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ON THE c-SEPARABLE AND DOMINANT SETS OF VARIABLES FOR THE FUNCTIONS

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In this paper we investigate some properties of the c-separable and dominant sets which are introduced immediately. We use some notations and terminology from [1,2,3].

Let f be a function, R_f - the set of all essential variables for f and S_f - the set of all separable sets of f.

Definition 1. A set M, $M \subseteq R_f$ is called c-separable for f with respect to $N = \{x_{i_1}, x_{i_2}, \ldots, x_{i_s}\} \subseteq R_f$, if for every s-values $c_{i_1}, c_{i_2}, \ldots, c_{i_s}$ of the variables in N, the subfunction of f which is obtained with these values, depends on all variables of M i.e. $M \subseteq R_f(x_{i_1}=c_{i_1}, x_{i_2}=c_{i_2}, \ldots, x_{i_s}=c_{i_s})$

When M is a c-separable set for f with respect to $R_f \setminus M$, it is called c-separable for f. The set of all c-separable sets for f with respect to N will be denoted by $S^*_{f,N}$ and $S^*_f = \{K | K \in S^*_{f, R_f \setminus K}\}$.

Definition 2. A set $M = \{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\} \subseteq R_f$, is called dominant set over the set N, $N \subseteq R_f$ for f, if there exist m - values $c_{i_1}, c_{i_2}, \ldots, c_{i_m}$ of the variables in M such that

$$N \cap R_f(x_{i_1} = c_{i_1}, x_{i_2} = c_{i_2}, \ldots, x_{i_m} = c_{i_m}) = \emptyset$$

and M is a minimal set with respect to this property.

When this equation is true and M is a dominant set over N, it is said that M dominates over N with the values $c_{i_1}, c_{i_2}, \ldots, c_{i_m}$.

The set of all dominant sets over N will be denoted by $L_{N,f}$ and $D_{N,f} = \{x_\alpha \in R_f \mid (\exists M)\ x_\alpha \in M \wedge M \in L_{N,f}\}$.

The proofs of the next lemmas follow immediately from the Definitions 1 and 2.

Lemma 1. If $M_i \in S^*_f$, $i \in I$, then $\bigcup_{i \in I} M_i \in S^*_f$.

Lemma 2. If $M \in S^*_{f,N}$ then for every N_1, $N_1 \subsetneq N$ the set M belongs to S^*_{f,N_1}.

Lemma 3. If $M \in S^*_{f,N}$ then for every M_1, $M_1 \subseteq M$ the set M_1 belongs to $S^*_{f,N}$.

Lemma 4. Let $M \subsetneq R_f$ and $N = \{x_{j_1}, x_{j_2}, \dots, x_{j_s}\}$ R_f. If there exist the values $c_{j_1}, c_{j_2}, \dots, c_{j_s}$ such that

$$M \cap R_f(x_{j_1} = c_{j_1}, x_{j_2} = c_{j_2}, \dots, x_{j_s} = c_{j_s}) = \emptyset$$

then there is a subset N_1 of N such that $N_1 \in L_{M,f}$.

Theorem 5. If $M \in L_{N,f}$, $N \in L_{p,f}$ and $M \cap N = \emptyset$ then there exists M_1 such that $M_1 \subseteq M$ and $M_1 \in L_{p,f}$.

Proof. We can suppose without loss of generality that $M = \{x_1, x_2, \dots, x_m\}$ and $N = \{x_{i_1}, x_{i_2}, \dots, x_{i_s}\}$.

Let $c_1, c_2, \dots, c_m$ and $c_{i_1}, c_{i_2}, \dots, c_{i_s}$ be some values which M dominates over N and N dominates over P. If

$$f_1 = f(x_1 = c_1, x_2 = c_2, \dots, x_m = c_m)$$

then for every s-values $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_s}$ of the variables in N we obtain

$$f_1 = f_1(x_{i_1} = \alpha_{i_1}, x_{i_2} = \alpha_{i_2}, \dots, x_{i_s} = \alpha_{i_s}).$$

Hence

$$f_1 = f_1(x_{i_1} = c_{i_1}, x_{i_2} = c_{i_2}, \dots, x_{i_s} = c_{i_s})$$

and by $M \cap N = \emptyset$ it follows

$$f_1 = f(x_1=c_1,\ x_2=c_2,\dots,x_m = c_m,\ x_{i_1} = c_{i_1},\ x_{i_2} = c_{i_2},\dots,x_{i_s} = c_{i_s})$$

and $P \cap Rf_1 = \emptyset$. By Lemma 4 it follows that there is M_1 such that $M_1 \subseteq M$ and $M_1 \in L_{p,f}$.

The condition $M \cap N = \emptyset$ is essential which may be seen from the following example. Let

$$f = x_1^0\ x_3 + x_2x_4x_5 + x_1x_4x_5 (mod\ 3)$$

where $x_1^0 = \begin{cases} 1 \text{ if } x_1 = 0 \\ 0 \text{ if } x_1 \neq 0 \end{cases}$

If $M = \{x_1, x_2\}$, $N = \{x_1, x_4\}$ and $P = \{x_3, x_5\}$ then $M \in L_{N,f}$ $N \in L_{p,f}$ but there isn't any set M_1 such that $M_1 \subseteq M$ and $M_1 \in L_{p,f}$.

Lemma 6. For every x_α, $x_\alpha \in R_f$, the set $\{x_\alpha\}$ belongs to $L_{\{x_\alpha\},f}$.

Proof. For every value c_α of the variable x_α it holds true $\{x_\alpha\} \cap R_f\ (x_\alpha = c_\alpha) = \emptyset$.

But $\{x_\alpha\}$ hasn't any nonempty proper subset and by Theorem 5 it follows $\{x_\alpha\} \in L_{\{x_\alpha\},f}$.

Theorem 7. If $x_\alpha \in R_f$ and $x_\beta \in D_{\{x_\alpha\},f}$ then $\{x_\alpha, x_\beta\} \in S_f$.

Proof. We can suppose without loss of generality that

$$M = \{x_\beta,\ x_3, x_4,\dots,x_m\} \in L_{\{x_\alpha\},f}.$$

If $x_\alpha = x_\beta$ then the theorem is trivial. Now, let $x_\alpha \neq x_\beta$. If we suppose that $x_\alpha \in M$ then by Lemma 6 it follows $M \notin L_{\{x_\alpha\},f}$. It is a contradiction. Hence $x_\alpha \notin M$.

Let c_β, c_3, $c_4,\dots,c_m$, $c_{m+1},\dots,c_n$ be $n-1$-values of the variables in $R_f \setminus \{x_\alpha\}$, $|R_f| = n$, such that $x_\alpha \in Rf_1$ and $x_\alpha \notin Rf_2$, where

$$f_1 = f(x_3 = c_3,\ x_4 = c_4, \ldots, x_m = c_m,\ x_{m+1} = c_{m+1}, \ldots, x_n = c_n)$$

and

$$f_2 = f(x_\beta = c_\beta,\ x_3 = c_3,\ x_4 = c_4, \ldots, x_m = c_m).$$

This choice of $c_\beta, c_3, c_4, \ldots, c_m, c_{m+1}, \ldots, c_n$ is possible because $x_\alpha \in R_f$ and $M \in L_{\{x_\alpha\},f}$. On the supposition that $\{x_\alpha, x_\beta\} \notin S_f$ we obtain $f_1 = f_1(x_\beta = c'_\beta)$ for every c'_β. In particular when $c'_\beta = c_\beta$ it follows $x_\alpha \in Rf_1(x_\beta = c_\beta)$ i.e. $x_\alpha \in Rf_2$. This a contradiction. The theorem is proved.

Corollary. If $x_\alpha \in D_{\{x_\beta\},f}$ or $x_\beta \in D_{\{x_\alpha\},f}$ then $\{x_\alpha, x_\beta\} \in S_f$.

Theorem 8. If $M \in L_{N,f}$ and there is a value c_α of the variable x_α such that $M \not\subseteq R_f(x_\alpha = c_\alpha)$ then $x_\alpha \in D_{N,f}$.

Proof. Let $M = \{x_1, x_2, \ldots, x_m\}$. If $x_\alpha \in M$ then the theorem is trivial. Now, let $x_\alpha \notin M$ and c_α be a value of the variable x_α such that $M \cap Rf_1 \neq M$, where $f_1 = f(x_\alpha = c_\alpha)$. We can suppose without loss of generality that $x_1 \notin Rf_1$. Let $c_1, c_2, \ldots, c_m$ be m - values of the variables in M such that

$$N \cap Rf(x_1 = c_1,\ x_2 = c_2, \ldots, x_m = c_m) = \emptyset$$

Then for every m - values $c'_1, c'_2, \ldots, c'_m$ of the variables $x_1, x_2, \ldots, x_m$ it holds true

$$N \cap Rf(x_2 = c'_2,\ x_3 = c'_3, \ldots, x_m = c'_m) \neq \emptyset \text{ and } f_1 = f_1(x_1 = c'_1).$$

This equation implies

$$N \cap R_f\,(x_\alpha = c_\alpha,\ x_2 = c_2, \ldots, x_m = c_m) = \emptyset$$

By Lemma 4 there is a subset M_1 of M' such that $M_1 \in L_{N,f}$ where $M' = \{x_\alpha, x_2, x_3, \ldots, x_m\}$.

Now, if $x_\alpha \notin M_1$ then $M \notin L_{N,f}$. This is a contradiction.

Hence $x_\alpha \in M_1$. The theorem is proved.

Corollary. For every essential variable of the function f, $\{x_\alpha\} \in S_f^*$ if and only if $D_{\{x_\alpha\},f} = \{x_\alpha\}$.

Theorem 9. For every N, $N \subseteq R_f$ the set $D_{N,f}'$ is a c-separable set for f.

Proof. If $N = \emptyset$ then obviously $D_{N,f}' \in S_f^*$. Now, let $N \neq \emptyset$ and we can suppose without loss of generality that $R_f = \{x_1, x_2, \ldots, x_n\}$ and $D_{N,f} = \{x_1, x_2, \ldots, x_p\}$, $p \leq n$. Moreover, we suppose that there are n-p - values $c_{p+1}, c_{p+2}, \ldots, c_n$ of the variables in $R_f \setminus D_{N,f}$ such that $D_{N,f} \not\subseteq Rf_1$, where

$$f_1 = f(x_{p+1} = c_{p+1}, x_{p+2} = c_{p+2}, \ldots, x_n = c_n).$$

Again, we can suppose without loss of generality that $x_p \notin Rf_1$ and

$$M = \{x_p, x_{i_2}, x_{i_3}, \ldots, x_{i_m}\} \in L_{N,f}.$$

Then for every m-1 - values $c_{i_2}, c_{i_3}, \ldots, c_{i_m}$ of the variables in $M \setminus \{x_p\}$ it holds true

$$N \cap R_f(x_{i_2} = c_{i_2}, x_{i_3} = c_{i_3}, \ldots, x_{i_m} = c_{i_m}) \neq \emptyset.$$

Now, we suppose that there are the values $c'_{i_2}, c'_{i_3}, \ldots, c'_{i_m}, c'_{p+1}, \ldots, c'_n$ such that $N \cap Rf_2 = \emptyset$ where

$$f_2 = f(x_{i_2} = c'_{i_2}, x_{i_3} = c'_{i_3}, \ldots, x_{i_m} = c'_{i_m}, x_{p+1} = c'_{p+1}, x_{p+2} = c'_{p+2}, \ldots, x_n = c'_n).$$

By Lemma 4 there is a subset M_1 of M' such that $M_1 \in L_{N,f}$, where $M' = \{x_{i_2}, x_{i_3}, \ldots, x_{i_m}, x_{p+1}, \ldots, x_n\}$. By $M \in L_{N,f}$ we obtain $M_1 \not\subseteq D_{N,f}$ which is a contradiction. Consequently, for every m+n-p-1 -values $\alpha_{i_2}, \alpha_{i_3}, \ldots, \alpha_{i_m}, \alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_n$

of the variables in $(R_f | D_{N,f}) \cup (M | \{x_p\})$ it holds true

$$N \cap R_f(x_{i_2}=\alpha_{i_2}, x_{i_3}=\alpha_{i_3}, \dots, x_{i_m}=\alpha_{i_m}, x_{p+1}=\alpha_{p+1}, x_{p+2}=\alpha_{p+2}, \dots, x_n=\alpha_n) \neq \emptyset.$$

But $f_1 = f_1(x_p = \alpha_p)$ for every α_p and there exist m-values $c''_p, c''_{i_2}, c''_{i_3}, \dots, c''_{i_m}$ of the variables in M such that

$$N \cap Rf(x_p = c''_p, x_{i_2} = c''_{i_2}, x_{i_3} = c''_{i_3}, \dots, x_{i_m} = c''_{i_m}) = \emptyset$$

and

$$N \cap Rf(x_p = c''_p, x_{i_2} = c''_{i_2}, \dots, x_{i_m} = c''_{i_m}, x_{p+1} = c_{p+1}, \dots, x_n = c_n) = \emptyset$$

This is a contradiction. The theorem is proved.

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A függvények változóinak c-szeparábilis és domináns halmazairól.

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Összefoglaló

A szerző bevezeti a c-szeparábilis és domináns halmazok fogalmát és néhány ezen fogalmakat jellemző tételt bizonyit be.

Об c-сепарабельных и доминантных множествах переменных для функций

С. Штраков

Резюме

Автор дает определение c-сепарабельных и доминантных множеств и доказывает несколько теорем, которые характеризуют эти множества.