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STRUCTURAL CHARACTERISTICS OF ONE CLASS OF BOOLEAN FUNCTIONS

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The terminology used in this paper is from [1-15]. The paper treats the Boolean functions f, which essentially depend on at least six (seven) variables and for which there are variables x_i and x_j such that $\{x_i, x_j\} \in S_f$ and the subgraph of f with the elements of $R_f \backslash \{x_i, x_j\}$ as apexes has the shape of an elementary chain, opened or closed.

The set of all essential variables of f and all spearable sets of arguments of f is denoted by R_f and S_f, respectively.

$S_{f,2}$ stands for the set of all separable two-element sets of arguments of f.

The number of all separables pairs of the function $f(x_1, \ldots, x_n)$ wherein x_i takes part, will be referred to as the order of the variables x_i for the above mentioned function.

Theorem I. If the set $\{x_i, x_j\}$ is separable for a Boolean function $f(x_1, \ldots, x_n)$ which essentially depends on $(n \geq 6)$ variables and the subgraph of f with the elements of the set $\{x_1, \ldots, x_n\} \backslash \{x_i, x_j\}$ as apexes has the shape of an elementary open chain, then one of the variables x_i, x_j is of order $n-1$ for f, and the other variable is of order not smaller than $n-4$.

Proof. Under the conditions stated in the theorem for the function $f(x_1, \ldots, x_n)$, let us accept that $\{x_{n-1}, x_n\} \in S_f$ and that the subgraph of f with apexes $x_1, x_2, \ldots, x_{n-2}$ has the shape of an elementary open chain, where for every $i = 1,\dots,n-3$

$$\{x_i, x_{i+1}\} \in S_f.$$

At least one of the variables x_{n-1}, x_n is of order $n-1$ for f. Without restricting the subject examination let us accept that x_{n-1} is of an order $n-1$ for f. We shall prove that x_n is of an order not smaller than $n-4$.

We shall do it by using the method of mathematical induction.

We shall prove the theorem when $n = 6$. We must prove that x_6 is of an order not smaller than 2. Let us assume the opposite. So x_6 must be of order 1 for f, moreover $\{x_5, x_6\} \in S_f$.

Let us denote by α the value of x_5 for which it is true that

$$x_6 \in R_{f(x_5=\alpha)}$$

Then it must be true that

$$R_{f(x_5=\bar{\alpha})} = \{x_1, x_2, x_3, x_4\},$$

and

$$S_{f(x_5=\bar{\alpha}),2} = \{\{x_1,x_2\},\{x_2,x_3\},\{x_3,x_4\}\},$$

which is impossible, according to theorem 21 from [14]. Let us prove the theorem for $n=7$. We must prove that x_7 is of order not smaller than the 3. Let us suppose it is not true, that is x_7 is of order 1 or 2 for $f(x_1,\dots,x_7)$.

The variable x_7 cannot be of order 1 for $f(x_1,\dots,x_7)$. Indeed, if $f(x_6 = \alpha)$ depends essentially on x_7, it means that

$$S_{f(x_6=\bar{\alpha}),2} = \{\{x_1,x_2\},\{x_2,x_3\},\{x_3,x_4\},\{x_4,x_5\}\},$$

which is not possible.

So x_7 must be of order 2 for $f(x_1,\dots,x_7)$ where $\{x_6,x_7\} \in S_f$.

Thus x_7 forms exactly one separatable pair for f with a variable from the set $\{x_1,\dots,x_5\}$. Let us accept that $\{x_7,x_5\} \in S_f$. Let α be a value for x_5 for which $f_1 = f(x_5 = \alpha)$ depends essentially on x_7. According to theorem 26 from [14],

$$R_{f_1} = R_f \setminus \{x_5\}.$$

From theorem 21 and 26 from [14] it follows that

$$\{x_6,x_7\} \in S_{f_1},$$

and the subgraph of f_1 with the apexes $x_1,\dots,x_4$ has the shape of elementary open chain. Moreover x_7 must be of order 1, for f_1 with respect to the separable pairs, which contradicts to the already proved case of the theorem when $n = 6$.

By analogy we come to a contradiction if we accept that $\{x_1,x_7\} \in S_f$. Let us accept that

$$\{x_4,x_7\} \in S_f.$$

Let C_2 be a value of x_2 for which

$$x_1 \in R_f, \quad f_2 = f\ (x_2 = c_2).$$

Then

$$R_{f_2} = R_f \setminus \{x_2\},$$

and x_1 will be of order 1 for f_2 and

$$\{x_1,x_6\} \in S_{f_2}.$$

Let C_6 be a value for x_6 such that the function $f_2\ (x_6 = c_6)$ depends essentially on x_1. Then

$$R_{f_3} = R_f \setminus \{x_1,x_2,x_{n-1}\},$$

where

$$f_3 = f_2\ (x_6 = \bar{c}_6).$$

The function f_3 must have three variables of order 1. But this is impossible (see [4] and [14]).

By analogy we come to a contradiction if we accept that

$$\{x_2,x_7\} \in S_f.$$

Let us accept that

$$\{x_3, x_7\} \in S_f.$$

Let C_3 be a value for x_3 such that the function $f_4 = f(x_3 = c_3)$ essentially depends on x_7. In this case x_6 must be of the order 5 for f_4 nd separable pairs for f must not be formed from the elements of the set $\{x_7\}, \{x_1, x_2\}, \{x_4, x_5\}$, which is impossible.

This proves the theorem when $n = 7$, too.
Let us accept that the theorem is true for some $n \geq 7$. We shall prove, that it is also true for the Boolean functions, which essentially depend on $n + 1$ variables and fulfil all the conditions of the theorem.

Let $f(x_1, \ldots, x_{n+1})$ be a Boolean function, which essentially depends on $n + 1$ $(n \geq 7)$ variables and $\{x_n, x_{n+1}\} \in S_f$. According to the conditions of the theorem the subgraph of f with the elements of the set $\{x_1, \ldots, x_{n-1}\}$ as apexes has the shape of an elementary open chain. For example let us accept that for every $i = 1, \ldots, n-2$,

$$\{x_i, x_{i+1}\} \in S_f.$$

At least one of the variables x_n, x_{n+1}, is of order n. For example, let x_n be of order n for the function $f(x_1, \ldots, x_{n+1})$, with respect tothe separable pairs. We shall prove that x_{n+1} is of order not smaller than $n - 3$ for $f(x_1, \ldots, x_{n+1})$.

We shall prove that under the conditions given in the theorem $\{x_1, x_{n+1}\} \in S_f$ or $\{x_{n-1}, x_{n+1}\} \in S_f$.

Let us assume this is not true, i.e.

$$\{x_1, x_{n+1}\} \notin S_f \quad \text{and} \quad \{x_{n-1}, x_{n+1}\} \notin S_f.$$

Let C_2 be a value for x_2 such that

$$x_1 \in R_{f_5}, \; f_5 = f(x_2 = c_2).$$

Then

$$R_{f_5} \supset \{x_3, \ldots, x_{n-1}\}.$$

The variable x_1 must be of order 1 for f_5. Thus $x_n \in R_{f_5}$. We shall prove that $x_{n+1} \in R_{f_5}$. If we assume the opposite and choose α in such a way that

$$x_1 \in R_{f_5}(x_n = \alpha),$$

then the function

$$f_6 = f_5 \ (x_n = \bar{\alpha}),$$

will depend essentially on $n-3$ variables and its graph will have the shape of an elementary open chain; and this is impossible. Therefore $x_{n+1} \in R_{f_5}$.

Since we have assumed that $\{x_{n-1}, x_{n+1}\} \notin S_f$, so $\{x_{n-1}, x_{n+1}\} \notin S_{f_5}$. Let α be a value for x_n such that

$$x_1 \in R_{f_5}(x_n = \alpha).$$

Then the function

$$f_6 = f_5(x_n = \bar{\alpha})$$

will essentially depend on $n-2$ variables. So x_{n+1} will be of order 1 for f_6 and $\{x_{n-1}, x_{n-2}\} \in S_{f_6}$. Therefore it must be true that $\{x_3, x_{n-2}\} \in S_{f_6}$, and $\{x_3, x_{n-2}\} \in S_f$ where $n \geqq 7$. The last statement contradicts to the initially given conditions.

We proved that

$$\{x_1, x_{n+1}\} \in S_f \quad \text{or} \quad \{x_{n-1}, x_{n+1}\} \in S_f.$$

Let us discuss the case when $\{x_1, x_{n+1}\} \in S_f$. Let C_1 be a value for x_1 such that

$$\{x_n, x_{n+1}\} \in S_{f_7}, \quad f_7 = f(x_1 = c_1).$$

But

$$R_{f_7} = \{x_2, \ldots, x_{n+1}\}$$

and the subgraph of f_7 with the elements of the set $\{x_2, \ldots, x_{n-1}\}$ as apexes has the shape of elementary open chain.

From the inductive assumption it follows that one of the variables x_n, x_{n+1} is of order $(n - 1)$ for f_7, and the other one is of an order not smaller than $n-4$.

Therefore the variable x_{n+1} will be of order not smaller than $x-4$ for f_7. Thus x_{n+1} will be of order not smaller than $n-3$ for f.

By analogy we can prove the theorem if $\{x_{n-1}, x_{n+1}\} \in S_f$. The theorem is proved. □

Is it possible to strenghten theorem 1 in the sense that, under the conditions of the theorem, one of the variables x_i, x_j is of order $n-1$ $(n \geq 6)$ and the other one is at least of order $n-3$ for f, in regard to the separable pairs? The answer is NO.

For example, let us take the Boolean function

$$f = x_1(x_2x_3 + \bar{x}_3x_4) + \bar{x}_1(x_4x_5 + x_5\bar{x}_6) \quad (mod\ 2).$$

The subgraph of f with apexes x_2, x_3, x_4, x_5 has the shape of an elementary open chain. The pair $\{x_1, x_6\}$ is separable for f, x_1 is of order 5 and x_6 is of order 2 for f in regard to the separable pairs.

Theorem II. If the set $\{x_i, x_j\}$ is separable for the Boolean function $f(x_1, \ldots, x_n)$, which essentially depends on n $(n \geq 7)$ variables and the subgraph of f with the elements of the set $\{x_1, \ldots, x_n\} \setminus \{x_i, x_j\}$ as apexes has the shape of an elementary closed chain, then one of the variables x_i, x_j is of order $n-1$ for f, and the other one is or order not smaller than $n-4$.

Proof. Let us assume that the function $f(x_1, \dots, x_n), (n \geq 7)$, satisfies the conditions of the theorem. We may assume that

$$\{x_{n-1}, x_n\} \in S_f,$$

and that the subgraph of f with the elements of the set $\{x_1, \dots, x_{n-2}\}$ as apexes has the shape of a closed elementary chain, and for every $i = 1, \dots, n-3$

$$\{x_i, x_{i+1}\} \in S_f \text{ and } \{x_{n-2}, x_1) \in S_f.$$

At least one of the variables x_{n-1}, x_n is of order $n-1$ for f. Let us accept that x_{n-1} is of order $n-1$ for f. We shall prove that x_n is of order not smaller than $n-4$ for f.

It is impossible for x_n to be of order 1 for f. Let us assume the opposite. If α is a value of x_{n-1} such that

$$x_n \in R_{f(x_{n-1}=\alpha)}.$$

then the graph of $f(x_{n-1} = \bar{\alpha})$ must have the shape of a closed elementary chain, which is impossible.

From what we have assumed it follows that there is a variable x_i, $i \in \{1, \dots, n-2\}$ such that $\{x_i, x_n\} \in S_f$. Let x_i be such a variable and let c_i be a value such that

$$\{x_{n-1}, x_n\} \in S_{f(x_i=c_i)}.$$

But in this case

$$R_{f(x_i=c_i)} = R_f \setminus \{x_i\},$$

and the subgraph of $f(x_i = c_i)$ with the elements of the set $\{x_1, \dots, x_{n-2}\} \setminus \{x_i\}$ as apexes has the shape of an elementary open chain. According to theorem 1 the variable x_n must be of order not smaller than $n-5$ for $f(x_i=c_i)$. Then x_n will be of order not smaller than $n-4$ for f.

Thus the theorem is proved.

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Ö S S Z E F O G L A L Á S

A BOOLE FÜGGVÉNYEK EGY OSZTÁLYÁNAK STRUKTURÁLIS TULAJDONSÁGAIRÓL

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A szerzők a cikkben $n(\geq 6)$ változós Boole függvényeknek egy osztályára adnak jellemzést. Bebizonyitják, hogy a függvény változói közül egynek a rendje $n-1$ és egy másiknak a rendje legalább $n-4$.

О СТРУКТУРАЛЬНЫХ СВОЙСТВАХ ОДНОГО КЛАССА БУЛЕВЫХ ФУНКЦИЙ

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В статье дается характеризация одного класса Булевых функций $n(\geq 6)$ переменных. Доказывается, что для функций из одного класса одна из двух переменных имеет порядок $n-1$ и порядок другой не меньше $n-4$.