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# QUALITATIVE PROPERTIES OF THE HALF-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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The subject of our lecture is the second order differential equation of the form

(1) $$y'' + p\, f(y,y') = 0$$

for $y = y(t)$ where the function $p = p(t)$ is piecewise continuous on the interval $I \subset (-\infty,\infty)$ and $f(u,v)$ satisfies the following relations

(2) $$F(u) = u\, f(u,1) > 0 \quad \text{and continuous for} \quad u \in R \setminus \{0\},$$

(3) $$f(cu,\ cv) = c\, f(u,v), \qquad c \in R, (u,v) \in R \times R.$$

The existence and the uniqueness of the solutions of the initial value problem for (1) is also assumed. The property (3) ensures that if $y = y(t)$ is a solution of (1) and $c$ is some constant then the function $c\, y(t)$ is also a solution of (1). Thus the differential equation (1) resembles the linear differential equation indicating the name of half-linear one (see [1]). The special case $f(u,v) = u^{\overset{*}{n}} |v|^{1-n}$ with $n > 0$ was studied in [2] and [3]. Here we make an attempt to deal the oscillatory case (in view of the inequality (2) ) as generally as possible.

Let $S = S(\varphi)$ be the solution of the following differential equation

(4) $$\ddot{S} + f(S,\dot{S}) = 0$$

with the initial conditions $S(0) = 0$, $\dot{S}(0) = 1$. If

$$\hat{\pi} = \int_{-\infty}^{\infty} \frac{dt}{1+F(t)} < \infty$$

then $S(\varphi)$ is an oscillatory function with the period $2\hat{\pi}$, and there are quantities $\pi_+, \pi_-, K$ depending only of $f$ with the properties

$$\pi_+ + \pi_- = \hat{\pi}$$

(5) $$\dot{S}(\pi_+) = \dot{S}(-\pi_-) = 0, \quad \dot{S}(\varphi)>0 \text{ for } -\pi_- <\varphi<\pi_+,$$

$$S(\varphi) = -K\, S(\varphi-\hat{\pi}), \qquad K > 0.$$

If $\varphi \neq \varphi_i = \pi_+ + i\hat{\pi}$, $i=0,\pm 1,\ldots,$ then we define the function $T = T(\varphi)$ by

$$T(\varphi) = \frac{S(\varphi)}{\dot{S}(\varphi)} \quad .$$

Owing to (5) the function $T$ is periodic with the period $\hat{\pi}$ and it has discontinuities at $\varphi = \varphi_i$ and it varies from $-\infty$ to $\infty$ as $\varphi$ varies from $\varphi_i$ to $\varphi_{i+1}$. Moreover $T$ fulfils the differential equation

(6) $$\dot{T} = 1 + F(T),$$

where the function $F$ is nonnegative due to (2).

Let us consider the curve $i$ on the plane $(x_1, x_2)$ given in parametric from by $(\dot{S}(\varphi), S(\varphi))$ for $-\infty < \varphi < \infty$. Since $K$ is in general not equal to $1$, the curve $i$ is winding round the origin $(0,0)$. Let $\varphi = \varphi_0$ be fixed. Then the points of the halfray starting from the origin and crossing the point $P_0(\dot{S}(\varphi_0), S(\varphi_0))$ are $x_1 = \rho\,\dot{S}(\varphi_0)$, $x_2 = \rho\,S(\varphi_0)$ with $\rho > 0$. Hence for the point $P_0$ of $i$ we have $\rho = 1$. Thus we have a generalized polar transformation on the plane $(x_1, x_2)$

$$x_1 = \rho\,S(\varphi)$$

(7)

$$x_2 = \rho\,S(\varphi)$$

which depends on the equation (1) in the above sense.

Let us consider a solution $y = y(t)$ of (1). This solution defines a curve $c$ on the plane $(x_1, x_2)$ given by $(y'(t), y(t))$ for $t \in I$. We fix the values $\varphi_0$ and $\rho_0$ by the initial conditions $y'(t^0) = \rho_0\,\dot{S}(\varphi_0)$, $y(t^0) = \rho_0\,S(\varphi_0)$. Then by (7) we have two welldefined functions $\varphi(t)$ and $\rho(t)$ as polar coordinates of the curve $c$

$$y'(t) = \rho(t)\,\hat{S}(\varphi(t))$$

(8)

$$y(t) = \rho(t)\,S(\varphi(t) .$$

which is a generalization of the Prüfer transformation. The functions $\varphi(t)$ and $\rho(t)$ satisfy first order differential equation system

$$\varphi' = \frac{1}{1+F(T)} + p(t)\,\frac{F(T)}{1+F(T)}$$

(9)

$$\rho' = \rho\,\frac{f(T,1)}{1+F(T)}\,(1-p).$$

It is interesting to remark that the system (9) is of triangular form because the right hand side of the first equation depends only on the unknown $\varphi$ but not on $\rho$, while in the second equation both the unknowns occur. This property of the system (9) enables us to generalize several properties known for linear second order differential equations (see e.g. in [4]).

We start with a simple lemma.

*LEMMA.* Let the functions $p_1(t)$, $p_2(t)$ be piecewise continuous and let the relation

$$p_1(t) \leq p_2(t) \quad \text{for} \quad t \in I \tag{10}$$

be valid. If $\varphi_1(t), \varphi_2(t)$ are the solutions of the differential equations

$$\varphi_j' = \frac{1}{1+F(T(\varphi_j))} + \varphi_j(t) \frac{F(T(\varphi_j))}{1+F(T(\varphi_j))} \tag{11}$$

with the initial condition $\varphi_j(t^o) = \varphi_{jo}$ $(j=1,2)$. Then the inequality

$$\varphi_1(t) \leqq \varphi_2(t) \quad \text{for} \quad [t_o, \infty] \cap I$$

holds. In the case $\varphi_{1o} = \varphi_{2o}$ we have also

$$\varphi_1(t) \geqq \varphi_2(t) \quad \text{for} \quad (-\infty, t_o] \cap I.$$

Let $y = y(t)$ be a solution of (1), and $t_1, t_2, \ldots$ be consecutive zeros of $y(t) = 0$. Then by (8) we have $S(\varphi(t_i)) = 0$, hence $\varphi(t_i) \equiv 0(mod\hat{\pi})$. By (9) $\varphi'(t_i) = 1 > 0$ thus $\varphi(t)$ is strictly increasing in the neighbourhood of $t = t_i$. Therefore $\varphi(t_{i+1}) = \varphi(t_i) + \hat{\pi}$ and $\varphi(t_i) < \varphi(t) < \varphi(t_{i+1})$

for $t_i < t < t_{i+1}$. Similarly we have for the zeros $t_1', t_2', \ldots$ of $y'(t) = 0$ that $\varphi(t_i') \equiv \pi_+ (mod\ \hat{\pi})$. If $p(t)$ is positive on the interval $(t_i, t_{i+1})$ then by (9) the function $\varphi$ is strictly increasing hence there is exactly one value $t_i' \in (t_i, t_{i+1})$ where $y(t)$ has local extremal value and

$\varphi(t_i') = \varphi(t_i) + \pi_+$.

This observation and the Lemma has many applications.

*Theorem 1.* Let $t_1$ and $t_2$ be two consecutive zeros of a nontrivial solution $y(t)$ of (1). Then every solution $\bar{y}(t)$ different from $cy(t)$, where $c$ is any constant, vanishes once and only once in $(t_1, t_2)$.

Usually we recall this theorem saying that the zeros of linearly independent solutions of (1) are interlacing. In the next theorem we formulate a stronger version of this interlacing property.

*Theorem 2.* Let $y(t)$ be a nontrivial solution of (1) and $J = (\tau_1, \tau_2)$ be an interval such that $J \subset I$ and $y(t)\, y'(t) \neq 0$ on $J$, $y(\tau_j)\, y'(\tau_j) = 0$ for $j = 1,2$. Let $\bar{y}(t)$ be other linearly independent solution of (1) satisfying either $\bar{y}(\tau_1) y(\tau_1) > 0$, $\bar{y}'(\tau_1) y'(\tau_2) > 0$ if $yy' < 0$ or $\bar{y}(\tau_1) y(\tau_2) > 0$, $\bar{y}'(\tau_1)\, y'(\tau_1) > 0$ if $yy' > 0$. Then there is a value $t^* \in J$ with $\bar{y}(t^*) = 0$ in the first case or $\bar{y}'(t^*) = 0$ in the second case.

Concerning the strong interlacing property we conjecture that if the solutions of differential equation

$$y'' + p(t)\, g(y, y') = 0$$

has the property given in Theorem 2 then the equation is half-linear, i.e. the function $g(y, y')$ satisfies the homogenity relation of type (3). In fact we could prove this conjecture only in the case when $g(y,y') = h(y)\, k(y')$. We have found that $g = y^{\overset{*}{n}} |y'|^{1-n}$, which was treated already in [3]. On the other hand similar statement is not true in general if we assume only the common interlacing property formulated by Theorem 1.

Now we want to compare the solutions of the differential equations

$$(12_j) \qquad y_j'' + p_j\, f(y_j, y_j') = 0, \quad j=1,2.$$

We say that the equation $(12_2)$ is a Sturmian majorant to $(12_1)$ if the inequality (10) holds.

*Theorem 3.* Let $t^o \in I$ and $j \in \{1,2\}$ and let the solution $y_j = y_j(t)$ of $(12_j)$ satisfy the initial conditions $y_j(t^o) = y_j^o \neq 0$, $y_j'(t^o) = y_j'^o$ with

$$\frac{y_1'^o}{y_1^o} = \frac{y_2'^o}{y_2^o} .$$

If $t_o, t_1$ are two consecutive zeros of $y_2(t) = 0$ such that $t_o < t^o < t_1$, and if $(12_2)$ is a Sturmian majorant to $(12_1)$, then

$$\frac{y_1(t)}{y_1^o} \geqq \frac{y_2(t)}{y_2^o} \quad \text{for} \quad t_o \leq t \leq t_1.$$

*Theorem 4.* Let the conditions be the same as in Theorem 3 with the only exception that we assume now

$$\frac{y_1'^o}{y_1^o} \geqq \frac{y_2'^o}{y_2^o} .$$

Denote $t_{j_1}$, $t_{j_2}, \ldots$ the zeros of $y_j(t) = 0$ on $(t^o, \infty) \cap I$. Then $t_{1k} \geq t_{2k}$ for $k = 1, 2, \ldots$ . If the coefficient $p_1(t) > 0$ then similar statement is true for the zeros of $y_j'(t) = 0$, too.

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