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ON THE NOTION OF REGULAR SINGULARITY

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Singular partial differential equations have been intensively studied in the last time (see [1],[2],[3],[4]). The various assumptions making menagable such equation resemble in many respect to the theory of ordinary analytic differential equations.

In the sequel we want to point out that such singularities arise on a very natural way from quite different problems.

The notion of regular singularity is classical within the field of ordinary analytic differential equations. The word 'singularity' refers to the property that the highest derivative of the unknown function can not be expressed as analytic function of the lower order terms while 'regular' refers to the form of solutions which may not be analytic at the singularity under consideration, though they can be combined from analytic function in a relatively simple manner, see for example the Bessel-functions of second kind which are of the form

$$u(x) + \log x \cdot v(x)$$

$u(x)$ and $v(x)$ analytic. As a further important phenomenon at a regular singularity, it may be mentioned, that the usual 'order of the equation = number of initial conditions' law fails to hold, the sign = must be substituted by '$\geq$'.

In the sequel, starting from different problems, we shall deduce a class of partial different problems, we shall of partial differential equations which can be considered as admitting regular singular singularity.

1. THE BUILDING IN INITIAL DATA METHOD

In the ordinary case looking for solutions of the simple Cauchy problem

$$y' = f(x,y)$$

$$y(0) = y.$$

the substitution

$$y(x) = y_0 + xv(x)$$

gives the equation

$$xv' + v = f(x, y_0 + xv).$$

Under suitable conditions on f this is a regular singular problem for the unknown function v. In the field of PDE-s the building in technique cannot be always applied.

In the case of the classical Cauchy-Kovalewski initial value problems the initial values in same special way can be built in [4], the problem of finding eigenfrequencies of a rectangular membrane seems not to be reducible.

2. NATURAL BOUNDARY CONDITIONS

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with piecewise smooth boundary and let us try to minimize the functional

$$I(u) = \int_\Omega [a(x)\ (grad\ u)^2 + b(x)u^2]\ dx$$

Taking the first variation , after some elementary differential geometry we get the elliptic equation

$$a\Delta u + (grad\ a - b)u = \dot{0}$$

and the *natural boundary condition*

$$a(x)\ grad\ u \perp \partial\Omega$$

that is, the vector $a(x)$. *grad u* should be parallel to the normal vector of the boundary surface $\partial\Omega$, except the points were $a(x)$ vanishes. However, in such point s the above elliptic equation becomes singular what tipically prohibits prescription of boundary conditions of the above type.

3. SYMMETRY

Again, the phenomenon can be best illuminated by an example. Using spherical coordinates the form of the Laplace equation in R^3 will be

$$\frac{\partial}{\partial r}(r^2 \frac{\partial \Phi}{\partial r}) + \frac{1}{sin\nu} \frac{\partial}{\partial \nu}(sin\nu \frac{\partial \Phi}{\partial \nu}) + \frac{1}{sin^2\nu} \frac{\partial^2 \Phi}{\partial \Phi^2} = 0$$

Here we have a regular singularity at the boundary $r = 0$ [2]. On the other hand, mere differentiability of the original solution at the origin of the (x,y,z) space is possible only if

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=0} = 0$$

4. ASYMPTOTICS

Finally we mention, that scattering behaviour for a given operator can also be built in yielding a regular singular problem. More generally, certain prescribed asymptotic properties of the solution can be built in using the change of scale

$$s \to e^{-t} .$$

Thus, the system of ordinary differential equations

$$y' = f(y)$$

has a unique stable solution for $t \to \infty$ if and only if the transformed regular singular problem

$$sv'(s) + f(v)s)) = 0$$

has a unique bounded solution at $t = 0$.

5. THE EQUATION

A great variety of the above discussed problems can be reduced to a system of linear partial differential equation of the form

$$\sum_{K=o}^{N} A_k(x)(tD_t)^k \vdash \sum_{j=o}^{N} C_{N-j}t(D_t t)^j u = f.$$

Here $x = (x_1, x_2, \ldots, x_m) \in C^m, t$ denotes a scalar, $D_t = \partial/\partial t$ the A_k-s and C_{N-j}-s are $n \times n$ matrices, the entries of the latters are differential polinomials [4]. Under suitable restriction it can be proved, that the "number" of solution is determined by the characteristic polynomial of the symbol i.e. the matrix pencil

$$\sum_{K=o}^{N} A_k(x) \lambda^k .$$

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