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OPTIMIZATION OF PARAMETERS IN LINEAR SYSTEMS OF CERTAIN TYPE

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INTRODUCTION

One of the classical problems in the examination of stabil systems characterisable with linear differential equation is to find the solution with the maximal degree of stability of the differential equation where the parameters $K_1,\dots,K_n$ (on which the coefficients depend) are chosen from a given domain.

II. FORMULATION OF THE PROBLEM

Consider the linear differential equation

$$x^{(n)}+g_1(K_1,\dots,K_n)x^{(n-1)}+\dots+g_n(K_1,\dots,K_n)x=0. \quad (1)$$

of which the characteristic equation is

$$\lambda^n+g_1(K_1,\dots,K_n)\lambda^{n-1}+\dots+g_n(K_1,\dots,K_n)=0 \quad (2)$$

Obviously g_i $i=1,\dots,n$ are positive in the domain of stability, since this is a necessary condition for the stability.

We suppose that there is a one-to-one correspondence between g_i $i=1,\ldots,n$ and K_i $i=1,\ldots,n$. We solve then the problem

$$\min_{(g_1,\ldots,g_n)\in\Phi} \quad \max_{i=1,\ldots n} \quad Re\ \lambda_i \tag{3}$$

where $\lambda_i = u_i + iv_i$, $i=1,\ldots,n$. are the roots of the characteristic equation, a u_i are the real, and v_i the imaginary parts, $\Phi = \Gamma \cap U$ where U is the domain of stability, and Γ is the map of the domain Ω into the domain of coefficients, where we look for the solution, and $\Omega \subset R^n$ is the given domain in the space of parameters. We suppose that $\Gamma = \Gamma_1\times\ldots\times\Gamma_n$ and $\Gamma_i = [\underline{\Gamma}_i, \overline{\Gamma}_i]$, $\underline{\Gamma}_i \leq \overline{\Gamma}_i$. Consider a sequence of polinomials of increasing degree where each polinomial contains the roots of the previous one. Let K_i^j be the i-th coefficient of the polinomial of degree j. $(i=1,\ldots,j)$. Using the relations between the coefficeints and the roots we get that the coefficients of the polinomials of the sequence can be expressed in terms of the coefficeints of the previous polinomials and the new root in the following way:

$$K_1^1 = -\lambda_1$$

$$K_1^2 = K_1^1 - \lambda_2$$

$$K_2^2 = -K_1^1\lambda_2$$

$$\vdots$$

$$K_1^j = K_1^{j-1} - \lambda_j$$

$$K_2^j = K_2^{j-1} - K_1^{j-1}\lambda_j$$

$$\vdots$$

$$\vdots$$

$$K_i^j = K_i^{j-1} - K_{i-1}^{j-1}, \lambda_j$$

$$\vdots$$

$$K_j^j = -K_{j-1}^{-1} \lambda_j$$

Lemma 1.

In the stability domain the coefficients K_i^j of a polinomial with real coefficients of degree j are polinomials with positive real coefficients of v_k, $k=1,\ldots,n$. (the coefficients are functions of u_k, $k=1,\ldots,n$ of course)

Proof: We use mathematical induction with respect to the number of roots. For $j=1,2$ the statement is trivial. Let us suppose that lemma 1 is proved for $j=n-1$ i.e.

$$K^{n-1}_i = f^{n-1}_i + p^{n-1}_i(v_k^2) \qquad k=1,\ldots,n \tag{4}$$

where f^{n-1}_i does not depend on v_k and $p^{n-1}_i(v_k^2)$ is always positive.

Let us first suppose that n is odd. In that case λ_n is real, as K_i^n are real. We have $u_n < 0$, since we are in the domain of stability as we supposed it. In this case our statement trivially follows from the statement for $n-1$, as

$$K_i^n = K^{n-1}_i - K^{n-1}_{i-1} \lambda_n$$

and so

$$K_i^n = f^{n-1}_i + p^{n-1}_i(v_k^2) - (f^{n-1}_{i-1} + p^{n-1}_{i-1}(v_k{}^2))u_n = f_i^n - f^{n-1}_{i-1}u_n + p^{n-1}_i(v_k^2) + u_n\, p^{n-1}_{i-1}(v_k^2) =$$

$$= f_i^n + p_i^n$$

If n is even, then we rewrite the equality (4) so, that we use the coefficients with upper index $n-2$. As

$$K_i^{n-1} = K_i^{n-2} - K_{i-1}^{n-2} \lambda_{n-1}$$

$$K_{i-1}^{n-1} = K_{i-1}^{n-2} - K_{i-2}^{n-2} \lambda_{n-1}$$

so we have

$$K_i^n = -K_{i-1}^{n-2}\lambda_{n-1} + K_i^{n-2} - (K_{i-1}^{n-2} - K_{i-2}^{n-2} \lambda_{n-1})\lambda_n = K_i^{n-2} - K_{i-1}^{n-2} \lambda_n - K_{i-1}^{n-2} \lambda_{n-1} +$$

$$+ K_{i-1}^{n-2} \lambda_{n-1}\lambda_i$$

As n is even, and the polinomial has real coefficients, it follows that

$$\lambda_{n-1} = u_{n-1} + i\ v_{n-1}, \quad \lambda_n = u_n + iv_n$$

and $u_n = u_{n-1}, \quad v_n = -v_{n-1}$

In this way

$$K_i^n = K_i^{n-2} - K_{i-1}^{n-2}\ 2u_n + K_{i-1}^{n-2}(u_n^2 + v_n^2)$$

from which our statement follows, as $u_n < 0$.

Lemma 2.

The domain of stability is not bounded in the space of coefficients.

Proof: Let us define $g_i = (-1)^j \sum_{i_s \subset N} \prod_{i_\ell \in I_s} \lambda_{i_\ell} \qquad j=1,\dots,n$

and I_s runs through the subsets with j elements of $N = \{1,\dots,n\}$,

Let us suppose that $g = \{g_1,\dots,g_n\}$ is in the domain of stability, that is $Re\ \lambda_i < 0 \quad i=1,\dots,n$. Then for $c>0$ we have $Re\ c\ \lambda_i < 0$. So the corresponding point in the space of coefficients

$$g^* = (g_1^*,\dots,g_n^*) \qquad \text{where} \quad g_j^* = c^j\ g_j \quad j=1,\dots,n \ .$$

is in the domain of stability too, so it is unbounded.
Definition: The upper boundary of the n dimensional rectangle Γ is the set

$$\{x \mid x \in \Gamma,\ \exists\, i \quad x_i = \overline{\Gamma}_i\}$$

The lower boundary is defined in an analogous way.

Lemma 3.

The optimum is on the common part of the upper boundary of Γ and the domain of stability.
Proof: As the domain of stability is open, the optimum, if it exists, is in the interior of this domain. Let us suppose, that the optimum is not on the upper boundary of Γ. It is obvious from the relations between the coefficients and the roots that in the expression $g_i = f_i + p_i(v_k^2)$

$$f_i = (-1)^i \sum_{i_s \subset N} \ \prod_{i_\ell \in I_s} u_{i_\ell} \qquad i=1,\dots,n$$

are positive too. It follows that when increasing u_i the upper boundary may be reached. As the coefficients are continuous monotonically increasing functions of $u_i \quad i=1,\dots,n$ if we increase that u_i which is according to its module, the least we can reach the upper boundary, which contradicts to the hypothesis.

Lemma 4.

If $\underline{\Gamma}_i = 0 \quad i=1,\dots,n$ then in the optimal case $v_i = 0$, $i = 1,\dots,n$.

Proof: Suppose that in the optimal point $v_i \neq 0$. Then the coefficients, corresponding to the roots u_i, $v_i = 0$ are in the interiour of the domain Φ. As the real parts did not change, these new roots are optimal too, which contradicts lemma 3.

Remark: In the case $\underline{\Gamma}_i \neq 0$ the point $v_i = 0$ may fall in the exterior of the domain.

Lemma 5.

In the case $\underline{\Gamma}_i = 0, \quad i=1,\dots,n$ there exists a global optimum. In this case the real parts are equal.
Proof: Let us suppose that they are not equal. Let u_1 be the greatest and u_2 the least with respect to their module of real parts. As

$$K_j = (-1)^j \, \Sigma \, u_{i_1} \dots u_{i_j} = u_1 (-1)^j \, \Sigma \, u_{i_1} \dots u_{i_{j-1}} +$$

$$+ u_2 (-1)^j \sum_{i_\ell \neq 1,2} u_{i_1} \dots u_{i_{j-1}} + u_1 u_2 (-1)^j \sum_{i_\ell \neq 1,2} u_{i_1}, \dots, u_{i_{j-2}} =$$

$$= - (u_1 + u_2)(-1)^{j-1} \, \Sigma \, u_{i_1} \dots u_{i_{j-1}} + u_1 u_2 (-1)^{j-2} \, \Sigma \, u_{i_1} \dots u_{i_{j-2}}$$

it is obvious that $u_1^* = u_2^* = -\sqrt{u_1 u_2}$ gives a value g which is better, then the hypothetically "optimal" one. We have proved the following:

Theorem

Let us consider a control system, described by differential equation with real coefficients

$$x^{(n)} + g_1 x^{(n-1)} + \ldots + g_n x = 0$$

Then the solution of the equation with the greatest degree of stability according to the conditions $g_i < \Gamma_i$ is the following

$$x(t) = (c_1 + c_2 t + \ldots + c_n t^{n-1}) e^{ut}$$

where u is the least real root of which the corresponding coefficients are in Γ and c_i are constants, depending on the boundary conditions.

Algorithm

As in the case of optimal solution for some value

$$g_i = (-1)^i \binom{n}{i} u^i$$

the algorithm is working in the following way: for each j we determine

$$u_j = \sqrt[j]{\frac{g_j}{\binom{n}{j}(-1)^j}}$$

Then we determine the quantities g_j using the relations between the roots and the coefficients, and we look whether these values are in Φ . If not, then we reject them. If yes then we take this solution in the set of the possible solutions. If we have already examined it for $j = 1,\ldots,n$ the quantities g_j, then then we choose the best from possible solutions (max.n)

Example:
Let us consider the control system,

The characteristic equation of this system is

$$x^3 + K_1 x^2 + (1+K_2)x + K_3 = 0$$

In some cases the following table shows the solution:

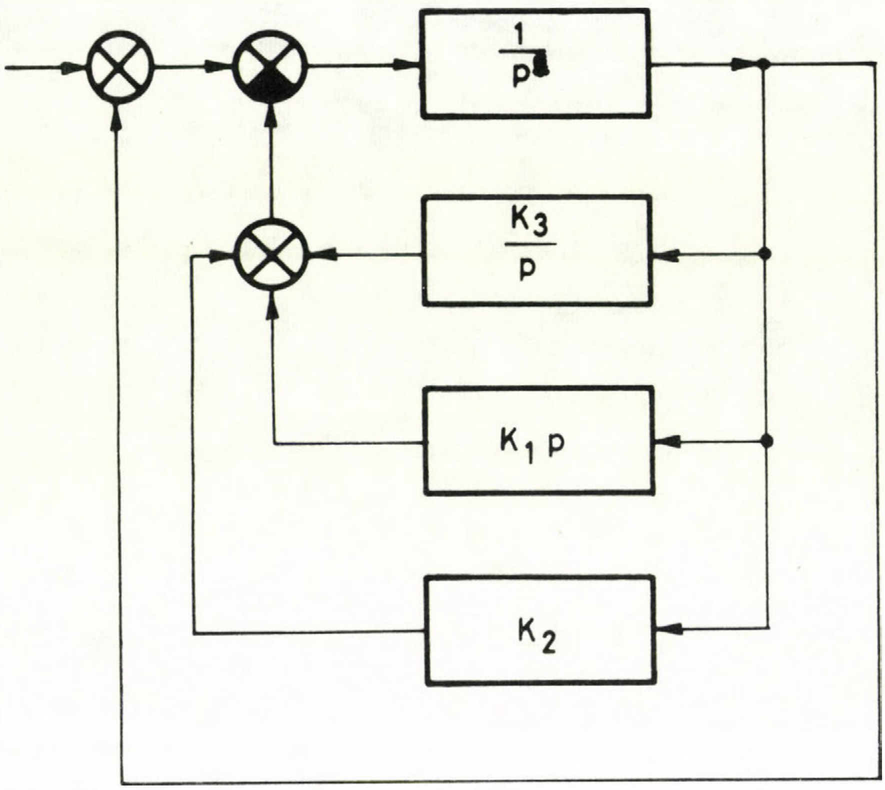
$\bar{\Gamma}_1$	$\bar{\Gamma}_2$	$\bar{\Gamma}_3$	root	opt.	coefficients		
3	3	1	-1	1	3	3	1
6	11	6	-1.8571	1	5.4514	9.9058	6
3	6	3	-1	1	3	3	1

REFERENCES

1. Morris Marden: The geometry of zeros of a polinomial in a complex variable 1949.

2. Zajac E.E.: Bounds of the Decay rate of damped linear systems "Quart. of Appl. Math." 1963. v.20 No.4.

3. Jindrich Spal: Algebraic Approach of the Root-Loci Method Kybernetika Čislo 5. Ročnik 6/1970


1
p
K3
p
K1 p
K2