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MINKOWSKI'S CONVEX BODY THEOREM AND THE MEASURE OF COVERING R^n BY A SET

Béla UHRIN

Computer and Automation Institute
Hungarian Academy of Sciences
1502 Budapest, Pf.63, Hungary

Let Λ R^n be a (geometric) lattice and let $A:=(a_1,a_2,\ldots,a_n)$, $a_i \in R^n$, $i=1,\ldots,n$, be a basis of Λ, i.e. $\Lambda=\{u\in R^n: u=\Sigma_{i=1}^n u_i a_i, u_i$ *integer*, $i=1,\ldots,n\}$. The set $P:=\{x\in R^n: x = \Sigma_{i=1}^n \lambda_i a_i$, $0\leq\lambda_i<1$, $i=1,\ldots,n\}$ is the unit cell of A. Denote by $\mu(S)$ the Lebesgue measure of S R^n and by $|S|$ the cardinality of the finite set S. The zero vector of R^n is denoted by θ. By $S+H$ we mean the algebraic (or Minkowski) sum of S, H R^n, i.e. $S+H :=\{x\in R^n: x=s+h, s\in S, h\in H\}$. The number $d\Lambda:=|det(a_1,\ldots,a_n)|$ is the determinant of Λ, we assume that $d\Lambda>0$.

A generalized form of Minkowski's convex body theorem asserts that if $K\subset R^n$ is convex and θ-symmetric (i.e. $K=-K$) and $(\mu(\frac{1}{2}K)/d\Lambda)>1$, then $|K\cap\Lambda|>1$.

The aim of this note is to show that it is not the ratio $(\mu(\frac{1}{2}K)/d\Lambda)$ that decides upon the number of lattice points being in K, but rather the ratio $(\mu(\frac{1}{2}K)/c(\frac{1}{2}K,\Lambda))$, where $c(\frac{1}{2}K,\Lambda)$ $(\leq d\Lambda)$ is a number that might be called "the measure of covering R^n by $\frac{1}{2}K$".

Before the exact definition of this number, let us recall that the pair $\{S,\Lambda\}$, $S\subset R^n$, is called an "(S,Λ)-covering of R^n " if

$$(1) \qquad \bigcup_{u\in\Lambda} (S + u) = R^n.$$

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The ratio $(\mu(S)/d\Lambda)$ is called the density of (S,Λ)-covering (see [1], p. 182.).

Definition 1. We say that $\{S,\Lambda\}$ is an "almost (S,Λ)-covering of R^n" if

$$(2) \qquad \mu(R^n \setminus \bigcup_{u\in\Lambda} (S+u))=0. \qquad \square$$

The condition (2) is equivalent to

$$(3) \qquad \mu(P\setminus \bigcup_{u\in\Lambda} ((S+u)\cap P))=0$$

(an excercise for the reader).

Denote

$$(4) \qquad S_P := \bigcup_{u\in\Lambda} ((S+u)\cap P).$$

Assertion 1. Let $A:=(a_1,\ldots,a_n)$ and $A' := (a_1',\ldots,a_n')$ be two bases of Λ and P and P' be unit cells of A and A', respectively. Then

$$(5) \qquad \mu(S_P) = \mu(S_{P'}),$$

consenquently

$$(6) \qquad \mu(P\setminus S_P) = \mu(P' \quad S_P') . \qquad \square$$

Proof:

$$\mu(S_P) = \mu(\bigcup_{v\in\Lambda} ((\bigcup_{u\in\Lambda} ((S+u)\cap P))\cap(P'+v))) =$$

$$= \mu(\bigcup_{w\in\Lambda} \bigcup_{r\in\Lambda} ((S+r)\cap P'\cap(P+w))) = \mu(S_{P'})$$

and $\qquad \mu(P) = \mu(P') = d\Lambda.$ ∎

Definition 2. The measure $\mu(S_P)$ is called "the measure of covering R^n by S" and is denoted by $c(S,\Lambda)$ ((5) shows that it depends only on S and Λ). □

Assertion 2. The pair $\{S,\Lambda\}$ is an "almost (S,Λ)-covering of R^n" if and only if $c(S,\Lambda)=d\Lambda$. □

Proof: Trivial consequence of definitions. ■

This assertion justifies the name given to $c(S,\Lambda)$ in Def.2. Denote by φ_Λ the canonical map of R^n onto the torus group $T:=R^n/\Lambda$. There is an interesting connection between $c(S,\Lambda)$ and φ_Λ, namely we have.

Assertion 3. For any L-measurable set $S\subset R^n$

$$(5') \qquad c(S,\Lambda) = \bar{\mu}(\varphi_\Lambda(S)),$$

where $\bar{\mu}$ is the measure on T generated by μ. □

Proof: Denote by ψ_P the isomorphism of T onto P. The measure $\bar{\mu}$ on T is then defined as $\bar{\mu}(H) := \mu(\psi_P(H))$, H T (this does not depend on P). Now, we can see easily that $S_P=\psi_P(\varphi_\Lambda(S))$. ■

Now, we have

Theorem 1. Let $K\subset R^n$ be convex and θ-symmetric. If $(\mu(\frac{1}{2}K)/c(\frac{1}{2}K,\Lambda))>m$ (where m is a positive integer), then K contains at least m pairs of non-zero lattice points u_i, $-u_i\in\Lambda$, $u_i\neq\theta$, $i=1,\dots,m$, such that all members of the set $H:=\{u_1,\dots,u_m,-u_1,\dots,-u_m\}$ are mutually different (i.e. $|H|=2m$). □

Proof: We shall prove that

$$(7) \qquad \tfrac{1}{2}(1+|K\cap\Lambda|) \geq (\mu(\tfrac{1}{2}K)/c(\tfrac{1}{2}K,\Lambda)).$$

The proof of (7) depends on the following two trivial facts:

- For any sets $S, H \subset R^n$ of finite cardinality

$$(F1) \qquad |S+H| \geq |S| + |H| - 1.$$

- For any Lebesgue-measurable set $S \subset R^n$

$$(F2) \qquad \mu(S) = \int_P |(S-x) \cap \Lambda| \, dx .$$

As to the proof of (7), first we can easily see that

$$(8) \qquad (\tfrac{1}{2}K)_P := \bigcup_{u \in \Lambda} ((\tfrac{1}{2}K+u)) \cap P) = \{x \in P : (\tfrac{1}{2}K-x) \cap \Lambda \neq \emptyset\}.$$

Secondly, for any $x \in (\frac{1}{2}K)_P$ we have

$$(9) \qquad K \cap \Lambda = ((\tfrac{1}{2}K-x)-(\tfrac{1}{2}K-x)) \cap \Lambda \supseteq ((\tfrac{1}{2}K-x) \cap \Lambda)-((\tfrac{1}{2}K-x) \cap \Lambda).$$

[Here we used the convexity and θ-symmetry of K, i.e. $\frac{1}{2}K-\frac{1}{2}K=K$. The assumption $\mu(K)>0$ implies, using (F2), that $\mu((\frac{1}{2}K)_P)>0 (\Rightarrow (\frac{1}{2}K)_P \neq \emptyset)$.]
Using (F1), the relation (9) implies

$$(10) \qquad |K \cap \Lambda| \geq 2|(\tfrac{1}{2}K-x) \cap \Lambda|-1.$$

Integrating both sides of (10) over $(\frac{1}{2}K)_P$, using (8), (F2) and taking into account that $\mu((\frac{1}{2}K)_P)=c(\frac{1}{2}K, \Lambda)$, we get (7). To finish the proof, (7) imply

$$(11) \qquad q := |K \cap \Lambda| \geq 2m.$$

Write $K \cap \Lambda$ in the lexicographically increasing order, say $u_1<u_2<\ldots<u_m<u_{m+1}<\ldots<u_{q-m}<u_{q-m+1}<\ldots<u_{2m}<\ldots<u_q$. Clearly $\theta \in K \cap \Lambda$, i.e. $u_r=\theta$ for some $1 \leq r \leq q$.

This means that there are exactly $r-1$ elements $u_k<\theta$ and exactly $q-r$ elements $u_i>\theta$. Assume that $r \leq m$. Inequality (11) implies $q-r \geq q-m \geq m$. But $K \cap \Lambda$ is θ-symmetric, hence $q-r \geq m$ implies that at least m elements $u_k<\theta$ belong to $K \cap \Lambda$ (all $-u_i$) and this is a contradiction (because there are exactly $r-1 \leq m-1$ such elements only). We come similarly to a contradiction if we assume that $r \geq q-m+1$. Hence

$$(12) \qquad m+1 \leq r \leq q-m,$$

showing that we have at least m mutually different elements $u_i < \theta$, so $-u_i > \theta$ are also mutually different and also different from u_i. ■

<u>Remark.</u> The above idea can be used to prove that the cardinality of any θ-symmetric set S R^n containing θ is odd. Indeed, assume $|S|=2k$. Writing S in the lexicographic order, the index of θ cannot be $\leq k$, hence it is $\geq k+1$ that is again a contradiction. □

Using the same method of proof as in Theorem 1 we can prove:

<u>Theorem 2.</u> Let $S \subset R^n$ be any L-measurable set. Then

$$(13) \qquad \tfrac{1}{2}(1+|(S-S)\cap \Lambda|) \geq (\mu(S)/c(S,\Lambda)),$$

consenquently, if $(\mu(S)/c(S,\Lambda))>m$, then $S-S$ contains m pairs of mutually different non-zero lattice points $u_i, -u_i \in \Lambda$, $i=1,\dots,m$, such that the cardinality of $H:=\{u_i, -u_i,\ i=1,\dots,m\}$ is $2m$. □

This theorem contains Thm 1 as a special case. The first generalization of Minkowski's theorem of this type is due to Blichfeldt (see |1|). It is clear that $c(S,\Lambda) \leq d\Lambda$, hence (7) and (13) yield substantial sharpening of Minkowski's and Blichfeldt's result, respectively. Many examples can be found such that $c(\tfrac{1}{2}K,\Lambda) \ll d\Lambda$ (i.e. $\{\tfrac{1}{2}K,\Lambda\}$ is far from being an $(\tfrac{1}{2}K,\Lambda)$-covering of R^n) and $\mu(K) \ll 2^n d\Lambda$, but their ratio is great enough to ensure many lattice points in K.

A deatiled discussion and further development of the proof of Minkowski's theorem via (9), (10) and (7) can be found in [2], [3] and [4].

REFERENCES

[1] C.G.LEKKERKERKER, "Geometry of Numbers", Wolters-Noordhoff Groningen, North-Holland, Amsterdam, 1969.

[2] B.UHRIN, A note to Minkowski's convex body theorem, Matematikai Lapok, <u>28</u> /4/, /1980/, 323-326, /in Hungarian/

[3] B.UHRIN, Some useful estimations in geometry of numbers, Period. Math. Hungar., <u>11</u> /2/, /1980/, 95-103.

[4] B.UHRIN, A generalization of Minkowski's convex body theorem, J. of Number Theory, <u>13</u> /2/, /1981/, 192-209.

A Minkowski-féle konvex-test tétel és az R^n-nek egy adott halmazzal vett lefedési mértéke

Uhrin Béla

<u>Összefoglaló</u>

Legyen Λ R^n egy geometriai rács, P az alapcellája és $d\Lambda = \mu(P)$ a determinánsa. Egy tetszőleges S R^n Lebesgne mérhető halmaz esetében az $c(S,\Lambda)=\mu(\underset{u\in\Lambda}{U} ((S+u)\ P))$ mértéket az "R^n-nek az S halmas által vett lefedési mértékének" nevezzük. Bebizonyitjuk, hogy az $S-S$ (algebrai) differencia-halmazban levő rácspontok számánál nem $\mu(S)/d\Lambda$ hányados számit (ahogyan eddig ismert volt), hanem a $\mu(S)/c(s,\Lambda)$ hányados. Ez élesiti Minkowski ill. Blichfeldt klasszikus eredményét.

ТЕОРЕМА МИНКОВСКОГО О ВЫПУКЛЫХ ТЕЛАХ И МЕРА ПОКРЫТИЯ R^n МНОЖЕСТВОМ

Б. Ухрин

Резюме

В статье доказывается, что число точек решетки $\Lambda \subset R^n$ содержащихся в /алгебраической/ разнице S-S множества $S \subset R^n$ измеряется числом $\mu(S)/c(S,\Lambda)$, а не числом $\mu(S)/d\Lambda$. Здесь $c(S,\Lambda) \leq d\Lambda$ есть мера покрытия R^n множеством S, определенная как $c(S,\Lambda)=\mu\left(\bigcup_{u\in\Lambda}((S+u)\cap P)\right)$. Этот результат улучшает классические результаты Минковского и Блихфелдта.