MTA SZTAKI Közlemények 34/1986 pp.91-104

ON STRONG OPERATIONS

Vu Duc Thi

MTA STTAKI

§1. INTRODUCTION

The families of strong dependencies were introduced and investigated in [1,2]. In this paper we define strong operations and investigate the properties of strong operations. Based on these properties we give some combinational results which are related to the families of strong dependencies.

First we give some necessary definitions, and in §2. formulate our results.

§1. DEFINITIONS

Definition 1.1. Let $R=\{h_1,\dots,h_m\}$ be a relation over the finite set of attributes Ω, and $A,B\subseteq\Omega$. Then we say that B strongly depends on A in R (denote $A \xrightarrow[R]{s} B$) if

$$(\forall h_i,h_j\in R)((\;a\in A)(h_i(a)=h_j(a))\rightarrow(\forall b\in B)(h_i(b)=h_j(b)));$$

B functionally depends on A in R (denote $A \xrightarrow[R]{f} B$) iff

$$(\forall h_i,h_j\in R)((\forall a\in A)(h_i(a)=h_j(a))\rightarrow(\forall b\in B)(h_i(b)=h_j(b))).$$

Let

$$S_R=\{(A,B)\;:\;A \xrightarrow[R]{s} B\}\;.$$

S_R is called the full family of strong dependencies.

Definition 1.2. Let R_1, R_2 be two relations over Ω. We say that R_1 and R_2 are s-equivalent if $S_{R_1}=S_{R_2}$.

R_1 is an irredundant relation if for all $R'\subset R_1$: $S_{R'}\neq S_{R_1}$.

Definition 1.3. Let Ω be a finite set, and denote $P(\Omega)$ its power set. Let $Y = P(\Omega)\times P(\Omega)$. We say that Y satisfies the S-axioms iff for any $A,B,C,D\subseteq\Omega$, $a\in\Omega$.

(S1) $(\{a\}, \{a\})\in Y$;

(S2) $(A,B)\in Y$, $(B,C)\in Y$, $B\neq\emptyset \rightarrow (A,C)\in Y$;

(S3) $(A,B)\in Y$, $C\subseteq A$, $D\subseteq B \rightarrow (C,D)\in Y$;

(S4) $(A,B)\in Y$, $(C,D)\in Y \rightarrow (A\cap C, B\cup D)\in Y$;

(S5) $(A,B)\in Y$, $(C,D)\in Y \rightarrow (A\cup C, B\cap D)\in Y$.

It is clear that S_R satisfies the S-axioms.

Definition 1.4. Let $Y = P(\Omega)\times P(\Omega)$.

We say that Y satisfies the C-axiom iff there is a family of subsets of Ω, $\{E_i : i=1,\dots,\ell; \bigcup_{i=1}^{\ell} E_i = \Omega\}$ such that

(i) for any $\underset{E_i\cap A\neq\emptyset}{A,B\subseteq E_i} \rightarrow (A,B)\in Y$;

(ii) $(C,D)\in Y$, $C\cap E_i\neq\emptyset \rightarrow D\subseteq E_i$.

§2. RESULTS

Theorem 2.1. Let $Y = \underline{P}(\Omega)\times P(\Omega)$. Then Y satisfies the S-axioms iff Y satisfies the C-axiom.

Proof. First we suppose that Y satisfies the S-axioms. Then by (S1), (S3), and (S5) for each $a\in\Omega$ we can construct an E_i ($E_i\subseteq\Omega$) so that $(\{a\},E_i)\in Y$, and $\forall E': E_i\subset E'$ implies $(\{a\},E')\notin Y$. It is obvious that $a\in E_i$, and we obtain n such E_i-s, where $|\Omega|=n$. Thus, we have the set $E=\{E_i: i=1,\dots,n; \bigcup_{i=1}^{n} E_i=\Omega\}$. We assume that $A=\{a_1,\dots,a_k : a_j\in\Omega, j=1,\dots,k\}\neq\emptyset$, and B_1 is a set such that $(A,B_1)\in Y$, $\forall B_2: B_1\subset B_2$ implies $(A,B_2)\notin Y$. According to the construction of E, it is clear that for each a_j there is an $E_{ij} \in E$ so that $(\{a_j\}, E_{ij}) \in Y$. By (S4) we have $(\bigcup_{j=1}^{k} a_j, \bigcap_{j=1}^{k} E_{ij}) = (A, \bigcap_{j=1}^{k} E_{ij}) \in Y$. By the definition of B_1 we obtain $\bigcap_{j=1}^{k} E_{ij} \subseteq B_1$. On the otherhand, by $(A, B_1) \in Y$, and by (S3) we have $(\{a_j\}, B_1) \in Y$ for all j $(j=1,\dots,k)$. Consequently, $B_1 \subseteq \bigcap_{j=1}^{k} E_{ij}$, i.e. $B_1 = \bigcap_{j=1}^{k} E_{ij}$. It is obvious that $\bigcap_{E_i \cap A \neq \emptyset} E_i \subseteq \bigcap_{j=1}^{k} E_{ij}$. Thus, for all B $(B \subseteq \bigcap_{E_i \cap A \neq \emptyset} E_i)$: $B \subseteq B_1$.

Consequently, $(A,B) \in Y$. If $(C,D) \in Y$, $C \cap E_i \neq \emptyset$, then we assume that $a_1 \in C \cap E_i$. On the otherhand, suppose that a is an attribute such that $(\{a\}, E_i) \in Y$, and $\forall E' : E_i \subset E'$ implies $(\{a\}, E') \notin Y$. By $a_1 \in E_i$, and (S3) $(\{a\}, \{a_1\}) \in Y$ holds. By (S3), and $a_1 \in C$ we obtain $(\{a_1\}, D) \in Y$.

Consequently, by (S2), and $a_1 \neq \emptyset$ $(\{a\}, D) \in Y$ holds. According to the definition of E_i we have $D \subseteq E_i$. Thus, Y satisfies the C-axiom. It can be seen that by convention $\bigcap \emptyset = \Omega$, for all $B (B \subseteq B)$ we have $(\emptyset, B) \in Y$. The proof of the reverse direction is easy and so will be omitted. The theorem is proved.

Now, we define thefollowing operation.

Definition 2.2. Let Ω be a finite set. The mapping $F: P(\Omega) \to P(\Omega)$ is called a strong operation over Ω if for every $a, b \in \Omega$, and $A \subseteq \Omega$, the following properties hold:

(i) $a \in F(\{a\})$,

(ii) $b \in F(\{a\}) \to F(\{b\}) \subseteq F(\{a\})$,

(iii) $F(A) = \bigcap_{a \in A} F(\{a\})$.

Remark 2.3. It is easy to see the following elementary properties of strong operations.

For $A, B \in P(\Omega)$: $F(A \cup B) = F(A) \cap F(B)$. By convention $\bigcap \emptyset = \Omega$ we obtain $F(\emptyset) = \bigcap \emptyset = \Omega$.

For $A \subseteq B$ $F(B) \subseteq F(A)$.

Definition 2.4. Let $Y = P(\Omega) \times P(\Omega)$. We say that Y is an s-family over Ω, if Y satisfies the S-axioms.

Lemma 2.5. Let S be an s-family over Ω. We define the mapping F_S: $P(\Omega) \to P(\Omega)$ as follows: $F_S(A)=\{a\in\Omega: (A,\{a\})\in S\}$. Then F_S is a strong operation. Conversely, if F is a strong operation, then there is exactly one s-family S so that $F=F_S$, where $S=\{(A,B): A,B\in P(\Omega):B\subseteq F(A)\}$.

Proof. Suppose that S is an s-family. It is obvious that $\forall a\in\Omega: a\in F_S(\{a\})$. By Theorem 2.1. S satisfies the C-axiom, and so we have $(C,D)\in S$, and $C\cap E_i\neq\emptyset$ implies $D\subseteq E_i$. It can be seen that in Theorem 2.1 for any $a\in\Omega$, $F_S(\{a\})\in \{E_i: i=1,\dots,n, /\Omega/=n\}$. Consequently, $(b,F_S(b))\in S$, $b\in F_S(\{a\})$, i.e. $(A,F_S(A))\in S$, $\forall a\in A$: $A\cap F_S(a)\neq\emptyset$ imply $F_S(A) = F_S(\{a\})$. Thus, $F_S(A) \subseteq \bigcap_{a\in A} F_S(\{a\})$. On the other hand, by (S5) in the S-axioms we obtain $\forall a\in A$: $(\{a\}, F_S(\{a\}))\in S$ implies $(A,\bigcap_{a\in A} F_S(\{a\}))\in S$, i.e. $\bigcap_{a\in A} F_S(\{a\})\subseteq F_S(A)$. Consequently, $F_S(A)=\bigcap_{a\in A} F(\{a\})$ holds.

Conversely, assume that F is a strong operation over Ω and $S=\{(A,B):B\subseteq F(A)\}$. We have to show that S is an s-family. By Theorem 2.1 we prove that S satisfies the C-axiom. We set $E=\{F(\{a\}): a\in\Omega, /\Omega/=n\}$. By the definition of S it is obvious that $B\subseteq \bigcap_{F(\{a\})\cap A\neq\emptyset} F(\{a\})$ implies $(A,B)\in S$ by $\bigcap_{F(\{a\})\cap A\neq\emptyset} F(\{a\}) \subseteq F(A)$. On the other hand if $(C,D)\in S$, and $C\cap F(\{a\})\neq\emptyset$, then we assume that $b\in C\cap F(\{a\})$, hence by (ii) $b\in F(\{a\})$ implies $F(\{b\}) \subseteq F(\{a\})$. It is obvious that $D\subseteq F(C)=\bigcap_{d\in C} F(\{d\})$. By $b\in C$, and $\bigcap_{d\in C} F(\{d\})\subseteq F(\{b\})$ we obtain $D\subseteq F(\{a\})$. It is clear that $\forall A\subseteq\Omega$: $(\emptyset,A)$, $(A,\emptyset)\in S$. It can be seen that $F=F_S$. Now, we suppose that there is a s-family S' so that $F_{S'}=F$. By the definition of S and F we obtain $S'\subseteq S$. If $(A,B)\in S$, then $B\subseteq F(A)=F_{S'}(A)$. By the definition of $F_{S'}$, we have $(A,B)\in S'$. Consequently, $S'=S$ holds. The proof is complete.

Remark 2.6. Clearly, if F_1 and F_2 are strong operations $(F_1\neq F_2)$, then $S_1\neq S_2$, where for i=1,2 : $S_i=\{(A,B): A,B\subseteq\Omega: B\subseteq F_i(A)\}$.

Definition 2.7. Let S be an s-family over Ω. We say that a relation R represents S iff $S_R=S$.

In [2] the equality sets of relation are defined as follows.

Definition 2.8. Let $R=\{h_1,\ldots,h_m\}$ be a relation over Ω. Denote by E_{ij} the set $\{a\in\Omega : h_i(a)=h_j(a),\ 1\le i<j\le m\}$. We set $E=\{E_{ij} : 1\le i<j\le m\}$.

Now, we give a necessary and sufficient condition for a relation R to represent an s-family.

Theorem 2.9. Let S be an s-family, and R be a relation over Ω. Then R represents S iff for each $a\in\Omega$:

$$\begin{aligned} F_S(\{a\}) &= \bigcap_{a\in E_{ij}} E_{ij} \quad \text{if } \exists E_{ij} : a\in E_{ij} \\ F_S(\{a\}) &= \Omega \quad \text{otherwise.} \end{aligned} \tag{1}$$

Proof. By Lemma 2.5 $S_R=S$ holds if and only if $F_{S_R}=F_S$. Consequently, first we show that

$$F_{S_R}(\{a\}) = \bigcap_{a\in E_{ij}} E_{ij} \quad \text{if} \quad E_{ij}: a\in E_{ij} ,$$

and in other case $F_{S_R}(\{a\})=\Omega$ holds. Clearly, $F_{S_R}(\{a\}) = \{b\in\Omega : \{a\} \overset{s}{\underset{R}{\rightarrow}} \{b\}\}$. Accordint to the definition of strong dependency we know that for any $a\in\Omega$: $\{a\} \overset{s}{\underset{R}{\rightarrow}} B \leftrightarrow \{a\} \overset{f}{\underset{R}{\rightarrow}} B$, where $a\neq\emptyset$. Let us denote by T the set $\{E_{ij} : a\in E_{ij}\}$. It is obvious that if $T=\emptyset$, then $\{a\} \overset{f}{\underset{R}{\rightarrow}} \Omega$, i.e. $F_{S_R}(\{a\})=\Omega$. If $T\neq\emptyset$ holds, then we set $A = \bigcap_{a\in E_{ij}} E_{ij}$. If $T=E$ holds (E is the set of all equality sets of R), then it is obvious that $\{a\} \overset{f}{\underset{R}{\rightarrow}} A$. If $T\subset E$ holds, then for E_{ij}: $E_{ij}\notin T$, $h_i(a)\neq h_j(a)$. Consequently, we have also $\{a\} \overset{f}{\underset{R}{\rightarrow}} A$. Denote A' the set with the following properties:

(i) $\{a\} \overset{f}{\underset{R}{\rightarrow}} A'$,

(ii) $A'\subset A''$ implies $\{a\} \overset{f}{\underset{R}{\rightarrow}} A''$.

It can be seen that $A'=A$. According to the definition of F_{S_R} we obtain $F_{S_R}(\{a\}) = \bigcap_{a\in E_{ij}} E_{ij}$. Thus, if $S_R=S$ holds, then

$$F_S(\{a\}) = \bigcap_{a\in E_{ij}} E_{ij} \quad \text{if } \exists\, E_{ij} : a\in E_{ij} ,$$

$$F_S(\{a\}) = \Omega \quad \text{otherwise.}$$

Conversely, if F_S satisfies (1), then according to the above part for any $a \in \Omega$ we have $F_S(\{a\}) = F_{S_R}(\{a\})$. Because F_S and F_{S_R} are strong operations, and by Lemma 2.5. We obtain $\forall A \subseteq \Omega$: $F_S(A) = F_{S_R}(A)$. Consequently, $F_S = F_{S_R}$ holds. The proof is complete.

We say that a relation R represents a strong operation F iff $F = F_{S_R}$. Based on Theorem 2.9, the next Corollary is obvious.

Corollary 2.10. Let F be a strong operation, R be a relation over Ω. Then R represents F iff

$$F(\{a\}) = \bigcap_{a \in E_{ij}} E_{ij} \quad \text{if } \exists E_{ij} : a \in E_{ij} \ ,$$

$$F(\{a\}) = \Omega \quad \text{otherwise ,}$$

where $a \in \Omega$.

Clearly, from a relation R we can construct the set of all equality sets of R. Consequently, a following corollary is also obvious.

Corollary 2.11. Let R be a relation, and F be a strong operation over Ω. Then there is an effective algorithm, that decide whether R represent F or not. This algorithm requires time polynomial in the number of rows and columns of R.

Based on Theorem 2.9. We going to construct an effective algorithm, which determines an irredundant relation.

Algorithm 2.12. Let $R = \{h_1, \dots, h_m\}$ be a relation over $\Omega = \{a_1, \dots, a_n\}$.

Step 1: From relation R we construct

$$E = \{E_{ij} : E_{ij} \text{ is an equality set of R} \quad 1 \leq i < j \leq m\} \ .$$

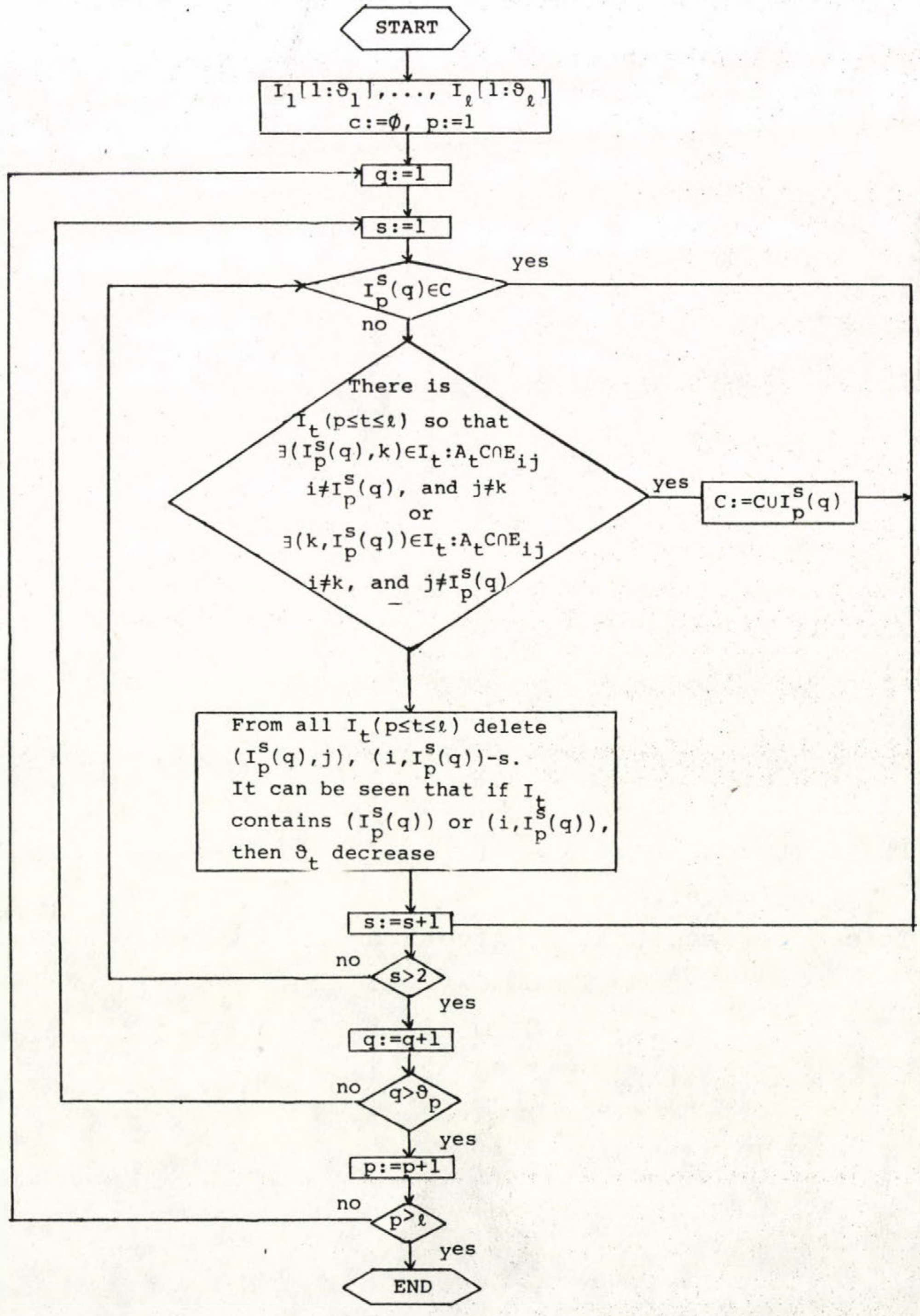
If there is not an E_{ij} so that $E_{ij} \neq \emptyset$ / then we choose any pair h_i, h_j . It is obvious that $R' = \{h_i, h_j\}$ is an irredundant relation such that $S_{R'} = S_R$. Now, we assume that $a_{t_1}, \dots, a_{t_\ell}$ are attributes such that $E_{ij} : a_{t_q} \in E_{ij}$, where $q = 1, \dots, \ell$.

Step 2: We construct sets of indexpairs, as follows: $I_1=\{(i,j): a_{t_1}\in E_{ij}\},\dots,I_\ell=\{(i,j): a_{t_\ell}\in E_{ij}\}$. Let $A_q=\bigcap_{a_{t_q}\in E_{ij}} E_{ij}$,

$$q = 1,\dots,\ell.$$

It is obvious that $\ell\le n$. Denote by ϑ_i the number of elements of I_i, $i=1,\dots,\ell$.

Denote $I_i^1(j)$, and $I_i^2(j)$ the first and second indices of jth pair in I_i, $i=1,\dots,\ell$; $1\le j\le\vartheta_i$. After that we perform the following bloc-scheme.



Then $R'=\{h_i : i\in C\}$ is an irredundant relation such that $R'\subseteqq R$, and $S_{R'}=S_{R'}$ i.e. $S_{R'}$ and S_R are s-equivalent.

Proof. It is clear that by Theorem 2.9., and Lemma 2.5. we have $S_{R'}=S_R$. It can be seen that in Step2 the Algorithm formulated in the bloc-scheme deletes all redundant rows of R. Thus, R′ is an irredundant relation.

The proof is complete.

Remark 2.13. It can be seen that Algorithm 2.12 requires time polynomial in the number of rows and columns of R.

Example 2.14. Let $\Omega=\{1,2,3,4,5,6\}$ and

$$R=\{h_1,h_2,h_3,h_4,h_5,h_6,h_7\}$$

be a relation over Ω.

Attributes:	1	2	3	4	5	6
	1	0	0	2	2	2
	2	0	0	3	3	0
	3	3	3	2	1	0
R =	1	1	2	4	3	3
	4	2	4	1	0	1
	5	2	4	5	4	4
	6	4	5	5	5	5

Clearly, $E_{12}=E_{56}=\{2,3\}$, $E_{23}=E_{67}=\{4\}$, $E_{14}=\{1\}$, $E_{23}=\{6\}$, and $E_{24}=\{5\}$.

Consequently, $I_1=\{(1,4)\}$, $I_2=\{(1,2),(5,6)\}$, $I_3=\{(1,2),(5,6)\}$, $I_4=\{(1,3),(6,7)\}$, $I_5=\{(2,4)\}$, and $I_6=\{(2,3)\}$.

It is obvious that $A_2=\{1\}$, $A_2=\{2,3\}$, $A_3=\{2,3\}$, $A_4=\{4\}$, $A_5=\{5\}$, and $A_6=\{6\}$.

It can be seen that by Algorithm 2.12 we obtain $C=\{1,2,3,4\}$, i.e.

	1	0	0	2	2	2
R′=	2	0	0	3	3	0
	3	3	3	2	1	0
	1	1	2	4	3	3

Definition 2.15. Let $I \subseteq P(\Omega)$, and I closed under intersection. Let $M \subseteq P(\Omega)$. Denote M^+ the set $\{\cap M' : M' \subseteq M\}$. We say that M generates I if $M^+ = I$.

By convention $\cap\emptyset = \Omega$, i.e. M^+ always contains Ω; so Ω is never required in M. It is obvious that $\Omega \in I$.

J.Demetrovics in [2] showed that for a given family M of subsets of Ω there is exactly one family N, which generates M^+, and has minimal cardinality.

Lemma 2.16. [2] Let $M \subseteq P(\Omega)$ be a family over Ω. Let $N = \{A \in M : (\forall B, C \in M)(A = B \cap C \rightarrow A = B \text{ or } A = C)\}$. Then N generates M^+ and if N′ generates M^+, then $N \subseteq N'$. It is possible that $\emptyset \in N$.

By Remark 2.3 we obtain $F(A \cup B) = F(A) \cap F(B)$ for $A, B \in P(\Omega)$, where F is a strong operation. Thus, the set $\{F(A) : A \subseteq \Omega\}$ closed under intersection. It is easy to see that

The set $\{F(\{a\}) \neq \Omega : a \in \Omega\}$ generates $\{F(A) : A \subseteq \Omega\}$. It is possible that $a \neq b$, but $F(\{a\}) = F(\{b\})$. It is known (see [1,2]) that if S is an s-family over Ω, then there is a relation R over Ω such that $S_R = S$. However, here we construct for a given s-family S a simple concrete relation R so that R represents S.

Proposition 2.17. Let S be an s-family over $\Omega = \{a_1, \dots, a_n\}$. Let $\{F_S \{a\} : i = 1, \dots, n\} = \{A_1, \dots, A_k : A_i \neq A_j, 1 \le i \le k, 1 \le j \le k, i \neq j, k \le n\}$. We set $T = \{A_1, \dots, A_k\}$, and $N = \{A \in T : A \neq \Omega, (\forall B, C \in T)(A = B \cap C \rightarrow A = B \text{ or } A = C)\}$. Suppose that $N = \{B_1, \dots, B_\ell\}$, $(\ell \le k)$. Then we set $R = \{h_0, h_1, \dots, h_\ell\}$ as follows: for all $a \in \Omega$: $h_0(a) = 0$.

$$\text{for each } i \ (i = 1, \dots, \ell) : h_i(a) = \begin{cases} 0 & \text{if } a \in B_i \ , \\ i & \text{otherwise} \ . \end{cases}$$

Then R represents S, i.e. $S_R = S$.

Proof. By Theorem 2.9 we prove that for each $a \in \Omega$:

$$F_S(\{a\}) = \bigcap_{a \in E_{ij}} E_{ij} \quad \text{if } \exists\, E_{ij} : a \in E_{ij} \ ,$$

$$F_S(\{a\}) = \Omega \text{ otherwise.}$$

It is easy to see that if $F_S(\{a\})=\Omega$, then there is not an E_{ij} so that $a\in E_{ij}$ (by the construction of R). If $F(\{a_{i_t}\})=B_t$, then by (ii) in Definition of strong operation we have $\forall B_k : a_{i_t} \in B_k$ implies $F_S(\{a_{i_t}\})=B_t \subseteq B_k$. Consequently,

$$F_S(\{a_{i_t}\}) = \bigcap_{a\in E_{i_t}} E_{i_t} = E_{o_t} = B_t.$$

For $a_{i_t}\in\Omega$: $F(\{a_{i_t}\}) = B_{j_1}\cap\ldots\cap B_{j_t}$. We obtain that for any $B_k(a_{i_t} \in B_k)$ $F(\{a_{i_t}\}) \subseteq B_k$ by (ii). Consequently,

$$F(\{a_{i_t}\}) = \bigcap_{a\in E_{ij}} E_{ij} = \bigcap_{q=1}^{t} E_{o_{j_q}} .$$

It can be seen that by (ii) for any a $(a\in\Omega)$ so that $F(\{a\}) = F(\{a_{i_t}\})$. We have

$$F(\{a\}) = \bigcap_{a\in E_{ij}} E_{ij} = E_{o_t} \quad \text{or} \quad F(\{a\}) = \bigcap_{a\in E_{ij}} E_{ij} = \bigcap_{q=1}^{t} E_{o_{j_q}}$$

The proof is complete.

Based on Proposition 2.17, if can be seen that if F is a strong operation over Ω, then there is an effective algorithm (this algorithm requires time polynomial in $/\Omega/$), which determines a relation (it is analogous to R in Proposition 2.17) or that this relation represents F.

Definition 2.18. Let S be an s-family over Ω. Let $Q(S)=\min\{m: S_R=S, /R/=m, R \text{ is a relation over } \Omega\}$. Thus, $Q(S)$ is the number of rows of minimal relation which represents the s-family S.

Corollary 2.19. Let S be an s-family over Ω.

Let

$T=\{F_S(\{a\}): a\in\Omega\}$ and $N=\{A\in T: A\neq\Omega, (\forall B,C\in T)(A=B\cap C \to A=B \text{ or } A=C)\}$.

Then if $/N/=0$ i.e. $T=\{\Omega\}$, then $Q(S)=2$. If $/N/\geq 1$, then $\sqrt{2\log_2 /N/} < Q(S) \leq /N/+1 \leq /\Omega/+1$.

Proof. According Theorem 2.9 if R represents S, then

$$F_S(\{a\}) = \bigcap_{a\in E_{ij}} E_{ij} \quad \text{if } \exists E_{ij}: a\in E_{ij} ,$$

$$F_S(\{A\}) = \Omega \text{ otherwise} .$$

It is easy to see that if $T=\{\Omega\}$, then for all E_{ij} we obtain $E_{ij}=\emptyset$. Consequently, $Q(S)=2$.

It is clear that according to the definition of strong operation N determines the family $\{F_S(\{a\}): a\in\Omega\}$. It is obvious that $/\Omega/\geq/N/$ and for $\forall A_i, A_j\in N$ $(A_i\neq A_j)$ we have

$$\{(i,j): 1\leq i<j\leq m,\ /R/ = m,\ A_i \subseteq E_{ij}\} \neq$$

$$\{(i,j): 1\leq i<j\leq m,\ /R/ = m,\ A_j \subseteq E_{ij}\} .$$

Consequently, $/N/\leq 2^{\binom{m}{2}}$. Thus, $\sqrt{2\log_2/N/}<m$. By Proposition 2.17 $Q(S)\leq/N/+1$. The proof is proved.

The next corollary is obvious.

Corollary 2.20. Let $Q(n)=\max\ \{Q(S):$ S is an s-family over Ω, $/\Omega/=n\}$. Then $Q(n)\leq n+1$.

Definition 2.21. Let

$T\subseteq P(\Omega)$ and let $N=\{A\in T:\ A\neq\Omega, (\forall B,C\in T)(A=B\cap C\rightarrow A=B \text{ or } A=C)\}$.

Then we say that T is s-semilattice if T closed under intersection, $\Omega\in T$ and N satisfies (1): for all $A\in N$

$$(\exists a\in A)(A_i\in N \text{ and } A\not\subset A_i \rightarrow a\notin A_i) .$$

Theorem 2.22. Let F be a strong operation over Ω. Let

$T_F= \{F(A): A\in P(\Omega)\}$ and

$N_F=\{A\in T_F:\ A\neq\Omega,\ (\forall B,\ C\in T_F)\ (A=B\cap C \rightarrow A=B \text{ or } A=C)\}$.

Then T_F is a s-semilattice. Conversely, if T is any s-semilattice, then there is exactly one strong operation so that $T=T_F$. Where for each a $(a\in\Omega)$

$$F(\{a\}) = \bigcap_{\substack{a\in A_i \\ A_i\in N}} A_i \quad \text{if } \exists A_i\in N: a\in A_i ,$$
$$F(\{a\}) = \Omega \quad \text{otherwise} .$$

Proof. It is obvious that for arbitrary strong operation we have $\forall A,B\in P(\Omega): F(A\cup B)= F(A)\cap F(B)$, $F(\emptyset)=\Omega$ and $A\subseteq B\rightarrow F(B)\subseteq F(A)$. Consequently $\Omega\in T_F$ and T_F closed under intersection. Now we assume that $A\in N_F$. If there is not attribute a so that $F(\{a\})=A$, then if $A=F(B)$ $(/B/\geq 2)$, then $A = \bigcap_{b_i\in B} F(\{b_i\})$. This contradicts the definition of N_F. Consequently, there is attribute a $(a\in\Omega)$ so that $F(\{a\})=A$. It is obvious that $a\in A$. Clearly, N_F satisfies (1).

Conversely, we assume that T is a s-semilattice over Ω. Then we set for each a $(a\in\Omega)$

$$F(\{a\}) = \bigcap_{\substack{b\in A_i \\ A_i\in N}} A_i \quad \text{if } \exists A_i\in N: a\in A_i ,$$
$$F(\{a\}) = \Omega \quad \text{otherwise.}$$

Clearly, for all $A\in N$ ($\exists a\in A$: $A_i\in N$ and $A\not\subset A_i \rightarrow a\notin A_i$) we obtain $F(\{a\})=A$ for different set $A(A\in N)$ is easy to see that there is attribute a so that $F(\{a\})=A$. Consequently, $\forall A\in N$: $a\in\Omega$: $F(\{a\})=A$. Now we prove that F is a strong operation. According to the construction of F it is clear that $a\in F(\{a\})$ and if there is $A_i\in N$ so that $a\in A_i$, then $F(\{a\})\in N^+$. If $b\in F(\{a\})$, then

$$F(\{b\}) = \bigcap_{\substack{a\in A_i \\ A_i\in N}} A_i \subseteq \bigcap_{\substack{a\in A_i \\ A_i\in N}} A_i = F(\{a\}).$$

It is obvious that the set $\{F(\{a\}): b\in\Omega\}$ determines the set $\{F(A): A\in P(\Omega)\}$. Consequently, F is a strong operation and $T=T_F$. If we suppose that there is strong operation F' so that $T=T_{F'}$. Then for all a $(a\in\Omega)$ there is b $(b\in\Omega)$ such that $F(\{a\})=F'(\{b\})$. It is obvious that $a\in F'(\{b\})$. Consequently, we have $F'(\{a\}) \subseteq F(\{a\})$. On the other hand, there is attribute c $(c\in\Omega)$ so that $F'(\{a\})=F(\{c\})$. By $a\in F(\{c\})$ we obtain

$F(\{a\}) \subseteq F'(\{a\})$.

The proof is proved.

<u>*Definition 2.23.*</u> Let S be a s-family over Ω, and $(A,B) \in S$. We say that (A,B) is a maximal right side dependency of S if $\forall B' : B \subseteq B'$, $(A,B') \in S \rightarrow B=B'$.

Denote by M(S) the set of all maximal right side dependencies of S. We say that B(B Ω) is a maximal right side of S if there is an A such that $(A,B) \in M(S)$. Denote U(S) the set of all maximal right sides of S.

<u>*Corollary 2.24.*</u> Let S be a s-family over Ω. Then U(S) is a s-semilattice. Conversely, if T is a s-semilattice, then there is exactly one s-family so that T=U(S). By Lemma 2.5 and Theorem 2.22 this corollary is obvious.

ACKNOWLEDGEMENTS

The author would like to take this opportunity to express deep gratitude to Professor Dr.J.Demetrovics for his hel, valuable comments and suggestions.

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Az erős operációkról

Vu Duc Thi

Összefoglaló

A szerző az erős operációk tulajdonságait vizsgálja. Néhány uj kombinatorikus eredményt is ad, amelyek az erős függőségek családjaira vonatkoznak.

Сильные операции

Ву Дык Тхи

Р е з ю м е

В настоящей работе изучается связь между сильными операциями и сильными зависимостями.