi), \quad 0 \leq \xi \leq 1,$$

where

(3.33) $$C(\xi) := \{ \tau \in T : k_{\omega}^{(\lambda)}(\tau) \geq \xi \}.$$

Hence

(3.34) $$\int_T k_{\omega}^{(\lambda)}(\tau)d\tau \geq \int_0^1 \mu_{n-k}(A(\xi) \boxplus (1-\lambda)B(\xi))d\xi.$$

By this (3.24) is proved (we apply (3.8) in the sharpest case $\beta = \frac{-\alpha}{1+k\alpha}$). ■

4. CONCLUDING REMARKS

1. The case $k = n-1$ of (3.8) has been principally proved in [7] (more exactly, the "sup" was used instead of "ess-sup"). The "ess-sup"-case of these inequalities needs some additional care.
For any $0 \leq k \leq n$, a weaker form of (3.8) has been first formulated and proved in [10].

For the domain $-\infty \leq \alpha \leq -\frac{1}{n}$ a following inequality has been also proved in [7]:

$$(4.1) \quad \int_{R^n} \sup_{\lambda x+(1-\lambda)y=t} M_\alpha^{(\lambda)}(f(x),q(y))dt \geq$$

$$\geq \min\left\{ \lambda^{n+(1/\alpha)} \int_{R^n} f(x)dx, (1-\lambda)^{n+(1/\alpha)} \int_{R^n} g(x)dx \right\}$$

(under the assumption the f and g are such that the function $\sup M_\alpha^{(\lambda)}$ ... is integrable).
Using the inequality (2.13), we can see easily that in each of the domains $-1/k \leq \alpha \leq -\frac{1}{k-1}$, $k = 0,1,2,\ldots,n-1$, the inequality (3.8) gives results that are "from both sides" sharper than (4.1).

The inequality (1.23') has been successfully applied in many branches of mathematics: stochastic programming (the case $\alpha = 0$, [14],[15]); mathematical statistics ($\alpha \geq -\frac{1}{n}$, [16]); theory of probability ($\alpha \geq -\frac{1}{n}$, [12]); theory of diffusion equations ($\alpha = 0$; [5]).
Some principally new results concerning the convolution of unimodal functions has been proved using both (1.23') and (4.1) ([9]).
The conditions of equality in (1.23') has been investigated in [11] ($\alpha \geq -\frac{1}{n}$) and in [15] ($\alpha = 0$).
In fact, using another method of the paper [1], one can prove sufficient and necessary conditions of equality in (1.23') for $n = 1$, $\alpha \geq -1$ (see [17], p. 131). The proof of such conditions for the sharper inequality (1.15) seems to be a more difficult problem and this has been done only for upper semi-continuous functions f and q (see [17]).

2. As to lower estimations for $\nu(\lambda A \boxplus (1-\lambda)B)$ (or for $\nu_*(\lambda A+(1-\lambda)B)$, where ν_* is the inner ν-measure) only the case $k = n$ of (3.24) has been studied ([5],[7],[11],[12],[15]). For $\alpha \leq -\frac{1}{n}$ the inequality (4.1) was used in [7] to prove a lower estimation for $\nu_*(\lambda A+(1-\lambda)B)$. Using (2.13), our inequality (3.24) can be used to write in each of the domains $-\frac{1}{k} \leq \alpha \leq -\frac{1}{k-1}$, $k = 0,1,\ldots,n-1$, inequalities which are "from both sides" sharper than those in [7]. The case $f \equiv 1$ (L-measure) and $k = n-1$ of (3.24) (more exactly taking $\mu_{n*}(\lambda A+(1-\lambda)B)$ instead of $\mu_n(\lambda A \boxplus (1-\lambda)B)$ is essentially due to Bonnesen (this is the classical sharpening of the B-M-L inequality, see [6] for details).
The proper geometric content of inequalities is not quite clear yet, it is so even in the case $f \equiv 1$ (L-measure). For L-measure μ_n the quantities $m_k(A)$ are called in the geometry "inner transversal measures" ("innere Quermass", see e.g. [18]). An interesting theme of study would be to compare (3.24) (at least the case $f \equiv 1$) with some other results in geometry that use transversal measures (see [6] for more details).
The study of more general measures seems to be interesting as well. Let us recall that the function f satisfying (3.23) for all $0 \leq \lambda \leq 1$ we called essentially α-concave. If in (3.23) we take "sup" instead of "ess-sup" and $t \in R^n$ instead of a.e. $t \in R^n$ (let us call these functions α-concave), we get a more restricted class of functions.

Many important density functions in statistics are known to be α-concave. For example, the density functions of normal distribution, Wishart distribution, multidimensional β-distribution, Dirichlet-distribution are known to be 0-concave (log-concave, see [14],[15]), while those of the Pareto-distribution, Student-t-distribution, F-distribution are known to be α-concave for some $\alpha < 0$ ([11]).

3. The inequalities (3.7),(3.8) and (3.24) are to be considered as tools for getting new lower estimations for $\int h_{\alpha}^{(\lambda)}(t)dt$ and $\nu(\lambda A \boxplus (1-\lambda)B)$. Say, we can apply them successively for a series of "nested" subspaces S. One can imagine, that we would get a plenty of inequalities of a pretty complicated from (a simple example of this procedure can be found in [6]). Further research will show, whether these complicated (but very sharp) inequalities can be used in solving some interesting problems.

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Одна редукционная теорема для мер суммы двух множеств в R^n

Б. Ухрин

Резюме

Пусть μ_k есть Лебеговая мера в R^k, $0 \le k \le n$, и определим для двух измеримых множеств $A, B \subset R^k$ их естественную выпуклую комбинацию как $(\lambda A \oplus (1-\lambda) B) := \{z \in R^k : \mu_k(\lambda A \cap (z-(1-\lambda)B)) > 0\}$, $0 \le \lambda \le 1$. Пусть $S \subset R^n$ и $T \subset R^n$ такие линейные подпространства размерности k u$(n-k)$, что $S \oplus T = R^n$. Автор в предыдущей статье /Coll. Math. Soc. J. Bolyai, Vol 48, North-Holland, 1987, 551-571/ дал нижную оценку для $\mu_n(\lambda A \oplus (1-\lambda)B)$ используя естественные супремумы функций $\varphi(u) := \mu_k(A \cap (S+u))$, $\psi(u) := \mu_k(B \cap (S+u))$, $u \in T$, и μ_{n-k}-мер естественных выпуклых комбинаций верных множеств уровня этих функций. Статья распространяет этот результат на более общие меры ν_n, которые индуцированы неотрицательными функциями /определенными на R^n/ из хорошо определенных субклассов одновершинных /унимодальных/ функций /субклассц т.н. α-вогнутых функций/. Для доказательства результата автор сперва доказывает n-мерное расширение классического 1-мерного интегрального неравенства Хенстока и Мацбита /Henstock, Macbeat, Proc. London Math. Soc., Ser III., 3 (1953), 182-194/. Результат для ν_n, а также доказанное интегральное неравенство уточняют и обобщают все предыдущие результаты похожего типа.

EGY REDUKCIÓS TÉTEL ÖSSZEG-HALMAZOK MÉRTÉKEIRE AZ R^n-BEN

Uhrin B.

Összefoglaló

Legyen μ_k az R^k-ban levő Lebesgue-mérték, $0 \leq k \leq n$. Két $A,B \subset R^k$ L-mérhető halmazra definiáljuk $\lambda A \boxplus (1-\lambda)B := \{z \in R^k : \mu_k(\lambda A \cap (z-(1-\lambda)B)) > 0\}$, $0 \leq \lambda \leq 1$ (a halmazok "lényeges konvex kombinációja"). Legyenek $S \subset R^n$ és $T \subset R^n$ k- ill. $(n-k)$-dimenziós alterek, amelyek direkt összegben kifeszitik a teret. A szerző egy előbbi cikkében /Coll. Math. Soc. J. Bolyai, Vol 48, North-Holland, 1987, 551-571/ a $\mu_n(\lambda A \boxplus (1-\lambda)B)$ mértékre egy alsó becslést adott, amelyben a $\varphi(u) := \mu_k(A \cap (S+u))$, $\psi(u) := \mu_k(B \cap (S+u))$, $u \in T$, függvények lényeges supremumai ill. ezen függvények alsó szinthalmazaira vonatkozó lényeges konvex kombinációinak μ_{n-k} mértékei szerepelnek. Jelen cikkben a szerző ezt az eredményt olyan ν_n mértékre terjeszti ki, amelyeket az R^n-en definiált unimodális függvényosztály bizonyos jól definiálható alosztályaiban levő függvények generálnak (az u.n. α-konkáv függvények).
Az eredmény bizonyitásához a szerző először egy klasszikus 1-dimenziós integrál-egyenlőtlenség (Henstock, Macbeath, Proc. London Math. Soc., Ser III. 3 (1953), 182-194) n-dimenziós kiterjesztését bizonyitja be. Mind a bizonyitott n-dimenziós integrál-egyenlőtlenség, mind a ν_n-re vonatkozó eredmény az eddigi hasonló eredményeket élesiti és általánositja.