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# FINITELY GENERATED CLONES WITH LINEAR FUNCTIONS IN $P_3$ AND $P_5$

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## INTRODUCTION

Let $E_k$ be the set $\{0,1,\dots,k-1\}$ for $k \geq 2$, $P_k^{(n)} = \{f \mid f\colon E_k^n \to E_k\}$ for $n = 1,2,\dots$ and let $P_k = \bigcup_{n=0}^{\infty} P_k^{n}$, where $P_k^{o}$ is the set of constant functions. A set of functions $Z \subset P_k$ is a clone if it contains the projections (i.e. the functions $e_j(x_1\dots x_j\dots x_n) = x_j$, $j=1,2,\dots,n$) and all superpositions over $Z$.
An open problem is the following: under what conditions is an arbitrary $Z \subset P_k$ finitely generated? ($Z$ is finitely generated if there exists a finite subset $Z_n \subset Z$ from which all functions of $Z$ can be obtained by superpositions.)
It is known that the clone of the linear functions $L_p$ in $P_p$ ($p$ is a prime) is finitely generated (Demetrovics and Bagyinszki [1]), where $L_p = \{L \mid L(x_1,\dots,x_n) = a_o + \sum_{i=1}^{n} a_i x_i$, $n = 1,2\dots\}$ (addition and multiplication are carried out mod $p$ and $a_i$ are residue classes mod $p$).
We deal with finitely generated clones of $P_3$ and $P_5$. The purpose of this paper is to prove the following theorem:

A clone $Z$ of $P_3$ or $P_5$ is finitely generated if it contains a nontrivial n-ary linear function $(n \geq 2)$ and an unary non linear function.

For $P_3$ a more general result was proved by Marcenkov [2]: $Z \subset P_3$ is finitely generated if it contains an n-ary linear function and an arbitrary non linear function. (Lemma 5 and its corollary). We give another proof, our method works for $P_5$ too (this part of the theorem is a new result).
The following statements will be useful in the sequel. If

$Z \subset P_3$ $(Z \subset P_5)$ contains n-ary linear function then it contains also the function $f(x,y) = 2x + 2y$ $(F(x,y) = 2x + 4y)$ [1].

If $Z$ has a near-unanimity function then it is finitely generated (this is an immediate corollary of the results of Baker's-Pixley's [3]). The function $m\colon E_k^n \to E_k$ is a near-unanimity function if $m(y,x,\ldots,x) = m(x,y,x,\ldots,x) = \ldots = m(x,x,\ldots,x,y) = x$ for all $x,y \in E_k$.

For the proof of the $P_3$-part of the theorem is sufficient to construct with superpositions from the function $f(x,y)$ and from an arbitrary non linear unary function a three-variable near-unanimity function $M\colon E_3^3 \to E_3$, for which $M(x,x,y) = M(x,y,x) = M(y,x,x) = X$ hold. Moreover it is sufficient to construct the function $\wedge_0(x,y)$ and $\vee_0(x,y)$ from which one can obtain the function $M$ using the formula of [2]:

$$M(x,y,z) = \vee_0(\vee_0(\wedge_0(x,y), \wedge_0(x,z)), \wedge_0(y,z)) \tag{1}$$

where the Cayley tables of $\wedge_0$ and $\vee_0$:

$\wedge_0(x,y)$

| x \ y | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 2 | 0 | 0 | 2 |

$\vee_0(x,y)$

| x \ y | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 1 | 0 |
| 2 | 2 | 0 | 2 |

PROOF FOR FIRST PART OF THE THEOREM

First we construct the functions $\wedge_0$ and $\vee_0$. The unary non linear functions of $P_3$ can be given by the table:

| x | 0 | 1 | 2 |
|---|---|---|---|
| $a(x)$ | 0 | 0 | 1 |
| $b(x)$ | 0 | 1 | 0 |
| $c(x)$ | 1 | 0 | 0 |
| $d(x)$ | 0 | 0 | 2 |
| $e(x)$ | 0 | 2 | 0 |
| $\varphi(x)$ | 2 | 0 | 0 |
| $g(x)$ | 0 | 1 | 1 |
| $h(x)$ | 1 | 0 | 1 |
| $i(x)$ | 1 | 1 | 0 |
| $j(x)$ | 0 | 2 | 2 |
| $k(x)$ | 2 | 0 | 2 |
| $l(x)$ | 2 | 2 | 0 |
| $m(x)$ | 1 | 1 | 2 |
| $n(x)$ | 1 | 2 | 1 |
| $o(x)$ | 2 | 1 | 1 |
| $p(x)$ | 1 | 2 | 2 |
| $q(x)$ | 2 | 1 | 2 |
| $r(x)$ | 2 | 2 | 1 |

Among these functions $b$, $d$, $g$, $j$, $m$ and $q$ are isomorphic in the sense that all of them fix two elements and to the third they assign one of the fixed elements. Now we shall obtain $\wedge_0$ and $\vee_0$ from the functions $f(x,y)$ and $b(x)$. The function $b(x)$ is in the clone generated by either $a(x)$ or $c(x)$. In the groups of the functions $d$, $g$, $j$, $m$, and $q$ (in the table these groups are separated with lines) similar computations can be carried out. E.g. from $q$ can be produced (with the same steps applicated for $b$) the functions $\wedge_2(x,y)$

| | | |
|---|---|---|
| 0 | 2 | 2 |
| 2 | 1 | 2 |
| 2 | 2 | 2 |

and

$V_2(x,y)$ $\begin{array}{|ccc} 0 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{array}$ from which the function $M$ is formed according to (1).

THE COMPUTATION FOR b(x)

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $b(x)$ | 0 | 1 | 0 |

$f(x,y)$ $\begin{array}{|ccc} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{array}$ $\quad b(f(x,y))$ $\begin{array}{|ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}$

$B_1(x,y) = b(f(b(x),y))$ $\begin{array}{|ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}$

$B_2(x,y) = b(f(x,b(y)))$ $\begin{array}{|ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}$

$f(B_1(x,y),B_2(x,y))$ $\begin{array}{|ccc} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{array}$

$$\wedge_0(x,y) = f(f(B_1,B_2),b(f(x,y)))$$

$$\vee_0(x,y) = f(\wedge_0(x,y),f(x,y))$$

THE REDUCTION OF THE FUNCTIONS a(x) AND c(x) TO b(x)

$a(a(x)) = 0;$ $\quad a(f(x,0)) = b(x);$

$f(c(f(x,y)),\ c(c(f(x,y)))) = 2;$ $\quad c(f(x,2)) = b(x)$ and because of the isomorphisms stated the assertion is proved.

THE SECOND PART OF THE THEOREM CONCERNING $P_5$

The proof is similar to the previous case. We shall obtain from the function $F(x,y)$ and from an arbitrary non linear unary function of $P_5$ the functions

$$\wedge_o(x,y)\quad \begin{array}{|ccccc} 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&2&0&0\\ 0&0&0&3&0\\ 0&0&0&0&4 \end{array} \qquad \text{and} \qquad \vee_o(x,y)\quad \begin{array}{|ccccc} 0&1&2&3&4\\ 1&1&0&0&0\\ 2&0&2&0&0\\ 3&0&0&3&0\\ 4&0&0&0&4 \end{array}$$

From these functions, using (1), we get a three variable near--unanimity function $M$ of $P_5$.
The computation gives also the polynomial forms of the functions $\wedge_o$ and $V_o$.
First we derive from $F$ two other linear functions

$$F(2x + 4y,y) = 2(2x + 4y) + 4y = 4x + 2y$$

$$F(4x + 2y,y) = 2(4x + 2y) + 4y = 3x + 3y.$$

If the functions $G$, $H$, $K$ are elements of the clone $Z$ then the same is true for the following linear combinations of them:

$$4G + 4H + 3K = 3(3G + 3H) + 3K$$

$$G + H + 4K = 4(4G + 4H + 3K) + 2K$$

$$G + 2H + 3K = 3(2G + 4H) + 3K$$

$$2G + 2H + 2K = 4(3G + 3K) + 2K .$$

The proof is also a reduction to such "good" functions as $b(x)$ was in $P_3$. In $P_5$, there are two kinds of such unary non linear functions:

$k(x) = 0\ 1\ 3\ 2\ 4$ (it fixes three values of $x$ and to the fifth assigns one of the fixed values) $l_4(x) = 01233$ (it fixes four values of $x$ and to the fifth assigns one of the fixed values.

THE CONSTRUCTION OF THE FUNCTIONS $\wedge_o$ AND $\vee_o$ FROM F(x,y) AND k(x)

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | 4 |
| $k(x)$ | 0 | 1 | 3 | 2 | 4 |

(The polynomial form of $k(x)$ is $x^3$).

Let the function $F_1$ be $F_1(x,y) = k(x) + k(y) + 4k(3x + 3y)$

$F_1(x,y)$

| | | | | |
|---|---|---|---|---|
| 0 | 4 | 2 | 3 | 1 |
| 4 | 1 | 0 | 0 | 0 |
| 2 | 0 | 3 | 0 | 0 |
| 3 | 0 | 0 | 2 | 0 |
| 1 | 0 | 0 | 0 | 4 |

The polynomial form of $F_1(x,y)$ is $4(x^3 + x^2y + xy^2 + y^3)$.
The table of $3k(x) + 3k(y)$ is:

| | | | | |
|---|---|---|---|---|
| 0 | 3 | 4 | 1 | 2 |
| 3 | 1 | 2 | 4 | 0 |
| 4 | 2 | 3 | 0 | 1 |
| 1 | 4 | 0 | 2 | 3 |
| 2 | 0 | 1 | 3 | 4 |

Using these functions

$$\wedge_o(x,y) = F_1\ (k(3x + 3y),\ 3k(x) + 3k(y))$$

The polynomial form of $\wedge_o$ is $4(x^4y + x^3y^2 + x^2y^3 + xy^4)$. To construct $\vee_o$ it is necessary to take the cube of the function $F_1(x,y)$. $F_2(x,y) = k(F_1(x,y))$. Its table is

| | | | | |
|---|---|---|---|---|
| 0 | 4 | 3 | 2 | 1 |
| 4 | 1 | 0 | 0 | 0 |
| 3 | 0 | 2 | 0 | 0 |
| 2 | 0 | 0 | 3 | 0 |
| 1 | 0 | 0 | 0 | 4 |

With this function

$$\vee_o(x,y) = F_2(\wedge_o(x,y), F_2(x,y))$$

The polynomial form of $\vee_o$ is $3x^4y + 4x^3y^2 + 4x^2y^3 + 3xy^4 + x + y$.
Using the same iteration steps one can derive also the unanimity function $M(x,y,z)$ from the other 9 such functions of $P_5$ which fix three values of $x$ and interchange the another two values.
(E.g. from $u(x) = 0\ 1\ 4\ 3\ 2$ can be produced in the way pre-

sented for $k(x)$ the functions

$$\wedge_3(x,y) \quad \begin{array}{|ccccc} \hline 0 & 3 & 3 & 3 & 3 \\ 3 & 1 & 3 & 3 & 3 \\ 3 & 3 & 2 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 4 \end{array} \quad \text{and} \quad \vee_3(x,y) \quad \begin{array}{|ccccc} \hline 0 & 3 & 3 & 0 & 3 \\ 3 & 1 & 3 & 1 & 3 \\ 3 & 3 & 2 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 4 & 4 \end{array}$$

from which the function $M$ is formed using (1) ).

THE CONSTRUCTION FROM THE FUNCTIONS F(x,y) AND $1_4$(x)

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | 4 |
| $l_4(x)$ | 0 | 1 | 2 | 3 | 3 |

We need a lot of superpositions since the function $l(x)$ make only a little change on $x$.

$$F(x,y) = 2x + 4y \quad \begin{array}{|ccccc} \hline 0 & 4 & 3 & 2 & 1 \\ 2 & 1 & 0 & 4 & 3 \\ 4 & 3 & 2 & 1 & 0 \\ 1 & 0 & 4 & 3 & 2 \\ 3 & 2 & 1 & 0 & 4 \end{array}$$

$$l_4(F(x,y) = \quad \begin{array}{|ccccc} \hline 0 & 3 & 3 & 2 & 1 \\ 2 & 1 & 0 & 3 & 3 \\ 3 & 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 3 & 2 \\ 3 & 2 & 1 & 0 & 3 \end{array}$$

$$H(x,y) = 3x + 3y \quad \begin{array}{|ccccc} \hline 0 & 3 & 1 & 4 & 2 \\ 3 & 1 & 4 & 2 & 0 \\ 1 & 4 & 2 & 0 & 3 \\ 4 & 2 & 0 & 3 & 1 \\ 2 & 0 & 3 & 1 & 4 \end{array}$$

$l_4(H(x,y))$

| | | | | |
|---|---|---|---|---|
| 0 | 3 | 1 | 3 | 2 |
| 3 | 1 | 3 | 2 | 0 |
| 1 | 3 | 2 | 0 | 3 |
| 3 | -2 | 0 | 3 | 1 |
| 2 | 0 | 3 | 1 | 3 |

$A = F(F, l_4F)$

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 3 | 2 | 1 |
| 2 | 1 | 0 | 0 | 3 |
| 0 | 3 | 2 | 1 | 0 |
| 1 | 0 | 0 | 3 | 2 |
| 3 | 2 | 1 | 0 | 0 |

$B = F(H, l_4H)$

| | | | | |
|---|---|---|---|---|
| 0 | 3 | 1 | 0 | 2 |
| 3 | 1 | 0 | 2 | 0 |
| 1 | 0 | 2 | 0 | 3 |
| 0 | 2 | 0 | 3 | 1 |
| 2 | 0 | 3 | 1 | 0 |

$C = A(A,B)$

| | | | | |
|---|---|---|---|---|
| 0 | 2 | 0 | 0 | 0 |
| 1 | 1 | 0 | 3 | 1 |
| 0 | 1 | 2 | 2 | 2 |
| 2 | 3 | 0 | 3 | 3 |
| 0 | 0 | 0 | 0 | 0 |

$D = A(B,A)$

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 0 | 3 | 3 |
| 0 | 1 | 0 | 0 | 2 |
| 2 | 2 | 2 | 0 | 1 |
| 0 | 0 | 0 | 3 | 0 |
| 1 | 3 | 0 | 2 | 0 |

$E = H(C,D)$

| | | | | |
|---|---|---|---|---|
| 0 | 4 | 0 | 4 | 4 |
| 3 | 1 | 0 | 4 | 4 |
| 1 | 4 | 2 | 1 | 4 |
| 1 | 4 | 0 | 3 | 4 |
| 3 | 4 | 0 | 1 | 0 |

$G = C(E,C)$

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | 0 | 0 | 0 |
| 1 | 0 | 2 | 0 | 0 |
| 0 | 0 | 0 | 3 | 0 |
| 2 | 0 | 0 | 1 | 0 |

$J = G(G,C)$

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 |
| 3 | 0 | 2 | 0 | 0 |
| 0 | 0 | 0 | 3 | 0 |
| 1 | 0 | 0 | 3 | 0 |

$K = G(G,J)$

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 2 | 0 | 0 |
| 0 | 0 | 0 | 3 | 0 |
| 0 | 0 | 0 | 0 | 0 |

$L = F(K,F)$

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |
| 3 | 1 | 0 | 1 | 2 |
| 1 | 2 | 2 | 4 | 0 |
| 4 | 0 | 1 | 3 | 3 |
| 2 | 3 | 4 | 0 | 1 |

$M = H(L, l_4H)$

| | | | | |
|---|---|---|---|---|
| 0 | 2 | 4 | 3 | 3 |
| 3 | 1 | 4 | 4 | 1 |
| 1 | 0 | 2 | 2 | 4 |
| 1 | 1 | 3 | 3 | 2 |
| 2 | 4 | 1 | 3 | 2 |

$N = E(M,L)$

| | | | | |
|---|---|---|---|---|
| 0 | 4 | 0 | 3 | 4 |
| 3 | 1 | 3 | 4 | 0 |
| 1 | 0 | 2 | 4 | 3 |
| 4 | 3 | 4 | 3 | 1 |
| 2 | 1 | 4 | 1 | 4 |

$O = F(H,N)$

| | | | | |
|---|---|---|---|---|
| 0 | 2 | 2 | 0 | 0 |
| 3 | 1 | 0 | 0 | 0 |
| 1 | 3 | 2 | 1 | 3 |
| 4 | 1 | 1 | 3 | 1 |
| 2 | 4 | 2 | 1 | 4 |

$P = O(O,K)$

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 0 |
| 4 | 1 | 0 | 0 | 0 |
| 3 | 4 | 2 | 3 | 4 |
| 2 | 3 | 3 | 3 | 3 |
| 1 | 2 | 1 | 3 | 2 |

$Q = F(P,C)$

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 2 | 0 | 0 |
| 2 | 1 | 0 | 2 | 4 |
| 1 | 2 | 2 | 4 | 1 |
| 2 | 3 | 1 | 3 | 3 |
| 2 | 4 | 2 | 1 | 4 |

$R = E(K,Q)$

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 4 |
| 4 | 0 | 2 | 4 | 4 |
| 0 | 4 | 4 | 3 | 4 |
| 0 | 4 | 0 | 4 | 4 |

$S = R(R,L)$

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 |
| 4 | 0 | 2 | 4 | 0 |
| 0 | 0 | 4 | 3 | 4 |
| 0 | 4 | 0 | 0 | 4 |

$T = S(S,H)$

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 |
| 4 | 0 | 2 | 0 | 0 |
| 0 | 0 | 0 | 3 | 4 |
| 0 | 0 | 0 | 0 | 4 |

$\wedge_o = T(T,H)$

$U = 4x+4y+3\wedge_o$

| | | | | |
|---|---|---|---|---|
| 0 | 4 | 3 | 2 | 1 |
| 4 | 1 | 2 | 1 | 0 |
| 3 | 2 | 2 | 0 | 4 |
| 2 | 1 | 0 | 3 | 3 |
| 1 | 0 | 4 | 3 | 4 |

$W = \wedge_o(U,F)$

| | | | | |
|---|---|---|---|---|
| 0 | 4 | 3 | 2 | 1 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 2 | 0 | 0 |
| 0 | 0 | 0 | 3 | 0 |
| 0 | 0 | 0 | 0 | 4 |

$Z = \wedge_o(4x\ \ 2y,U)$

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 4 | 1 | 0 | 0 | 0 |
| 3 | 0 | 2 | 0 | 0 |
| 2 | 0 | 0 | 3 | 0 |
| 1 | 0 | 0 | 0 | 4 |

$V_o = 4W + 4Z + 3\wedge_o$.

For the following 4 functions (with 4 fixed values) the same computations can be carried out as for $l_4(x)$

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $l_o(x)$ | 1 | 1 | 2 | 3 | 4 |
| $l_1(x)$ | 0 | 2 | 2 | 3 | 4 |
| $l_2(x)$ | 0 | 1 | 4 | 3 | 4 |
| $l_3(x)$ | 0 | 1 | 2 | 4 | 4 |

E.g. from $l_o(x)$ can be produced in the manner applicated for $l_4(x)$ the functions

$$\wedge_4(x,y)\ \begin{array}{|ccccc} 0&4&4&4&4\\ 4&1&4&4&4\\ 4&4&2&4&4\\ 4&4&4&3&4\\ 4&4&4&4&4 \end{array} \qquad \text{and} \qquad \vee_4(x,y)\ \begin{array}{|ccccc} 0&4&4&4&0\\ 4&1&4&4&1\\ 4&4&2&4&2\\ 4&4&4&3&3\\ 0&1&2&3&4 \end{array}$$

From $x$ and from an arbitrary other function $g$ with 4 fixed values one can produce with an admissible linear combination those function $l_i(x)$ which has the same 4 fixed values as $g$. The proof will be complete if we show that from an arbitrary non linear unary function of $P_5$ and from $F(x,y)$ we can obtain a function which has one of the previous two properties of $k(x)$ and $l_4(x)$. We show this statement first for some types of functions.

a) If in the clone $\mathcal{Z}$ $f$ and $g$ two functions which differ from each other only at one value of $x$ then $3f+2g$ is 0 for exactly 4 values of the argument, i.e. $3f+2g+x$ has four fixed values.

b) <u>Functions with 3 fixed values</u>

Let $f$ be a function which fixed 3 values of $x$ and interchanges the other two values of $x$ : $y$ and $z$. (From $f$ we can already derive near-unanimity function). Let $g$ be $3f+3x$. It fixes the same three values as $f$. Let $g(y) = g(z) = 3y + 3z = a_0$ which is different from $y$ and from $z$. Next let $h$ be $4g+2x$. $h(y) = 4a_0 + 2y = 2z + 4y$. Let $h(y) = a_1$ from the equations it follows that $a_1 \neq a_0$ $a_1 \neq y$, $a_1 \neq z$ $h(z) = 4a_0 + 2z = 4z + 2y$. $h(z)$ differs from $a_0$, $z$, $y$ too. The table of $F(x,y)$ shows that $2z+4y \neq 4z+2y$ i.e. $h(z) \neq a_1$. Let $h(z) = a_2$. Finally let $j$ be $2g+4x$.

| | | | | | |
|---|---|---|---|---|---|
| $x$ | $a_0$ | $a_1$ | $a_2$ | $y$ | $z$ |
| $f(x)$ | $a_0$ | $a_1$ | $a_2$ | $z$ | $y$ |
| $g(x)$ | $a_0$ | $a_1$ | $a_2$ | $a_0$ | $a_0$ |
| $h(x)$ | $a_0$ | $a_1$ | $a_2$ | $a_1$ | $a_2$ |
| $j(x)$ | $a_0$ | $a_1$ | $a_2$ | $a_2$ | $a_1$ |

From $g(x)$, $h(x)$ or from $j(x)$ one can derive the function $f$ with the following linear combinations:

$$f(x) = 2g(x) + 4x \qquad f(x) = 3h(x) + 3x \qquad f(x) = 4j(x) + 2x.$$

Let $m_i(x) = a_0a_1a_2za_i$ $(i=0,1,2)$. For $i=0,1,2$, $m_i \circ m_i(x) =$ $= a_0a_1a_2a_ia_i$ and $m_i \circ m_i$ differs from $m_i$ only at one value of $x$. The same is true for the functions $n_i(x) =$ $= a_0a_1a_2a_iy$ $(i=0,1,2)$.

The remaining 6 functions with the same three fixed values are $a_0a_1a_2a_0a_1$, $a_0a_1a_2a_0a_2$, $a_0a_1a_2a_1a_0$, $a_0a_1a_2a_1a_0$, $a_0a_1a_2a_1a_1$, $a_0a_1a_2a_2a_0$ and $a_0a_1a_2a_2a_2$. One can reduce them with linear combinations to the functions $m_i$ and $n_i$.
(E.g. $3(a_0a_1a_2a_2a_2) + 3x = n_1(x)$ ).

To function with three fixed values can be reduced such a function $b$ which differs from $b \circ b$ only at two values of $x$. The function $3b + 2(b \circ b)$ is 0 for exactly 3 values of $x$, i.e. the function $3b + 2(b \circ b) + x$ of the clone $Z$ has three fixed values.

c) Functions with 2 fixed values

The following table shows the functions for which $a_0$ and $a_1$ are the two fixed values. One (two) point(s) behind the function $f$ means that $f \circ f$ differs from $f$ only at one (two) value(s) of $x$. (4) means that $f \circ f$ has 4 fixed values.

| x $a_0a_1a_2a_3a_4$ f(x) | x $a_0a_1a_2a_3a_4$ f(x) | x $a_0a_1a_2a_3a_4$ f(x) | x $a_0a_1a_2a_3a_4$ f(x) |
|---|---|---|---|
| $a_0a_1a_0a_0a_0$ | $a_0a_1a_1a_0a_0$ | $a_0a_1a_3a_0a_0$· | $a_0a_1a_4a_0a_0$· |
| $a_0a_1a_0a_0a_1$ | $a_0a_1a_1a_0a_1$ | $a_0a_1a_3a_0a_1$· | $a_0a_1a_4a_0a_1$· |
| $a_0a_1a_0a_0a_2$· | $a_0a_1a_1a_0a_2$· | $a_0a_1a_3a_0a_2$·· | $a_0a_1a_4a_0a_2$(4) |
| $a_0a_1a_0a_0a_3$· | $a_0a_1a_1a_0a_3$· | $a_0a_1a_3a_0a_3$·· | $a_0a_1a_4a_0a_3$·· |

| x $a_oa_1a_2a_3a_4$ f(x) | x $a_oa_1a_2a_3a_4$ f(x) | x $a_oa_1a_2a_3a_4$ f(x) | x $a_oa_1a_2a_3a_4$ f(x) |
|---|---|---|---|
| $a_oa_1a_oa_1a_o$ | $a_oa_1a_1a_1a_o$ | $a_oa_1a_3a_1a_o\cdot$ | $a_oa_1a_4a_1a_o\cdot$ |
| $a_oa_1a_oa_1a_1$ | $a_oa_1a_1a_1a_1$ | $a_oa_1a_3a_1a_1\cdot$ | $a_oa_1a_4a_1a_1\cdot$ |
| $a_oa_1a_oa_1a_2\cdot$ | $a_oa_1a_1a_1a_2\cdot$ | $a_oa_1a_3a_1a_2\cdot\cdot$ | $a_oa_1a_4a_1a_2$(4) |
| $a_oa_1a_oa_1a_3\cdot$ | $a_oa_1a_1a_1a_3\cdot$ | $a_oa_1a_3a_1a_3\cdot\cdot$ | $a_oa_1a_4a_1a_3\cdot\cdot$ |
| $a_oa_1a_oa_2a_o\cdot$ | $a_oa_1a_1a_2a_o\cdot$ | $a_oa_1a_3a_2a_o$(4) | $a_oa_1a_4a_2a_o\cdot\cdot$ |
| $a_oa_1a_oa_2a_1\cdot$ | $a_oa_1a_1a_2a_1\cdot$ | $a_oa_1a_3a_2a_1$(4) | $a_oa_1a_4a_2a_1\cdot\cdot$ |
| $a_oa_1a_oa_2a_2\cdot\cdot$ | $a_oa_1a_1a_2a_2\cdot\cdot$ | $a_oa_1a_3a_2a_2$(4) | $a_oa_1a_4a_2a_2$(4) |
| $a_oa_1a_oa_2a_3\cdot\cdot$ | $a_oa_1a_1a_2a_3\cdot\cdot$ | $a_oa_1a_3a_2a_3$(4) | q: $a_oa_1a_4a_2a_3$ |
| $a_oa_1a_oa_4a_o\cdot$ | $a_oa_1a_1a_4a_o\cdot$ | $a_oa_1a_3a_4a_o\cdot\cdot$ | $a_oa_1a_4a_4a_o\cdot\cdot$ |
| $a_oa_1a_oa_4a_1\cdot$ | $a_oa_1a_1a_4a_1\cdot$ | $a_oa_1a_3a_4a_1\cdot\cdot$ | $a_oa_1a_4a_4a_1\cdot\cdot$ |
| $a_oa_1a_oa_4a_2\cdot\cdot$ | $a_oa_1a_1a_4a_2\cdot\cdot$ | p: $a_oa_1a_3a_4a_2$ | $a_oa_1a_4a_4a_2$(4) |
| $a_oa_1a_oa_4a_3$(4) | $a_oa_1a_1a_4a_3$(4) | $a_oa_1a_3a_4a_3$(4) | $a_oa_1a_4a_4a_3$(4) |

For the function $p$ and $q$ $p \circ p = q$ and $q \circ q = p$ hold. So either none or both of them are in the clone $Z$. $3p + 3q$ takes 3 different values for $a_2$, $a_3$ and $a_4$ because $3a_3+3a_4$, $3a_4+3a_2$ and $3a_2+3a_3$ differ from each other. An easy computation shows that $3p+3q \neq x$. So $3p + 3q$ such a function at least with two fixed values which differs from $p$, $q$ and from the functions within the frames.

Let $f(x) = a_oa_1a_ia_ja_k$ $(i,j,k \in \{0,1\})$ one of the two valued functions of the table. The values $2a_i+4a_2$ and $4a_i+2a_2$ differ from $a_i$, $a_2$ and from each other. So at least one of them is $a_3$ or $a_4$. Therefore at least one of the functions $2f(x) + 4x$ and $4f(x) + 2x$ is such a function with two fixed values which differ from every function with the frames.

Finally if the function 4 differs from $h \circ h$ at three values of $x$ then the function $3h + 2(h \circ h) + x$ has two fixed values. Now we list the unary non linear functions of $P_5$ and reduce them to such functions which are in the lexicographical ordering more ahead.

| $x$ 0 1 2 3 4 <br> $f(x)$ | Reduction of f(x) |
|---|---|
| 0 0 a b c | Every of these 124 functions has one of the previous properties. |
| 0 1 a b c | Every of these 124 functions has at least two fixed values. |
| 0 2 a b c | $2x+4(02abc) = 00def$ |
| 0 3 a b c | $4x+2(03abc) = 00def$ |
| 0 4 a b c | $3x+3(04abc) = 00def$ |
| 1 0 a b c | $f \circ f = 01a'b'c$ |
| 1 1 a b c | Every has one of the listed properties |
| 1 2 0 a b | $f \circ f \circ f = 012a'b'$ |
| 1 2 1 a b | $f \circ f = 212cd$, $4(212cd)+2(121ab) = 030ef$ |
| 1 2 2 a b | $f \circ f = 222cd$, $4(222cd)+2(122ab) = 022ef$ |
| 1 2 3 a b | $f \circ f = 23acd$ $4(23acd)+2(123ab) = 01ghi$ |
| 1 2 4 a b | $f \circ f = 24bcd$ $4(24bcd)+2(124ab) = 00ghi$ |
| 1 3 a b c | $f \circ f = 3bdhi$ $2(3bdhi)+4(13abc) = 0jklm$ |
| 1 4 a b c | $f \circ f = 4cdhi$ $3(4cdhi)+3(14abc) = 0jklm$ |
| 2 a b c d | $3x+3(2abcd) = 1ghij$ |
| 3 a b c d | $4x+2(3abcd) = 1ghij$ |
| 4 a b c d | $2x+4(4abcd) = 1ghij$ |

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# ÖSSZEFOGLALÁS

## A $P_3$ ÉS $P_5$-BEN LEVŐ VÉGESEN GENERÁLT KLÓNOK JELLEMZÉS LINEÁRIS FÜGGVÉNYEKKEL

*Csikós M.*

A cikkben a szerző a következő tételt bizonyitja be:

A $P_3$ és $P_5$-ben egy klón akkor és csakis akkor végesen generált, ha tartalmaz egy nemtriviális n-szeres lineáris függvényt ($n \geq 2$) és egy egyszeres nemlineáris függvényt. Ez Demetrovics és Bagyinszki ill. Marčenkov idevágó eredményeit egésziti ki.

## КОНЕЧНО ПОРОЖДЕННЫЕ КЛОНЫ В $P_3$ И $P_5$ С ЛИНЕЙНЫМИ ФУНКЦИЯМИ

М. Чикош

В статье доказывается следующая теорема:

Клон содержащийся в $P_3$ или $P_5$ является конечно порожденным тогда и только тогда, если содержит n-арную линейную функцию ($n \geq 2$) и унарную нелинейную функцию. Этот результат добавляет новые информации к результатам Бадьински - Деметровича и Марченкова.