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ON THE PARTIAL ASYMPTOTIC STABILITY BY LJAPUNOV FUNCTION WITH SEMIDEFINITE DERIVATIVE

LÁSZLÓ HATVANI

Institute Bolyai (Szeged)

Consider the autonomous differential equation

$$\dot{x} = X(x) \qquad (x \in R^n,\ X(0) = 0), \tag{1}$$

where the function $X : R^n \to R^n$ is continuous. Denote by $x(t; x_o)$ any solution of (1) satisfying the initial condition $x(0; x_o) = x_o$. Let a partition $x = (y, z)$ be given, where $y \in R^m$, $z \in R^{n-m}$, $1 \leqq m \leqq n$. The zero solution of (1) is said to be *y-stable* if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\| x_o \| < \delta(\varepsilon)$ implies $\| y(t; x_o) \| < \varepsilon$ for $t \geqq 0$. We say that the zero solution of (1) is *asymptotically y-stable* if it is y-stable and there exists a positive number σ such that $\| x_o \| < \sigma$ implies $\| y(t; x_o) \| \to 0$ as $t \to \infty$ [1].

For a continuously differentiable function $V : R^n \to R$ we define the function

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V(x)}{\partial x_i} X_i(x),$$

which is called as the derivative of V with respect to (1).

Classical theorems in Ljapunov's second method concerning asymptotic stability require negative definiteness of the derivative of the Ljapunov function with respect to the system. In numerous applications this condition fails to be satisfied. For example, the derivative of the total mechanical energy of a conservative mechanical system under the action of dissipative forces with total dissipation is negative definite with respect to velocities only so we cannot conclude asymptotic stability for the equilibrium with respect to coordinates. E.A. Barabashin and N.N. Krasovskiĭ [2] replaced negative definiteness of the derivative of the Ljapunov function by the condition that it is negative semidefinite and its zero set contains no complete trajectory except the origin. Their method has been extended to study of partial asymptotic stability [3, 4]. However, in all extensions it is supposed that all of the uncontrolled coordinates z are bounded along every solution. We are going to give an extension using conditions which can be checked directly.

Theorem. Suppose that there exists a continuously differentiable function $V : R^r \to R$ with the following properties:

(i) V(0) = 0, and V is positive definite in y, i.e. there exists a continuous, strictly increasing function $a : R_+ \to R_+$ such that $a(0) = 0$, and

$$V(y, z) \geq a(||y||) \qquad (y, z) \in R^n;$$

(ii) if $c > 0$, $(y(t), z(t))$ is a solution of (1) and $(y(t), z(t)) \in \{x : \dot{V}(x) = 0, \ V(x) = c\}$ $(t \in R_+)$, then $y(t) \equiv 0$;

(iii) for every $c > 0$ the set of the points $y \in R^m$ for which there exists a sequence $\{(y_i, z_i)\}$ such that $y_i \to y$, $||z_i|| \to \infty$, $V(y_i, z_i) \to c$ as $i \to \infty$, consists at most of the origin.

Then the zero solution of (1) is asymptotically y-stable. This theorem can be applied to derive sufficient conditions for equilibrium of conservative mechanical system to be asymptotically stable with respect to coordinates in the presence of dissipative forces [5]. In order to illustrate these results, consider a point of mass equal to 1 moving in a constant field of gravity. Suppose the point is constrained to move on a surface of equation $u = y^2(1 + z^2)$, axis $\underline{u}$ is directed vertically upward, and furthermore, that it is subjected to viscous friction with total dissipation. This motion was investigated by K. Peiffer and N. Rouche [6]. They proved that the equilibrium $y = z = 0$ is asymptotically stable with respect to the velocity $\dot{y}$. Applying our theorem we can prove that the equilibrium is asymptotically stable with respect to the coordinate y as well.

Indeed, the total mechanical energy satisfies all conditions of the theorem. It is positive definite in y because the potential energy is $P = gy^2(1 + z^2)$. The dissipation is complete, consequently if $V = 0$ along a motion then velocities are equal to 0 identically, i.e. the point is in equilibrium. Therefore, $y(t) \equiv 0$, which means that the second condition is satisfied, too.
Finally, if

$$y_i \to y, \quad |z_i| \to \infty, \quad gy_i^2(1 + z_i^2) \to C \quad (i \to \infty),$$

then $y = 0$, so the third condition is also satisfied.

REFERENCES

[1] V.V. Rumjancev, On the stability of motion in a part of variables (in Russian), Vestnik Moskov. Univ. Ser. I Mat. Meh., 1957. No.4, 9-16.

[2] E.A. Barbashin, N.N. Krasovskiĭ, On the global stability of motion (in Russian), Dokl. Akad. Nauk SSSR, 86(1952), 453-456.

[3] A.S. Oziraner, On asymptotic stability and instability relative to a part of variables (in Russian), Prikl. Mat. Meh., 37(1973), 659-665.

[4] C. Risito, Sulla stabilitá asintotica parziale, Ann. Mat. Pura Appl. (4), 84(1970), 279-292.

[5] L. Hatvani, On the partial asymptotic stability and instability (to appear).

[6] K. Peiffer, N. Rouche, Ljapunov's second method applied to partial stability, J. Mécanique, 8(1969), 323-334.