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# ON THE STABILITY OF SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

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We shall investigate the asymptotic behaviour of solutions of differential equations in Banach spaces, comparing the norm of solutions with the power functions $t^{\alpha}$ ($\alpha$ real) as $t\to\infty$. All the operators, appearing in these equations are everywhere definded and bounded.

1. Consider in a Banach space $\mathfrak{B}$ the linear differential equation

$$(1) \qquad \dot{x} = \frac{dx}{dt} = A(t)x, \qquad (x\in \mathfrak{B})$$

where the operator $A(t):\mathfrak{B}\to\mathfrak{B}$ is linear for all $t\in[0,\infty)$ and $A$ is a locally Bochner integrable function of $t$ on $[0,\infty)$. It is known (see [1]) that the solution $x(t)$ of the equation (1) and the initial condition $x(0) = x_o\in\mathfrak{B}$ can be obtained by the formula $x(t) = U(t)\, x_o$, where $U(t):\mathfrak{B}\to\mathfrak{B}$ is the Cauchy operator of (1).

Let $x(t) = U(t)\, x_o$ $(x_o \neq 0)$ be a non-trivial solution of (1). The first characteristic number of $x(t)$ characterizes the exponential behaviour of $\|x(t)\|$, as $t\to+\infty$, wich was introduce by A.M. Lyapunov:

$$\kappa[x] = \lim_{t\to+\infty} \frac{\ell n\| x(t)\|}{t} .$$

We shall denote by $\kappa_s = \sup_{x\neq 0} \kappa[x]$, where $x(t)$ is a nontrivial solution of (1). The following relations are valid for the first characteristic numbers ([1]):

1., if $\sup_{t\geq 0} \int_t^{t+1} \| A(S)\| ds<\infty$, then $\kappa_s$ is finite;

2., for all equations (1) is valid, that $\kappa_s = \overline{\lim_{t\to\infty}} \frac{\ln \| U(t)\|}{t}$;

3., if the operator A is not depending on t, that is
$A(t) \equiv A = \text{const.}$, then $U(t) = e^{At}$ and

(2) $$\kappa_s = \lim_{t\to\infty} \frac{\ln \| e^{At} \|}{t} = \inf_{t>0} \frac{\ln \| e^{At} \|}{t} = \max \operatorname{Re} \sigma(A),$$

where $\sigma(A)$ is the spectrum of the operator A.

2. We can find some equations for which it is not sufficient to know the exponential behaviour of $\| x(t)\|$ as $t\to+\infty$, however we have to compare $\| x(t)\|$ with other functions of t as $t\to+\infty$, for example the power functions $t^{\beta}$ ($\beta$ real). The second characteristic number $\lambda[x]$ of $x(t)$ characterizes this growth property of $\| x(t)\|$ by the following way

$$\lambda[x]: = \lim_{t\to\infty} \frac{\ln(\| x(t)\| e^{-\kappa [x]t})}{\ln t},$$

where $\kappa[x] \neq \pm \infty$ is the first characteristic number of $x(t)$.

Let be the $\lambda_s = \sup_{x\neq 0} \lambda[x]$ and $\mu = \lim \frac{\ln(\| U(t)\| e^{-\kappa_s t})}{\ln t}$

We can derive the following relations between the numbers $\lambda_s$ and $\mu$.

Lemma 1. i) $\mu \leq \lambda_s$;

ii) If $A(t) \equiv A = \text{const}$, then $\mu\geq 0$ and if the equality

(3) $$\lim_{h\to 0+} \frac{\| I+hA\| -1}{h} = \kappa_s$$

is valid, then $\mu=0$.

Proof. i) If we assume that $\lambda_s<\mu$, then we can choose a $\rho$ such that $\lambda[x]\leq\lambda_s<\rho<\mu$. By the definition of $\lambda[x]$ there exist $N_{\rho,x_0}>0$ such that

$$\| x(t) \| = \| U(t)x_o \| \leq N_{\rho,x_o} e^{\kappa[x]t} t^{\rho} \leq N_{\rho,x_o} e^{\kappa_s t} t^{\rho}. \ (t \geq 1)$$

Thus the operator family $\{U(t)e^{-\kappa_s t} t^{-\rho} : t \in [1,\infty)\}$ is bounded for all $x_o \in \mathfrak{B}$ and from the Banach-Steinhaus theorem we obtain, that

$$\| U(t) \| \leq N_{\rho} e^{\kappa_s t} t^{\rho}.$$

It contradicts the definition of $\mu$ and inequality $\rho < \mu$.

ii) From the equality (2) we obtain

$$\inf_{t>0} \frac{\ln \| e^{(A-\kappa_s I)t} \|}{t} = \max \operatorname{Re} \sigma(A-\kappa_s I) = 0,$$

where $I$ is the identity operator on $\mathfrak{B}$. Thus $\| e^{(A-\kappa_s I)t} \| \geq 1$ and from this follows that $\mu = \overline{\lim\limits_{t\to+\infty}} \dfrac{\ln(\| \ell^{At} \| e^{-\kappa_s t})}{\ln t} \geq 0.$

We remark that the limit in the left hand side of the equation (3) exists for all operator $A$([2]). Consider the derivative of $\| e^{At} \|$ with respect to $t$

$$\frac{d^+ \| e^{At} \|}{dt} = \lim_{h\to 0+} \frac{\| e^{A(t+h)} \| - \| e^{At} \|}{h} = \lim_{h\to 0+} \frac{\| e^{At}(I+hA) \| - \| e^{At} \|}{h}$$

$$\leq \| e^{At} \| \kappa_s$$

Integrating this from 0 to $t$ we obtain, that $\| e^{At} \| \leq e^{\kappa_s t}$. and thus $\mu \leq 0$. From this follows with $\mu \geq 0$, that $\mu = 0$.

If $\mathfrak{B} = \mathcal{H}$ is a Hilbert space, then

$\lim\limits_{h\to 0+} \dfrac{\| I+hA \| - 1}{h} = \sup \operatorname{Re} W(A)$, where $W(A) = \{(Ax,x) : \| x \| = 1\}$ is the numerical range of $A$. Thus from the statement ii) of lemma 1. we obtain, that if the operator $A$ is convexoid in H, then $\mu = 0$. ($A$ is convexoid if the relation Conv $\sigma(A) = \overline{W(A)}$ is valid, conv $D$ denotes the convex hull of $D$, and $\overline{D}$ the closure of $D$). For example the normal operators, Toeplitz operators are convexoid (see [3]).

3. When $B=R^n$ the n dimensional Euclidian space, then $\mu$, $\lambda_s$, $\lambda[x]$ are always non-negative integers. In infinite dimensional spaces these are not true in general, these are illustrated by the following:

Examples

1. If we consider in the Hilbert space $\ell^2$

$$(4) \qquad \dot{x} = Ax, \qquad (x\in\ell^2)$$

where A is a unilateral weighted shift in $\ell^2$ with positive $\alpha_n$ weigths and $\alpha_n \to 0$ $(n\to\infty)$, then for the equation (4)

$$\lambda_s = \mu = +\infty.$$

2. If the operator A in (4) is the unilateral shift S in $\ell^2$, then there exists a subset H of $\ell^2$, which is dense in $\ell^2$ and if $x_o \in H$, then $\lambda[x]=-1/4$, where $x(t)=e^{St}x_o$.

Proof.

1. It is well known [3], that the operator A is quasynilpotent i.e. $\sigma(A) = \{0\}$. Thus we obtain from (2), that $\kappa_s=0$. Let $e_1 = (1,0,0,\dots)$ and we obtain

$$\alpha_1\dots\alpha_n \frac{t^n}{n!} \leq \left(\sum_{k=1}^{\infty} (\alpha_1\dots\alpha_k)^2 \frac{t^{2k}}{(k!)^2}\right)^{1/2} =$$

$$= \| e^{At} e_1 \| \leq \| e^{At} \| ,$$

thus

$$\lambda[e_1] = \lambda_s = \mu = +\infty.$$

2. If we denote $e_n = (0,\dots,0,\overset{n}{\tilde{1}},0,\dots)$, then

$$e^{St}e_n = (0,\dots 0,\overset{n}{\tilde{1}},\frac{t}{1!}, \frac{t^2}{2!} ,\dots),$$

thus we obtain

$$\| e^{St} e_n \|^2 = \sum_{k=0}^{\infty} \frac{t^{2k}}{(k!)^2} = J_0\,(2it) \quad (n=1,2,3,\dots)$$

where $J_o(x)$ is the zero order Bessel function ($i=\sqrt{-1}$). From the asymptotic behaviour of $J_o\,(2\,it)$ ([4])

$$J_o\,(2\,it) \sim K\,\frac{e^{2t}}{\sqrt{t}} \quad (t\to+\infty,\ K>0)$$

follows that

$$\lambda[e_n] = -1/4 \qquad (n=1,2,\dots).$$

It is evident that the subset $H = \{x\in\ell^2 : x=(x_1,\dots x_n,0,\dots)\}$ is dense in $\ell^2 (n = 1,2,\dots)$. We know, that $\kappa[x]=1$ for every $x(t)=e^{St}x_o$ ($x_o\in\ell^2$) ([5]) and from inequality

$$\| x(t)\| = \| e^{St}x_o\| \le \sum_{j=1}^{n} |x_j|\,\| e^{St} e_j\| = (\sum_{j=1}^{n} |x_j|)J_o(2it)$$

follows

$$\lambda[x] = -1/4 \text{ if } x_o = (x_1,x_2,\dots,x_n,0,0,\dots)\in H.$$

We can derive easily from this the following

Lemma 2. The zero solution of the equation

(5) $\quad \dot{x} = (S-I)x \qquad (x\in\ell^2)$

is asymptotically stable, but not exponentially.

Remark. This case in $R^n$ can not be occur.

Proof. The stability of zero solution (5) we can derive from inequality

$$\| x(t)\| = \| e^{(S-I)t}x_o\| \le e^{\| S\| t}e^{-t}\| x_o\| = \| x_o\| \quad (x_o\in\ell^2).$$

When $x_o\in H$, then $\kappa[x]=0$ and $\lambda[x]=-1/4$, thus $\| x(t)\|\to 0$, in the same way as $t^{-1/4}$, when $t\to\infty$. If $x\in\ell^2$, then there exists an

$x_0 \in H$ such that $\| x-x_0 \| \le \varepsilon/2$ and

$$\| e^{(S-I)t} x \| \le \| e^{(S-I)t} x_0 \| + \| e^{(S-I)t} (x-x_0) \| \le \varepsilon/2+\varepsilon/2=\varepsilon,$$

if t is sufficiently large.

Consider the following non-linear equation in $\ell^2$

$$(6) \qquad \dot{x}=(S-I)x + F(t,x) \qquad (F(t,0)\equiv 0).$$

Theorem. If $F(t,x)$ satisfies the conditions

$$\| F(t,x) \| \le f(t) \| x \|$$

and $\int_0^\infty f(s)ds \le K < \infty$, then the solution $x=0$ of (6) is asymptotically stable.

Proof. We obtain the solution $x(t)$ of (6) for which $x(0)=x_0$ by the integral equation

$$x(t)=e^{(S-I)t} x_0 + \int_0^t e^{(S-I)t} F(\tau,x(\tau))d\tau.$$

It follows from lemma 2., that

$$\| x(t) \| \le \varepsilon' + \int_0^t f(\tau) \| x(\tau) \| d\tau$$

valid if t is sufficiently large. Applying the Bellman lemma ([1]) we can derive

$$\| x(t) \| \le \varepsilon' e^{\int_0^t f(\tau)d\tau} \le \varepsilon' e^K < \varepsilon.$$

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