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CONGRUENCES ON CLOSED SETS OF SELF-DUAL FUNCTIONS IN MANY-VALUED LOGICS AND ON CLOSED SETS OF LINEAR FUNCTIONS IN PRIME-VALIED LOGICS

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A congruence on a closed set M in the k-valued logic P_k, $k\geq 2$, is an equivalence relation on M which is compatible with the operations (permutation and identification of variables, addition of fictitious variables and substitution) of P_k.

In this paper we prove that the congurences on closed sets of self-dual functions of P_k are determined by congruences on closed sets of non-self-dual functions.

Moreover, we determine all congruences on the closed sets of linear functions (see [1] and [8]) in prime-valued logic.

1. Basic concepts

Let E_k denote the set $\{0,1,\ldots,k-1\}$, where $k\geq 2$. Let P_k^n denote the set of all functions $f^n: E_k^n \to E_k$ $(n\geq 1)$ and put $P_k = \bigcup_{n\geq 1} P_k^n$. If there is no danger of confusion, the superscript n of the function f^n is omitted.

The set of all function of P_k^1 having exactly 1 values we denote by $P_k^{[1]}$.

The functions $e_i^n \in P_k$ $(1\leq i\leq n)$ defined by $e_i^n(x_1,\ldots,x_n)=x_i$ are called projections. The n-ary constant function with value a is denoted by c_a^n.

The operations on P_k are ζ, τ, Δ, ∇, $*$, which are defined by

$$(\zeta f)(x_1,\ldots,x_n)= f(x_2,x_3,\ldots,x_n,x_1)$$
$$(\tau f)(x_1,\ldots,x_n)= f(x_2,x_1,x_3,\ldots,x_n)$$
$$(\Delta f)(x_1,\ldots,x_{n-1})= f(x_1,x_1,x_2,\ldots,x_{n-1})$$
$$(\nabla f)(x_1,\ldots,x_{n+1})= f(x_2,x_3,\ldots,x_{n+1})$$

$$(f*g)(x_1,x_2,\dots,x_{n+m-1}) = f(g(x_1,\dots,x_m),x_{m+1},\dots,x_{n+m-1}),$$

where f^n, $g^m \in P_k$ (see [5] or [6]).

Superpositions over the set A_P_k are functions obtained from A by using the operations $\zeta,\tau,\Delta,\nabla,*$ finitely many times. The closure $[A]$ of a set $A\subseteq P_k$ is the set of all superpositions over A. A set A is said to be closed if $[A]=A$.

An equivalence relation κ on a closed set A is called a congruence on A iff $f\sim g\ (\kappa)$, $s\sim t\ (\kappa)$ imply $f*s\sim g*t\ (\kappa)$ and $\alpha f\sim\alpha g\ (\kappa)$ for all $\alpha\in\{\zeta,\tau,\Delta,\nabla\}$ and for all $f,g,s,t\in A$. A.I.Mal'cev showed in |5| that every closed set A_P_k has three trivial congruences κ_0, κ_a and κ_1:

$$f\sim g\ (\kappa_0) :\Longleftrightarrow f=g \wedge \{f,g\}\subseteq A$$

$$f^n\sim g^m\ (\kappa_a) :\Longleftrightarrow n=m \wedge \{f,g\}\subseteq A$$

$$f\sim g\ (\kappa_1) :\Longleftrightarrow \{f,g\}\subseteq A.$$

Let κ and κ' be two congruences on A. We write $\kappa\subseteq\kappa'$ iff $f\sim g\ (\kappa)$ implies $f\sim g\ (\kappa')$ for all f, $g\in A$.

For the other undefined notations we refer the reader to [1]-[8], particularly to [6].

For the proofs of our theorems we need the following lemma which is well known.

1.1 LEMMA ([2]). Let A be a closed set in P_k containing the projections. If κ is a congruence on A with $\kappa\not\subseteq\kappa_a$, then $\kappa=\kappa_1$.

2. Congruences on closed sets of self-dual functions of P_k

Let $s(x)=x+1 \bmod k$ and let S be the set of all functions of P_k preserving the relation $\{(a,s(a))\mid a\in E_k\}$. The functions of S are called self-dual functions. If k is a prime number then S is a maximal closed set of P_k ([7]). In [3] for a maximal closed set of self-dual functions it was proved that a function $f^n \in P_k$ is a function of S iff there exists a function $F^{n-1} \in P_k$ with the property

$$f(\tilde{x})=\sum_{i=0}^{k-1} j_i(x_1)\cdot s^i(F(s^{k-i}(x_2),\dots,x^{k-i}(x_n))) \bmod k, \quad (1)$$

where $s^i(x)$: $x+i \bmod k$; $j_i(x)=\begin{cases}1 & if\ x\ i\\ 0 & otherwise\end{cases}$ and

$F(x_1,\dots,x_{n-1})=f(0,x_1,\dots,x_{n-1})$. The proof in [3] does not use the property that k is prime. Therefore (1) is right for every k. Because of this property we can define a bijective mapping α of $S(\_P_k)$ onto P_k': $\bigcup_{n\geq 0}\{f^n:E_k^n\to E_k\}$ as follows:

$$\alpha : f \to F.$$

2.1 LEMMA. The mapping α has the following properties:

(i) For the operations $\hat{\zeta},\hat{\tau},\hat{\Delta},\hat{\nabla},\hat{*}$ defined by

$$(\hat{\zeta}f)(x_1,\dots,x_n)=f(x_1,x_3,x_4,\dots,x_n,x_2)$$
$$(\hat{\tau}f)(x_1,\dots,x_n)=f(x_1,x_3,x_2,x_4,\dots,x_n)$$
$$(\hat{\Delta}f)(x_1,\dots,x_{n-1})=f(x_1,x_2,x_2,x_3,\dots,x_{n-1})$$
$$(\hat{\nabla}f)(x_1,\dots,x_{n+1})=f(x_1,x_3,x_4,\dots,x_n)$$
$$(f\hat{*}g)(x_1,\dots,x_{n+m-2})=f(x_1,g(x_1,x_2,\dots,x_m),x_{m+1},\dots,x_{n+m-2})$$

is $\alpha(\hat{\zeta}f)=\zeta F$, $\alpha(\hat{\tau}f)=\tau F$, $\alpha(\hat{\Delta}f)=\Delta F$, $\alpha(\hat{\nabla}f)=\nabla F$ and $\alpha(f\hat{*}g)=F*G$, i.e. the algebra $\langle S;\hat{\zeta},\hat{\tau},\hat{\Delta},\hat{\nabla},\hat{*}\rangle$ is isomorphic to the algebra $\langle P_k';\zeta,\tau,\Delta,\nabla,*\rangle$.

(ii) For every closed subset $A(\neq\emptyset)$ of S is $\alpha(A)$ a closed set, $\alpha(A)\not\subseteq S$ and $A\subseteq\alpha(A)$.

PROOF. (i) is easy to check.
Let A be a closed subset of S. Then by (i) we get that $\alpha(A)$ is likewise a closed set. Assume $\alpha(A)\subseteq S$. Then

$$F(x_2,\dots,x_n)=s^i(F(s^{k-i}(x_2),\dots,s^{k-i}(x_n)))$$

for $i=0,1,\ldots,k-1$ and for every $f^n \in A$ (see [3]). Thus by (1) we get that the variable x_1 in every function $f \in A$ is fictitious. however, this is not possible. Therefore is $\alpha(A) \not\subseteq S$.
Let $f \in A$. Then is $\nabla f \in A$ and therefore $\alpha(\nabla f) = f \in \alpha(A)$, i.e. $A \subseteq \alpha(A)$.

2.2 THEOREM. Let A be a closed subset of S, κ a congruence on A and let $\alpha(\kappa)$ defined by

$$F \sim G(\alpha(\kappa)) :\Longleftrightarrow \alpha^{-1} F \sim \alpha^{-1} G \ (\kappa)$$

an equivalence relation on $\alpha(A)$. Then

(i) $\alpha(\kappa)$ is a congruence on $\alpha(A)$ and

(ii) $\alpha(\kappa)_{/A} =$, i.e. the congruences on A we can get by restriction of the congruences on $\alpha(A)$ to A.

PROOF. (i). Since κ is a congruence on A κ is also compatible with the operations $\hat{\zeta}, \hat{\tau}, \hat{\Delta}, \hat{\nabla}, \hat{*}$. Then by 2.1 (i) follows that $\alpha(\kappa)$ is a congruence on $\alpha(A)$.
(ii) By 2.1 (ii) is $A \subset \alpha(A)$ and therefore $\alpha(\kappa)_{/A}$ is a congruence on A. Let f and g be functions of A. If $f \sim g \ (\kappa)$ then $\nabla f \sim \nabla g \ (\kappa)$ and by definition of $\alpha(\kappa)$ is $\alpha(\nabla f) = f \sim g = \alpha(\nabla g) (\alpha(\kappa)_{/A})$, i.e. $\subseteq \alpha(\kappa)_{/A}$. If $f \sim g \ (\alpha(\kappa)_{/A})$ then by definition of $\alpha(\kappa)$ we get that $\alpha^{-1} f \sim \alpha^{-1} g \ (\kappa)$. Since f and g are functions of S is $\alpha^{-1} f = \nabla f$ and $\alpha^{-1} g = \nabla g$. Therefore is $\nabla f \sim \nabla g \ (\kappa)$ and $\Delta(\nabla f) = f \sim g = \Delta(\nabla g) \ (\kappa)$, i.e. $\alpha(\kappa)_{/A} \subseteq \kappa$. Thus $\alpha(\kappa)_{/A} =$.

3. Congruences on some closed subsets of $[P_k^1]$

In this section we prove a theorem which we need for the determination of the congruences on the closed subsets of linear functions.

Let $C \subseteq P_k^{[1]}$, G a subgroup os $\langle P_k^{[k]}; * \rangle$, where the functions of G preserve the set C, U a normal subroup of the group G and let μ be an equivalence relation on C which is preserved by the functions of G.

It is easy to check that the equivalence relation $^{U,\mu}$ on $[G\cup C]$ defined by

$$f^n \sim g^m(\kappa^{U,\mu}) :\Longleftrightarrow n = m \wedge (\exists i\ \exists f',g' \in G\cup C:\ f(\tilde{x}) = f'(x_i) \wedge g(\tilde{x}) = g'(x_i) \wedge (f'*U = g'*U \vee f' \sim g'(\mu)))$$

is a congruence on $[G\cup C]$.

3.1 THEOREM. Let G be a subgroup of $\langle P_k^{[k]};*\rangle$, $C \subseteq P_k^{[1]}$ and $G \subseteq Pol\ C$. Then exactly on $[G\cup C]$ there exist the congruences κ_0, κ_a, κ_1 and congruences of the type $\kappa^{U,\mu}$, where U is a normal subgroup of G and μ is an equivalence relation on G which is preserved by the functions of G.

PROOF. Let κ be a congruence on $[G\cup C]$ and $\kappa \neq \kappa_1$. Then by 1.1 is $\kappa \subseteq \kappa_a$. We have to ditinguish the following cases:

Case 1: There exist κ-congruent functions f^n and g^n with $\Delta^{n-1}f \in G$ and $\Delta^{n-1}g \in C$.

Then is $\Delta^{n-1}f \sim \Delta^{n-1}g(\kappa)$. Thus $e_1^1 \sim \Delta^{n-1}g =: c_a(\kappa)$, $a \in E_k$.

By this we have for every function $h^m \in [G\cup C]$:

$e_1^1 * h = h \sim c_a^m = c_a * h(\kappa)$, i.e. $\kappa = \kappa_a$.

Case 2: There exist κ-congruent functions f^n and g^n with $f(x_1,\dots,x_n) = f'(x_i), g(x_1,\dots,x_n) = g'(x_j)$, $\{f',g'\} \subseteq G$ and $i \neq j$.

Without loss of generality we can assume that $i=1$ and $j=2$. The inverse functions of f' and g' we denote by f'' and g'', respectively. Then we have

$f(f''(x_1), g''(x_2), x_2, \dots, x_2) = e_1^2(x_1,x_2)$
$\sim g(f''(x_1), g''(x_2), x_2, \dots, x_2) = e_2^2(x_1,x_2)\ (\kappa)$. Therefore is
$e_1^2(s(\tilde{x}), t(\tilde{x})) = s(\tilde{x}) \sim t(\tilde{x}) = e_2^2(s(\tilde{x}), t(\tilde{x}))\ (\kappa)$ for every s
and t of $[G\cup C]^m$, $m \geq 1$, i.e. $\kappa = \kappa_a$.

Case 3: Two n-ary functions f and g are κ-congruent if and only if either $\{f,g\}\subseteq[C]$ or there exist an i and $f',g'\in G$ with $f(x_1,\ldots,x_n)=f'(x_i)$, $g(x_1,\ldots,x_n)=g'(x_i)$.

In this case the congruence κ is exactly determined by $\kappa_{/G}$ and $\kappa_{/C}$.

As we know, the congruence on a group G are determined by a normal group U of G and $f\sim g(\ )$ iff $f*U=g*U$ for all f, $g\in G$.

Obviously, $_{/C}$ is an equivalence relation on C which is preserved by every function of G.

Thus $=\ ^{U},\ /C$.

4. Congruences on closed sets of linear functions in prime-valued logics

Let p be a fixed prime number. L denote the set of all linear functions over $<E_p;\ +,\ \cdot\ mod\ p>$ in P_p, i.e.

$$L:=\bigcup_{n\geq 1}\{f^n\in P_p\,|\,\exists a_i: f(\tilde{x})=a_0+\sum_{i=1}^{n} a_i\cdot x_i \ mod\ p\}.$$

In |1| it was proved that L has exactly the following closed subsets:

$$L\cap S=\bigcup_{n\geq 1}\{f^n\in L\,|\,a_1+a_2+\ldots+a_n=1\ mod\ p\},$$

$$L\cap Pol(a)=\bigcup_{n\geq 1}\{f^n\in P_p\,|\,\exists a_i: f(\tilde{x})=a+\sum_{i=1}^{n} a_i\cdot(x_1-a)\},$$

$a\in E_p$,

$L\cap S\cap Pol(0)$ and closed subsets A with $A\subseteq[L^1]$.

If $A\subseteq[L^{[1]}]$, then it is easy to see that the closed set A has only congruences of the type κ^{μ} and of the type κ^{μ}_{a} defined by

$$f^n \sim g^m(\kappa^{\mu}) :\Longleftrightarrow \Delta^{n-1} f \sim \Delta^{m-1} g\ (\mu) \quad \text{and}$$

$$f^n \sim g^m(\kappa_a^{\mu}) :\Longleftrightarrow n=m \wedge \Delta^{n-1} f \sim \Delta^{n-1} g(\mu),$$

where μ is an any equivalence relation on A^1.
If $A \not\subseteq [L^{[1]}]$ and $A \subseteq [L^1]$ then the congruences on A follow by theorem 3.1.

We denote by κ_c an equivalence relation defined by

$$f^n \sim g^m(\kappa_c) :\Longleftrightarrow n=m \wedge (\exists a: f(\tilde{x})=a+g(\tilde{x}) \bmod p).$$

Obviously, κ_c is a congruence on L. We will show that κ_c is the only nontrivial congruence on $A \subseteq L$ for $A \not\subseteq [L^1]$.

4.1 THEOREM ([4]). κ_0, κ_c, κ_a and κ_1 are the only congruences on L .

4.2 THEOREM. κ_0, κ_a and κ_1 are the only congruences on $L \cap Pol(a)$ for every $a \in E_p$.

PROOF. Clearly, the closed sets $L \cap Pol(a)$, $a \in E_p$, are mutually isomorphic. Therefore we can assume that $a=0$. Let κ be a congruence on $L \cap Pol(0)$. The following two cases are possible:

Case 1: $\kappa \not\subseteq \kappa_a$.
By 1.1 is $\kappa = \kappa_1$.
Case 2: $\kappa_0 \subset \kappa \subseteq \kappa_a$.

Then there exist κ -congruent functions f^n, g^n and an n-tuple $\tilde{a}=(a_1,\dots,a_n)$ with $f(\tilde{a}) \neq g(\tilde{a})$. Therefore is
$f(a_1 \cdot x, a_2 \cdot x, \dots, a_n \cdot x) =: a \cdot x \sim b \cdot x := g(a_1 \cdot x, a_2 \cdot x, \dots, a_n \cdot x)\ (\kappa)$,
where $a \neq b$.

The functions $h(x,y)=x-y \bmod p$ and $t(x)=(a-b)^{-1} \cdot x$ belong to $L \cap Pol(0)$. Thus we get $h(ax,ax)=c_0 \sim h(ax,bx) := h'(x)$ ( ) and $c_0^1 = t * c_0^1 \sim t * h' = e_1^1\ (\kappa)$. This implies that $c_0^1 * r^m = c_0^m \sim r = e_1^1 * r\ (\kappa)$ for every $r^m \in L \cap Pol(0)$, $m \geq 1$. Therefore $\kappa = \kappa_a$.

4.3 THEOREM. κ_0, κ_c, κ_a and κ_1 are the only congruences on $L \cap S$.

PROOF. Obviously, $\alpha(L \cap S) = L$. Therefore, using 2.2 and 4.1 we have the the theorem.

4.4 THEOREM. κ_0, κ_a and κ_1 are the only congruences on $L \cap S \cap Pol(0)$.

PROOF. The theorem follows from $\alpha(L \cap S \cap Pol(0)) = L \cap Pol(0)$, 2.2 and 4.2.

We also remark that the structure of the congruences becomes more complicated, if k is not a prime number. If k is square-free, then follows by [8] and by [4] (theorem 3.6) that every closed subset of L has only a finite number of congruences. But, if k is not square-free then there exist closed subsets of L with an infinite number of congruences. Finally we give an example for a closed subset with a such a property.

Let $Z := \bigcup_{n \geq 1} \{f^n \in P_4 \mid \exists a_i \in \{0,2\} : f(\tilde{x}) = \sum_{i=1}^{n} a_i \cdot x_i \bmod p\}$, let $r(f)$ be the number of the non-fictitious variables of the function f.

Further let χ_i be an equivalence relation defined by

$$f^n \sim g^m (\chi_i) :\Longleftrightarrow f = g \vee (n = m \wedge r(f) \leq i \wedge r(g) \leq i),$$

$i = 1, 2, \ldots$. If is easy to prove that χ_i for all $i \geq 1$ is a congruence on Z of P_4.

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