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FILTRATION OF GAS WITH ABSORPTION : FREE BOUNDARY AND ASYMPTOTIC BEHAVIOUR

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We shall study the solutions of the Cauchy problem for the equation

$$Lu \equiv -u_t + (u^{\mu})_{xx} - c\cdot u^{\nu} = 0 \qquad (1)$$

in $\quad R^2_+ = \{(t,x):\ 0 \leq t < \infty, \quad -\infty < x < \infty\}$

with initial value

$$u(0,x) = u_0(x), \qquad -\infty < x < \infty. \qquad (2)$$

Here $\mu > 1$, $\nu > 0$ and $c > 0$ are constants. The function $u_0(x)$ is nonnegative, continuous, and it has compact support. Let supp $u_0 = [-\ell, \ell]$, $\ell > 0$.

A great number of problems in physics leads to the equation (1). For example, it describes both the filtration of gas in porous medium and the process of heat conduction with absorption when the coefficients depend on the density or the temperature in polynomial way.

The equation (1) is parabolic when $u > 0$ and it degenerates and becomes of the first order when $u = 0$.

It has been known for a long time that the problem (1), (2) has no classical solutions in general (e.g. see [1]). For this reason the class of admissible solutions must be extended.

The definition for the generalized solution of the problem (1), (2) and the proof of its existence and uniqueness can be found in [3,6]. Some comparison theorems necessary to prove the following theorems 1-4 are stated there, too.

We shall investigate the solutions for sufficiently large t $(t \geq t_o > 0)$. Therefore we assume $\nu \geq 1$. This is connected with the well-known fact that if $\nu < 1$ then $u(t,x) = 0$ when $t \geq T_o > 0$ (see [3]). It is also known (e.g. see [2]) that in the case of the problem (1), (2) "the perturbation propagates with finite velocity": for any $t > 0$, the function $u(t,x)$ has compact support with respect to x (this fact is a simple corollary following from the results of the present paper). There exist two continuous curves $x = \zeta_i(t)$, $i=1,2$ such that $\zeta_1(t) < 0$ and $\zeta_2(t) > 0$ and

$$\operatorname{supp} u(t,x) = \{(t,x): t \geq 0, \ \zeta_1(t) \leq x \leq \zeta_2(t)\}.$$

The functions $\zeta_i(t)$ are often called free boundaries.

The basic results of this paper concern the functions $u(t,x)$ and $\zeta_i(t)$. One can find the detailed proofs in [5].

Let $u(t,x)$ be the generalized solution of the problem (1),(2).

Theorem 1. Let $\nu > \mu$. Then there exist positive constants α_i that depend only on the initial data and such that for $t \geq t_o$ the following inequalities hold:

1) $u(t,x) \leq a_1 t^{-1/(\mu+1)}$

2) $|\zeta_i(t)| \leq a_2 t^{1/(\mu+1)}, \qquad i=1,2.$

When $\nu < \mu + 2$ then

3) $u(t,x) \geq a_3 t^{-1/(\nu-1)}$ for small $|x|$,

4) $|\zeta_i(t)| \geq a_4 t^{(\nu-\mu)/2(\nu-1)}, \qquad i=1,2.$

When $\nu \geq \mu + 2$ then

5) $u(t,x) \geq a_5 t^{-1/(\mu+1)}$,

6) $|\zeta_i(t)| \geq a_6 t^{1/(\mu+1)}, \qquad i=1,2.$

Remark 1. When $\nu \to \mu + 2 - 0$ then the differences in exponents on the right hand sides in 1), 3) and 2), 4) tend to zero.

The first time the estimates for $\zeta_i(t)$ were proved by B.F. Knerr differently (see [7]).

Theorem 2. Let $\mu = \nu$. Then for $t \geq t_o$ the inequalities

1) $u(t,x) \geq a_7 t^{-1/(\mu-1)}$ for small $|x|$,

2) $|\zeta_i(t)| \geq a_8 (\ln t)^{1/2}, \qquad i=1,2$

3) $u(t,x) \leq a_9 t^{-1/(\mu-1)}$,

4) $|\zeta_i(t)| \leq a_{10} t^{1/(\mu+1)}$, $i=1,2$

hold with positive constants a_7, a_8, a_9 and a_{10}.

Remark 2. The inequality 2) means that for $\mu = \nu$ there is no localization of the perturbation i.e. there does not exist $L > 0$ such that supp $u(t,x) \subseteq \{(t,x): |x| \leq L\}$. The first time this fact was proved in 1973 by me and published in 1976 in my dissertation (see [4]). The estimate

$|\xi_i(t)| \geq a \, \ln^{1-\varepsilon}(\ln t)$ was obtained there, it is weaker than

2). First the estimate 2) was shown by B.F. Knerr [7] by a method different from that in [5].

Theorem 3. Let $1 < \nu < \mu$. Then for $t \geq t_0$ the inequalities

1) $u(t,0) \geq a_{11} t^{-1/(\nu-1)}$,

2) $u(t,x) \leq a_{12} t^{-1/(\nu-1)}$,

3) $|\zeta_i(t)| \leq a_{13}$, $i=1,2$

are valid.

Remark 3. The inequality 3) states - by definition - that in this case the localization of the perturbation takes place. This fact is known from [2].

We recall that in all above cases $u(t,x)$ tends to zero as a polynomial. The next theorem establishes the case $\nu = 1$ when this is not valid.

Theorem 4. Let $1 = \nu < \mu$. Then for $t \geq t_o$ the following inequalities hold:

1) $u(t,x) \geq a_{14}\, e^{-ct}$ for small $|x|$,

2) $u(t,x) \leq a_{15}\, e^{-ct}$,

3) $|\zeta_i(t)| \leq a_{16}$, $i=1,2.$

Here $c > 0$ is the constant given in the equation (1).

REFERENCES

[1] Barenblatt, G.I.: On the self-similar solutions of the Cauchy problem for nonlinear parabolic equation of non-stationary filtration, Prikl. Mat. i Mekh. 20(1956)

[2] Kalasnikov, A.S.: On the character of propagation the perturbations, Trudi Semin. I.G. Petrovski 1(1975)

[3] Kalasnikov, A.S.: The propagation of disturbances in problems of non-linear heat conduction with absorption USSR. Comp. Math. and Math. Phys., 14(1974)

[4] Kersner, R.: Some properties of generalized solutions of quasilinear degenerate parabolic equations, Thesis, 1976.

[5] Kersner, R.: On the behaviour of generalized solutions of degenerate quasilinear parabolic equations when t grows to infinity. Acta Math. Sci. Hungaricae 34(1979)

[6] Kersner, R.: Degenerate parabolic equations with general nonlinearities, Nonlin. Anal. Theory, Meth. Vol. 4. No6, 1980.

[7] Knerr, B.F.: The behaviour of the support ..., preprint 1977.