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A NOTE ON TRIANGULARIZATION OF SECOND-ORDER AUTONOMOUS DIFFERENTIAL EQUATIONS

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The purpose of this paper is to announce necessary and sufficient conditions for the triangularizability of the second-order equation

$$(*) \qquad x + k(x,\dot{x}) = 0.$$

The equation (*) is said to be triangularizable if it can be transformed into triangular form

$$\dot{x}_1 = f_1(x_1)\ , \quad x_2 = f_2(x_1,x_2).$$

All the results presented here are proved in [1]. In what follows we shall suppose that equation (*) has nondegenerous critical points, i.e. $k \in C^3(\mathbb{R}^2)$ and

(1) For every critical point c of (*) the eigenvalues λ_1,λ_2 of the Jacobian

$$\begin{bmatrix} 0 & 1 \\ -\partial_1 k & -\partial_2 k \end{bmatrix}$$

at c are different reals and $\lambda_1\lambda_2 = \partial_1\ k(c) \neq 0$. The equation (*) is said to be nondegenerous if its critical points are nondegenerous and the induced dynamical system $(\mathbb{R}^2, \varphi)$ satisfies the following conditions:

(A_0) $(\mathbb{R}^2, \varphi)$ is not a global knot point,

(A_1) The critical point set $C^\varphi \subset \mathbb{R}^2$ is discrete,

(A_2) If $q \in \mathfrak{J}^\eta_\varphi(p)$ (= η -prolongational limit set of $\{p\}$ at the direction $\eta \in \{-,+\}$) is valid for $p,q \in R^2 - C^\varphi$ then the set of trajectories

$$\{\varphi(r;\ R) \mid q \in \mathfrak{J}^\eta_\varphi(r),\ \ r \in \mathfrak{J}^\eta_\varphi(p)\}$$

is finite and for every noncritical point $p \in R^2$

$$\varphi(p;\ \mathbb{R}) \cap \overline{\Sigma^\varphi - \varphi(p;\ \mathbb{R})} = \emptyset$$

holds (Σ^φ = underlying set of Markus-separatrices). Still, a nondegenerous equation (*) may have a complicated behaviour outside the strips $\mathbb{R} \times [-n,n]$ parallel to the first axis and so we need some restrictions on these domains as follows:

(P_1) For every finite interval $I \subset \mathbb{R}$ the set

$$Zero(k) \cap I \times \mathbb{R} \subset \mathbb{R}^2$$

is bounded,

(P_2) For every finite interval $I \subset \mathbb{R}$ there exists a function

$$M_I : \mathbb{R} - (-L,L) \to \mathbb{R}_+$$

with some sufficiency large number L, for which

$$\int_L^{\infty} \frac{1}{M_I} = \int_{-\infty}^{-L} \frac{1}{M_I} = \infty$$

and

$$\sup_{s,t \in I} |k(t,x_2)-k(s,x_2)| \leq M_I(x_2)|x_2|, \quad |x_2| \geq L,$$

hold. Either (P_1) or (P_2) ensure that the induced dynamical system is parallelizable outside a sufficeintly wide strip $\mathbb{R} \times [-n,n]$.

Theorem 1. Let the nondegenerous equation (*) be triangularizable and denote $(\mathbb{R}^2,\varphi)$ its induced dynamical system. Then (Δ_1) $|L^{\eta}_{\varphi}(p)| \leq 1$ for every $p \in \mathbb{R}^2$ and $\eta \in \{-,+\}$, where $L^{\eta}_{\varphi}(p) = \eta$ -limit set of $\{p\}$,

(Δ_2) $(\mathbb{R}^2,\varphi)$ has no saddle with multiplicity > 2.

(Δ_3) $(\mathbb{R}^2,\varphi)$ has no invariant simple closed Jordan curve γ on which the parametrization of trajectories give rise to an orientation of γ.

(Δ_4) There are no points $p,q \in \mathbb{R}^2 - C^{\varphi}$ for which $p \in \mathfrak{J}^+_{\varphi}(q)$ and $q \in \mathfrak{J}^+_{\varphi}(p)$ hold.

For the low critical point case the converse of the statement above is also valid, namely:

Theorem 2. Assume that the nondegenerous equation (*) with $|C^{\varphi}| \leqq 2$ fulfils one of the conditions (P_1) or (P_2). Then (*) is triangularizable iff (Δ_i), $i=1,2,3$, hold.

As applications of the results above, by (Δ_3), the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \qquad \mu > 0,$$

is nontriangularizable and by an analysis of the asymptotic behaviour of the trajectories, the Emden-Fowler equation

$$\ddot{x} + (2\mu-1)\dot{x} + \mu(\mu-1)\ (x - sgnx \cdot |x|^n) = 0$$

is triangularizable, provided that $\sigma+n+1<0$, $n \geqq 3$, $n \in N$, where $\mu = -\frac{\sigma+2}{n-1}$

REFERENCES

[1] G. Tóth, Triangularization methods in the transformation theory of planar dynamical systems Period. Math. Hungar. Vol. 11(3),(1980), pp. 197-211, II. ibid. Vol. 11(4),(1980), pp. 298-308, III.ibid. (to appear)