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DOUBLY ORDERED LINEAR RANK STATISTICS

By

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1. Introduction

Let $N=m+n$, where m and n are positive integers. Let the real numbers $x_1, \ldots, x_N$ be pairwise different from each other. Let Rank x_k and rank x_k denote the rank of x_k within the sequences $x_1, \ldots, x_N$, and $x_1, \ldots, x_n$, respectively. In other words, if the rearrangement according to size $z_1 < \ldots < z_N$ of those numbers $x_k = z_{r_k}$, and if the rearrangement according to size $y_1 < \ldots < y_n$ of the numbers $x_1, \ldots, x_n$, $x_k = y_{i_k}$, then we say that x_k has rank r_k and i_k with respect to these orders, and we write $\text{Rank}\, x_k = r_k$, and $\text{rank}\, x_k = i_k$ $(k=1, \ldots, m)$, respectively.

Denote by $\Pi_m^{(N)}$ the set of all $(r_1, \ldots, r_m)$ chosen without repetition from the elements $1, \ldots, N$, and by P_m the set of all permutations of the elements $1, \ldots, m$ without repetition.

Let the distribution functions of the real random variables $X_1, \ldots, X_N$ be continuous. Then $P(X_j = X_k) = 0,\ j \neq k$.

Denote by $\{r_1, \ldots, r_m\}$ and $[i_1, \ldots, i_m]$ the vectors with the components $r_1, \ldots, r_m$ and $i_1, \ldots, i_m$, where $(r_1, \ldots, r_m) \in \Pi_m^{(N)}$, $(i_1, \ldots, i_m) \in P_m$, respectively.

DEFINITION 1.2. The vector $\{r_1, \ldots, r_m\}$ is said to be the outer-rank of the random variables $X_1, \ldots, X_m$ with respect to the random variables $X_1, \ldots, X_N$, if

$$\{\text{Rank}\, X_1, \ldots, \text{Rank}\, X_m\} = \{r_1, \ldots, r_m\}.$$

DEFINITION 1.1. The vector $[i_1, \ldots, i_m]$ is said to be the inner-rank of the random variables $Y_1, \ldots, Y_m$, if

$$[\text{rank}\, Y_1, \ldots, \text{rank}\, Y_m] = [i_1, \ldots, i_m].$$

Obviously, the random events $\{r_1, \ldots, r_m\}$ and $[i_1, \ldots, i_m]$ are independent if the random variables $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_m$ are independent, i.e. in this case

$$P(\{r_1, \ldots, r_m\}, [i_1, \ldots, i_m]) = P(\{r_1, \ldots, r_m\}) P([i_1, \ldots, i_m]).$$

By the help of this formula we get the following theorem on the basis of [3] (p. 369, Satz 10).

THEOREM 1.1. *Let the random variables $Z_1 = (X_1, Y_1), \ldots, Z_m = (X_m, Y_m)$ and the random variables $X_{m+1}, \ldots, X_N$ be given. If the random vector variables $(X_1, \ldots, X_N)$ and $(Y_1, \ldots, Y_m)$ are independent and if the joint distribution functions of the random variables $X_1, \ldots, X_N$ and the random variables $Y_1, \ldots, Y_m$ are symmetric functions of* *their variables, and they are continuous in each of the variables, then*

$$P(\{r_1, \ldots, r_m\}, [i_1, \ldots, i_m]) = \frac{1}{m!(n+1)\ldots(n+m)},$$

$$(r_1, \ldots, r_m) \in \Pi_m^{(N)}, \quad (i_1, \ldots, i_m) \in P_m.$$

The conditions of Theorem 1.1 will be satisfied if $X_1, \ldots, X_N$ and if $Y_1, \ldots, Y_m$ are samples with continuous distribution functions, and these random variables are independent.

Let the matrices

$$(1.1) \qquad A_j = \begin{pmatrix} a_{11}^{(j)} \ldots a_{1N}^{(j)} \\ \cdot \quad \ldots \quad \cdot \\ a_{m1}^{(j)} \ldots a_{mN}^{(j)} \end{pmatrix} \quad (j = 1, \ldots, m)$$

with real elements be given, and let $A = A_1 \ldots A_m$ be the $m \times mN$ matrix with blocks (1.1).

On the basis of Theorem 1.1 we give the following definition.

DEFINITION 1.3. The random variable $X_{m,n}^{(N)}$ is said to be a doubly ordered linear rank statistics generated by the matrix A if

$$(1.2) \qquad P(X_{m,n}^{(N)} = a_{i_1 r_1}^{(1)} + \ldots + a_{i_m r_m}^{(m)}) = \frac{1}{m!(n+1)\ldots(n+m)},$$

where $(r_1, \ldots, r_m)$ and $(i_1, \ldots, i_m)$ run over the sets $\Pi_m^{(N)}$ and P_m, respectively.

Let the random vector variables $Z_1 = (X_1, Y_1), \ldots, Z_m = (X_m, Y_m)$ and the random variables $X_{m+1}, \ldots, X_N$ be given. Suppose that the random variables $X_1, \ldots, X_N$, $Y_1, \ldots, Y_m$ are independent with continuous distribution functions. Suppose that $X_1, \ldots, X_m$ and $X_{m+1}, \ldots, X_N$ are samples with distribution functions $F(x)$ and $G(x)$, respectively. Then the doubly ordered linear rank statistics $X_{m,n}^{(N)}$ defined by (1.2) give us the possibility to decide on the acceptance or the rejection of the joint hypothesis

a) the second components of the random vector variables $Z_1, \ldots, Z_m$ have a common distribution function;

b) $F(x) = G(x)$, $x \in R_1$.

If all rows of the matrix A are equal, then $X_{m,n}^{(N)}$ give us the possibility to take a decision on the acceptance or rejection of the hypothesis b).

Denote by $\Psi_{m,n}^{(N)}(t)$ the characteristic function of the random variable $X_{m,n}^{(N)}$ defined by (1.2).

The aim of this paper is to investigate the characteristic function $\Psi_{m,n}^{(N)}(t)$. Beside the Introduction the paper contains two sections. In Section 2 the characteristic function $\Psi_{m,n}^{(N)}(t)$ will be approximated by the permanents of simpler characteristic functions. On the basis of this approximation theorem we give an asymptotic formula for $\Psi_{m,n}^{(N)}(t)$. The theorems of Section 3 are dealing with the construction of doubly ordered linear rank statistics with given limit distribution. To do this it is necessary to extend te well-known Koksma's inequality for arbitrary distributions.

2. The characteristic function of the doubly ordered linear rank statistics

First of all we prove the following theorem.

THEOREM 2.1. *Let $\Psi_{m,n}^{(N)}(t)$ be the characteristic function of the doubly ordered linear rank statistics defined by* (1.2). *Let*

$$\varphi_{jk}^{(N)}(t)=\frac{1}{N}\sum_{n=1}^{N}e^{ita_{jn}^{(k)}}\quad (j,k=1,\dots,m)$$

and

$$\Phi_m^{(N)}(t)=\begin{pmatrix}\varphi_{11}^{(N)}(t)\dots\varphi_{1m}^{(N)}(t)\\ \cdot\quad\dots\quad\cdot\\ \varphi_{m1}^{(N)}(t)\dots\varphi_{mm}^{(N)}(t)\end{pmatrix}. \tag{2.1}$$

Then

$$\left|\Psi_{m,n}^{(N)}(t)-\frac{N^m}{(n+1)\dots(n+m)}\frac{1}{m!}\operatorname{Per}\Phi_m^{(N)}(t)\right|\leq\frac{N^m}{(n+1)\dots(n+m)}-1$$

for $t\in R_1$.

PROOF. If $M=(a_{jk})$ is a square matrix of order n with complex numbers as its elements, then the permanent of M, denoted by $\operatorname{Per}M$, is defined as follows:

$$\operatorname{Per}M=\sum_{(i_1,\dots,i_m)}a_{1i_1},\dots a_{mi_m},$$

where $(i_1,\dots,i_m)$ runs over the full symmetric group.

On the basis of (1.2) we get that

$$\Psi_{m,n}^{(N)}(t)=\frac{1}{m!(n+1)\dots(n+m)}\sum_{(r_1,\dots,r_m)\in\Pi_m^{(N)}}$$

$$\sum_{(i_1,\dots,i_m)\in P_m}\exp\{it(a_{i_1r_1}^{(1)}+\dots+a_{i_mr_m}^{(m)})\}=\frac{1}{m!(n+1)\dots(n+m)}\cdot$$

$$\sum_{(r_1,\dots,r_m)\in\Pi_m^{(N)}}\operatorname{Per}\begin{pmatrix}e^{ita_{1r_1}^{(1)}}\dots e^{ita_{1r_m}^{(m)}}\\ \cdot\quad\dots\quad\cdot\\ e^{ita_{mr_1}^{(1)}}\dots e^{ita_{mr_m}^{(m)}}\end{pmatrix}.$$

Let $B=B_1\dots B_m$ the $m\times mN$ matrix with blocks $B_1,\dots,B_m$, where the B_j matrix is defined as follows. The elements of the j-th row are equal to one, while the remaining elements are equal to zero. Therefore the permanent of the $m\times m$ submatrix M of B is different from zero if and only if M has one column from each of the matrices $B_1,\dots;B_m$. The number of such submatrices of B is N^m and the permanent of these is equal to one.

Let

$$(2.2)\qquad A_j(t)=\begin{pmatrix} e^{ita_{11}^{(j)}}\dots e^{ita_{1N}^{(j)}} \\ \cdot \quad \dots \quad \cdot \\ e^{ita_{m1}^{(j)}}\dots e^{ita_{mN}^{(j)}} \end{pmatrix}\quad (j=1,\dots,m),$$

and let $A(t)=A_1(t)\dots A_m(t)$ be the $m\times m^N$ matrix with blocks (2.2).
We have

$$(2.3)\qquad \mathrm{Per}\,(A(t)B^*)=N^m\,\mathrm{Per}\,\Phi_m^{(N)}(t),$$

where $\Phi_m^N(t)$ is defined by (2.1). On the other hand, using the Cauchy-Binet expansion theorem,

$$(2.4)\qquad \mathrm{Per}\,(A(t)B^*)=m!(n+1)\dots(n+m)\Psi_{m,n}^{(N)}(t)+H(t),$$

where $H(t)$ is equal to the sum of the permanents of those $m\times m$ submatrices of $A(t)$, which have one and only one column from each of the matrices (2.2), but at least two columns have the same column index. The number of such matrices is equal to $N^m-(n+1)\dots(n+m)$. Since the moduli of the elements of these matrices are equal to one, the moduli of the permanents of these matrices are less than or equal to $m!$ Thus on the basis of (2.3) and (2.4) we get the statement of Theorem 2.1.

By the help of Theorem 2.1 we obtain easily the following theorem.

THEOREM 2.2. *Let the sequence* $\{X_{m,n}^{(N)}\}_{n=1}^{\infty}$ *of doubly ordered linear rank statistics be given, where* $X_{m,n}^{(N)}$ *is generated by the* $m\times mN$ *matrix* $A^{(N)}=A_1^{(N)}\dots A_m^{(N)}$ *with blocks*

$$A_j^{(N)}=\begin{pmatrix} a_{11}^{(j)}(N)\dots a_{1N}^{(j)}(N) \\ \cdot \quad \dots \quad \cdot \\ a_{m1}^{(j)}(N)\dots a_{mN}^{(j)}(N) \end{pmatrix}\quad (j=1,\dots,m).$$

Then uniformly in $t\in R_1$ *we have*

$$\lim_{n\to\infty}\left[\Psi_{m,n}^{(N)}(t)-\frac{N^m}{(n+1)\dots(n+m)}\frac{1}{m!}\,\mathrm{Per}\,\Phi_m^{(N)}(t)\right]=0,$$

where $\Psi_{m,n}^{(N)}(t)$ *is the characteristic function of* $X_{m,n}^{(N)}$.

3. Doubly ordered linear rank statistics with given limit distribution

This section consists of two parts. In the first one we extend the well-known Koksma's inequality for arbitrary distribution. In the second part we use this inequality to construct doubly ordered linear rank statistics with given limit distribution.

a) Let $H(a,b)$ be the set of the strictly monoton increasing continuous distribution functions $F(x)$, for which $a=\sup\{x\in R_1|F(x)=0\}$, $b=\inf\{x\in R_1|F(x)=1\}$, where $a<b$ are real numbers.

Let the sequence

$$(3.1)\qquad \omega=\{x_n\}_{n=1}^{\infty},\quad x_n\in[a,b)$$

be given. For a positive integer N and a subset E of $[a, b)$ let the counting function $A(E; N)$ be defined as the number of terms x_n, $1 \leqq n \leqq N$, for which $x_n \in E$.

DEFINITION 3.1. Let $F(x) \in H(a, b)$. The sequence (3.1) is said to be $F(x)$-distributed, if for every pair α, β of real numbers with $a \leqq \alpha < \beta \leqq b$ we have

$$\lim_{n \to \infty} \frac{1}{N} A([\alpha, \beta); N) = F(\beta) - F(\alpha).$$

As we said we shall use the $F(x)$-distributed sequences in the second part of this chapter to construct doubly ordered linear rank statistics with given asymptotic. Therefore in this first part of this chapter we compile the necessary definitions and theorems, which will be used in the above mentioned constructions. Definition 3.1 can be found in [2] (p. 54), but the following definitions and theorems only in the case of uniformly distributed sequences. The proofs of the following theorems are almost the same as in the case of uniformly distributed sequences. Therefore we shall refer only to the corresponding pages of the book [2].

DEFINITION 3.2. For a finite sequence $x_1, \ldots, x_N$ of real numbers, $x_n \in [a, b)$

$$D_N(F) = D_N(x_1, \ldots, x_N | F) = \sup_{a < \alpha \leqq b} \left| \frac{1}{N} A([a, \alpha); N) - F(\alpha) \right|$$

is said to be the discrepancy of the given sequence with respect to $F(x) \in H(a, b)$.

THEOREM 3.1. *Let $x_1 \leqq x_2 \leqq \ldots \leqq x_N$ be N numbers in $[a, b)$. Then their discrepancy with respect to $F(x) \in H(a, b)$ is given by*

$$D_N(F) = \max_{j=1, \ldots, N} \max \left(\left| F(x_j) - \frac{j}{N} \right|, \ \left| F(x_j) - \frac{j-1}{N} \right| \right) =$$

$$= \frac{1}{2N} + \max_{j=1, \ldots, N} \left| F(x_j) - \frac{2j-1}{2N} \right|.$$

PROOF. The proof is the same as in the case of uniformly distributed sequences ([2], p. 91, Theorem 1.4), but it is necessary to use that $F(x) \in H(a, b)$.

For a finite sequence $x_1, \ldots, x_N$ of real numbers, $x_n \in [a, b)$, let

$$D_N^*(F) = D_N^*(x_1, \ldots, x_N | F) = \sup_{a \leqq \alpha < \beta \leqq b} \left| \frac{1}{N} A([\alpha, \beta); N) - [F(\beta) - F(\alpha)] \right|.$$

LEMMA 3.1. *The sequence* (3.1) *is $F(x)$-distributed if and only if* $\lim_{N \to \infty} D_N^*(\omega | F) = 0$.

PROOF. The proof is the same as in the case of uniformly distributed sequences ([2], p. 89, Theorem 1.1) if we take into consideration that $F(x) \in H(a, b)$.

LEMMA 3.2. *The quantities $D_N(F)$ and $D_N^*(F)$ are related by the inequality*

$$D_N(F) \leqq D_N^*(F) \leqq 2D_N(F).$$

PROOF. The proof is the same as in the case of uniformly distributed sequences ([2], p. 91, Theorem 1.3).

As an immediate consequence of Lemmata 1 and 2, we get the following theorem.

THEOREM 3.2. *The sequence* (3.1) *is $F(x)$-distributed if and only if*

$$\lim_{N\to\infty} D_N(\omega|F) = 0.$$

LEMMA 3.3. *Let $x_1 \leqq \ldots \leqq x_N$ be given N points in $[a, b)$, and let g be a function of bounded variation on $[a, b]$. Then with $x_0 = a$, $x_{N+1} = b$, we have the identity*

$$\frac{1}{N}\sum_{n=1}^{N} g(x_n) - \int_a^b g(t)\,dF(t) = \sum_{n=0}^{N}\int_{x_n}^{x_{n+1}}\left(F(t)-\frac{n}{N}\right)dg(t),$$

where $F(x)\in H(a, b)$.

PROOF. (See [2], p. 143, Lemma 5.1.) Using integration by parts and Abel's summation formula, we get

$$\sum_{n=0}^{N}\int_{x_n}^{x_{n+1}}\left(F(t)-\frac{n}{N}\right)dg(t) = \int_a^b F(t)\,dg(t) - \sum_{n=0}^{N}\frac{n}{N}[g(x_{n+1})-g(x_n)] =$$

$$= [F(t)g(t)]_a^b - \int_a^b g(t)\,dF(t) + \frac{1}{N}\sum_{n=0}^{N-1} g(x_{n+1}) - g(b) =$$

$$= \frac{1}{N}\sum_{n=1}^{N} g(x_n) - \int_a^b g(t)\,dF(t),$$

because $F(b)g(b) = g(b)$, $F(a)g(a) = 0$.

In the following we prove an extension of the Koksma's inequality ([2], p. 143, Theorem 5.1).

THEOREM 3.3. *Let $F(x)\in H(a, b)$. Let g be a function of bounded variation $V(g)$ on $[a, b]$, and suppose we are given N points $x_1, \ldots, x_N$ on $[a, b)$ with discrepancy $D_N(F)$. Then*

$$\left|\frac{1}{N}\sum_{m=1}^{N} g(x_n) - \int_a^b g(t)\,dF(t)\right| \leqq V(g)\,D_N(F).$$

PROOF. Without loss of generality, we may assume that $x_1 \leqq \ldots \leqq x_N$. Thus, we can apply Lemma 3.3. For fixed n with $0 \leqq n \leqq N$, because $F(x)\in H(a, b)$, we have

$$\left|F(t)-\frac{n}{N}\right| \leqq \max\left(\left|F(x_n)-\frac{n}{N}\right|, \left|F(x_{n+1})-\frac{n}{N}\right|\right) \leqq D_N(F)$$

for $x_n \leqq t \leqq x_{n+1}$ by Theorem 3.1, and the desired inequality follows immediately.

COROLLARY 3.1. *Let* $F(x)\in H(a, b)$. *Let* g *be a function of bounded variation* $V(g)>0$ *on* $[a, b]$, *and suppose we are given the sequence* (3.1). *Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(x_n) = \int_a^b g(t)\, dF(t)$$

holds if and only if the sequence (3.1) *is* $F(x)$*-distributed.*

PROOF. Using Theorems 3.2 and 3.3, the statement follows immediately.

b) In this second part of the section we give a procedure to construct doubly ordered linear rank statistics with given limit distribution. To this construction we need some lemmata.

LEMMA 3.4. *If* f *is continuous function of bounded variation* $V(f)$ *on* $[a, b]$, *then* $V(e^{itf})\leqq|t|V(f)$, $t\in R_1$.

PROOF. See [1], Lemma 3.1.

Using Theorem 3.3 and Lemma 3.4, we get the following lemma.

LEMMA 3.5. *Let* f *be a continuous function of bounded variation* $V(f)$ *on* $[a, b]$, *and suppose we are given* N *points* $x_1, \ldots, x_N$ *in* $[a, b)$ *with discrepancy* $D_N(F)$ *with respect to the distribution function* $F(x)\in H(a, b)$. *If*

$$\varphi^{(N)}(t) = \frac{1}{N} \sum_{n=1}^{N} e^{itf(x_n)},$$

then

$$\left|\varphi^{(N)}(t) - \int_a^b e^{itf(x)}\, dF(x)\right| \leqq |t| V(f)\, D_N(F)$$

for $t\in R_1$.

LEMMA 3.6. *Let* f_j *be a continuous function of bounded variation* $V(f_j)$ *on* $[a, b]$, *and suppose we are given* N *points* $x_1^{(j)}, \ldots, x_N^{(j)}$ *in* $[a, b)$ *with discrepancy* $D_N^{(j)}(F_j)$ *with respect to the distribution function* $F_j(x)\in H(a, b)$, *where* $j=1, \ldots, m$. *Moreover we define the matrices*

$$\Phi_m(t) = \begin{pmatrix} \varphi_{11}(t) \ldots \varphi_{1m}(t) \\ \cdot \quad \ldots \quad \cdot \\ \varphi_{m1}(t) \ldots \varphi_{mm}(t) \end{pmatrix}, \tag{3.2}$$

and

$$\Phi_m^{(N)}(t) = \begin{pmatrix} \varphi_{11}^{(N)}(t) \ldots \varphi_{1m}^{(N)}(t) \\ \cdot \quad \ldots \quad \cdot \\ \varphi_{m1}^{(N)}(t) \ldots \varphi_{mm}^{(N)}(t) \end{pmatrix}$$

with the elements

$$\varphi_{jk}(t) = \int_a^b e^{itf_j(x)}\, dF_k(x) \quad (j, k = 1, \ldots, m) \tag{3.3}$$

and

$$\varphi_{jk}^{(N)}(t) = \frac{1}{N} \sum_{n=1}^{N} e^{itf_j(x_n^{(k)})} \quad (j, k = 1, \ldots, m), \tag{3.4}$$

respectively. Then

$$\frac{1}{m!}|\operatorname{Per}\Phi_m^{(N)}(t)-\operatorname{Per}\Phi(t)| \leqq \frac{|t|}{m}\left[\sum_{j=1}^{m} V(f_j)\right]\left[\sum_{j=1}^{m} D_N^{(j)}(F_j)\right]$$

for $t\in R_1$.

Proof. Let us introduce the notation

$$\Delta_j=\frac{1}{m!}\operatorname{Per}\begin{pmatrix}\varphi_{11}(t)\dots\varphi_{1j-1}(t) & \varphi_{1j}^{(N)}(t)-\varphi_{1j}(t) & \varphi_{1j+1}^{(N)}(t) & \dots\varphi_{1m}^{(N)}(t)\\ \cdot\ \dots\ \cdot & \cdot & \cdot & \dots\ \cdot\\ \varphi_{m1}(t)\dots\varphi_{mj-1}(t) & \varphi_{mj}^{(N)}(t)-\varphi_{mj}(t) & \varphi_{mj+1}^{(N)}(t) & \dots\varphi_{mm}^{(N)}(t)\end{pmatrix}$$

$$(j=1,\dots,m),$$

then we can easily verify that

$$\frac{1}{m!}[\operatorname{Per}\Phi_m^{(N)}(t)-\operatorname{Per}\Phi_m(t)]=\Delta_1+\dots+\Delta_m. \tag{3.5}$$

Since the moduli of the characteristic functions (3.3) and (3.4) are less than or equal to one, on the basis of Lemma 3.5 we get

$$|\Delta_j|\leqq\frac{(m-1)!}{m!}\sum_{k=1}^{m}|\varphi_{kj}^{(N)}(t)-\varphi_{kj}(t)|\leqq\frac{|t|}{m}V(f_j)\sum_{k=1}^{m}D_N^{(k)}(F_k)\quad (j=1,\dots,m).$$

Utilizing this inequality in (3.5), we obtain the statement of our lemma.

Using Lemma 3.6 we get the following theorem as a consequence of Theorem 2.1.

Theorem 3.4. *Let* f_j *be a continuous function of bounded variation* $V(f_j)$ *on* $[a,b]$, *and suppose we are given N points* $x_1^{(j)},\dots,x_N^{(j)}$ *in* $[a,b)$ *with discrepancy* $D_N^{(j)}(F_j)$ *with respect to the distribution function* $F_j(x)\in H(a,b)$, *where* $j=1,\dots,m$. *Let*

$$A_j^{(N)}=\begin{pmatrix}f_j(x_1^{(1)})\dots f_j(x_N^{(1)})\\ \cdot\ \dots\ \cdot\\ f_j(x_1^{(m)})\dots f_j(x_N^{(m)})\end{pmatrix}\quad (j=1,\dots,m). \tag{3.6}$$

Let us denote by $\Psi_{m,n}^{(N)}(t)$ *the characteristic function of the doubly ordered linear rank statistics* $X_{m,n}^{(N)}$ *generated by the matrix* $A=A_1^{(N)}\dots A_m^{(N)}$. *Then we get*

$$\left|\Psi_{m,n}^{(N)}(t)-\frac{N^m}{(n+1)\dots(n+m)}\frac{1}{m!}\operatorname{Per}\Phi_m(t)\right|\leqq \tag{3.7}$$

$$\leqq\frac{N^m}{(n+1)\dots(n+m)}\left[1+\frac{|t|}{m}\sum_{j=1}^{m}V(f_j)\cdot\sum_{j=1}^{m}D_N^{(j)}(F_j)\right]-1$$

for $t\in R_1$, *where the matrix* $\Phi_m(t)$ *is defined by* (3.2).

Theorem 3.5. *Let* f_j *be a continuous function of bounded variation* $V(f_j)$ *on* $[a,b]$, *and suppose we are given the sequence* $\omega_j=\{x_n^{(j)}\}_{n=1}^{\infty}$, $x_n^{(j)}\in[a,b)$ *with discrepancy* $D_N^{(j)}(F_j)$ *of the points* $x_1^{(j)},\dots,x_m^{(j)}$ *with respect to the distribution function* $F_j(x)\in$ $\in H(a,b)$ *for* $j=1,\dots,m$. *Let* $V(f_1)+\dots+V(f_m)>0$. *Denote by* $\Psi_{m,n}^{(N)}(t)$ *the charac-*

teristic function of the doubly ordered linear rank statistics $X_{m,n}^{(N)}$ *generated by the matrix* $A^{(N)}=A_1^{(N)}\dots A_m^{(N)}$, *where* $A_j^{(N)}$ *is defined by* (3.6). *Then*

$$\lim_{n\to\infty} \Psi_{m,n}^{(N)}(t) = \frac{1}{m!}\operatorname{Per}\Phi_m(t)$$

holds uniformly in any finite interval $t\in[-T, T]$ *if and only if the sequence* ω_j *is* $F_j(x)$*-distributed for* $j=1, \dots, m$, *where* $\Phi_m(t)$ *is defined by* (3.2).

PROOF. Since

$$\lim_{n\to\infty} \frac{N^m}{(n+1)\dots(n+m)} = 1,$$

the right hand side of (3.7) has the limit zero uniformly in any finite interval $t\in[-T, T]$ if and only if $\lim_{N\to\infty} D_N^{(j)}(F_j)=0$ $(j=1, \dots, m)$. Thus we get the statement of our theorem using Theorem 3.2.

Denote by Y_{jk} the random variable with characteristic function (3.3). Then Theorem 3.5 can be expressed in the following alternative.

Under the conditions of Theorem 3.5 the sequence of the random variables $\{X_{m,n}^{(N)}\}_{n=1}^{\infty}$ converges weakly to the mixture of the random variables $Y_{1i_1}+\dots+Y_{mi_m}$, $(i_1, \dots, i_m)\in P_m$ with weights $\frac{1}{m!}$ if and only if, the sequence ω_j is $F_j(x)$-distributed for $j=1, \dots, m$. (The random variables $Y_{1i_1}, \dots, Y_{mi_m}$ are independent.)

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