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THE STABILITY AND CONVERGENCE OF GENERAL ONE-STEP METHODS FOR THE NUMERICAL SOLUTION OF VOLTERRA FUNCTIONAL DIFFERENTIAL EQUATIONS

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INTRODUCTION

In this paper a generalization of Tavernini's results [1] on the numerical solution of one-step methods of Volterra functional differential equations is given for the case of variable stepsize.

Let us consider the initial value problem

(1) $y'(x) = F(x, y), \qquad a < x \leq b,$

(2) $y(x) = g(x), \qquad \alpha \leq x \leq a,$

where g is a given initial function continuous on $[\alpha, a]$ and the functional $F:[\alpha,b]\times C_n[a,b]\to R^n$ is such that the conditions of the existence and uniqueness theorem for the solution are fulfilled (Driver [2]), namely the following assumptions hold:

i) for fixed y, $F(x,y)$ is continuous in x for $x \in [a,b]$,

ii) the functional F satisfies the Lipschitz condition

$$|| F(x,y_1)-F(x,y_2) || \leq L \, || y_1 - y_2 ||^{[\alpha,x]}$$

for any $y_1, y_2 \in C_n[\alpha, b]$ and x in $[a,b]$ where L is constant.
Here for real x_1 and x_2 with $x_1 \leq x_2$, $C_n[x_1, x_2]$ denotes the space of continuous functions on $[x_1, x_2]$ and for $y \in C_n[x_1, x_2]$ the norm is defined as

$$\| y_1 - y_2 \|^{[\alpha, x]} = \max_{\alpha \leq s \leq x} \| y_1(s) - y_2(s) \| .$$

A consequence of ii) is that $F(x,y)$ is independent of $y(s)$ for $s > x$, i.e. F is a Volterra functional. Obviously, this class of functional differential equations contains for example the class of ordinary differential equations, delayed differential equations, etc.

NOTATION AND DEFINITIONS

Let us introduce the grid $\Delta_N = \{x_o, x_1, \ldots, x_N\} \in [a,b]$ and the norm $\| \Delta_N \| = \max_{o \leq i \leq N-1} h_i$ where $h_i = x_{i+1} - x_i$.

The solution of (1), (2) is always denoted by y.

Definition 1.

We call a general one-step method the following:

(3) $y(x_j + r h_j) = y(x_j) + r h_j \Phi(x_j, h_j, y_j, r)$, $x_i \in \{\Delta_N\}$, $r \in [0,1]$, $o < i \leq N$

(4) $y(x) = \tilde{g}(x)$, $x \in [\alpha, a]$,

where $\Phi: S \times C_n [\alpha,b] \times [0,1] \to R^n$ is the increment function of the method and $S = \{(x,\delta) \mid \alpha \le x \le b \text{ and } 0 < \delta \le b-x\}$ and $\tilde{g}$ is some continuous approximation to the initial function g.

In the following we suppose $\Phi(x,h,y,r)$ to be continuous in r for fixed x,y,h and let Φ satisfy the Lipschitz condition, namely

$$\|\Phi(x,h,y_1,r) - \Phi(x,h,y_2,r)\| \le K \|y_1 - y_2\|^{[\alpha,x]} ,$$

where K is constant.

It can be seen that the numerical solution of (1) is defined by the method (3) everywhere in the interval of integration and not only in the gridpoints as it is typical when solving ordinary and partial differential equations.

Definition 2.

The general one-step method (3) is called to be convergent, if for every gridpoint

$$\| y - \tilde{y} \| \to 0 \quad \text{if} \quad \|\Delta_N\| \to 0, \quad \| g - \tilde{g} \| \to 0$$

Definition 3.

The truncation error τ of the method (3) at $x_i + rh_i$ is defined as follows:

$$\tau(x_i,h_i,r) := y(x_i) + h_i \, \Phi(x_i,h_i,y,r) - y(x_i + rh_i).$$

Φ is supposed to be such that the truncation error satisfies the condition

$$|\tau\ (x_i, h_i, r)| \leq h_i \quad \varepsilon(x,r,h)$$

where $\varepsilon(x,r,h)$ is a given error function.

Definition 4.

The method (3) is called consistent if

$$\Phi(x,h,y,1) \to F(x,y), \qquad (||\Delta_N|| \to 0)$$

uniformly in x, or (an equivalent condition)

$$\varepsilon(x,1,h) \to 0 \qquad (||\Delta_N|| \to 0)$$

for all $x \in [a,b]$.

It might be regarded as a more natural generalization of the consistency, if we suppose, that

$$\varepsilon(x,r,h) \to 0 \qquad (||\Delta_N|| \to 0)$$

for all $r \in [0,1]$. However one can see that the error is simply accumulated at the gridpoints while in other points it is multiplied by rh_i.

Definition 5.

The global error of the method (3) is

$$||y - \tilde{y}||^{[\alpha,b]}$$

Next we define the function

$$\nabla(x,\delta) = \begin{cases} \frac{1}{\delta}\,[y(x+\delta)-y(x)], & \delta > 0, \\ y'(x) \qquad , & \delta = 0 \end{cases}$$

for all $(x,\delta) \in S$.

Finally, let us have the

Definition 6.

The method (3) is called to be stable, if there are constants $c, h^* > 0$ such that

$$\|y^* - y\|^{[a,b]} \leq c\ \{\|g^* - \tilde{g}\| + \sum_{j=1}^{N} \|\delta_j\|\}$$

provided that $\|\Delta_N\| \leq h^*$ and y^* is the solution of the perturbed recursion

$$y^*(x_i + rh_i) = y^*(x_i) + rh_i[\Phi(x_i, h_i, y^*, r) + \delta_{i+1}].$$

Theorems

In the remaining part of this paper we give some convergence and stability results for general one-step methods (3).

Theorem 1.

Assume that

i) the increment function Φ is such that the truncation error satisfies the inequality

$$\|\Phi(x,h,y,r) - \nabla(x,rh)\| \leq \varepsilon(x,r,h)$$

for all $x \in \{\Delta_N\}$, $r \in [0,1]$ and $0 < h \leq h^*$,

ii) ε is a monotone function of h and $\varepsilon(x,r,h) \leq \bar{\varepsilon}(x,r,h^*) \leq \varepsilon_1$ (ε_1 is constant)

iii) furthermore

$$\bar{\varepsilon}(x,1,h^*) \to 0 \qquad (||\Delta_N|| \to 0).$$

Then the error is bounded by

$$||y - \tilde{y}||^{[\alpha,b]} \leq [||g - \tilde{g}||^{[\alpha,a]} + h^*\varepsilon_1 + $$

$$+ \int_a^b \bar{\varepsilon}(x,1,h^*)dx] [1 + (b-a)Ke^{K(b-a)}].$$

By the help of the above estimation we obtain the following convergence and stability result.

Theorem 2.

If the general one-step method (3)

i) is consistent and

ii) the increment function Φ is uniformly bounded in x and r, for all $||\Delta_N|| \to 0$

$$||\Phi(x,h^*,y,r)|| \leq M, \qquad M \text{ - constant},$$

then it is convergent for any $\{\Delta_N\}_{N=1}^{\infty}$ such that

$$||\Delta_N|| = \max_{o<i\leq N-1} h_i = h^* \to 0 \quad \text{and} \quad ||g - \tilde{g}|| \to 0.$$

Theorem 3.

The general one-step methods (3) are stable.
The proofs of these theorems can be found in [3].

REFERENCES

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