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HOW TO CONSTRUCT A LARGE SET OF NON-EQUIVALENT FUNCTIONALLY CAMPY LATE ALGEBRAS

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Many authors have investigated certain classes of functionally complete algebras. The finite algebras $<A,t>$ and $<A,d>$ with the discriminator t and the dual discriminator d, respectively, all but 6 homogeneous algebras and other types of algebras having a 'large' automorphism-group were proved to be functionally complete. See Werner [6], Fried-Pixely [2], Csákány [1], Pálfy-Szabó-Szendrei [4]. In the present paper we are going to determine the number of non-equivalent functionally complete algebras. We call the algebras $<A,F>$ and $<A,B>$ equivalent if $[F]=[B]$ where $[F]$ denotes the set of all polynomials of $<A,F>$. We denote the set of all functions on A by O_A. In the following $E_\ell=\{0,1,\dots k-1\}$ will present the base set of the investigated algebras and $\mathfrak{C}$ denotes the cardinality of the continuum.

In the case $|A|=2$ we have a complete description of all algebras (See E. Post [5]), and so we know, that there are only finitely many non-equvalent functionally complete algebras with a two-element base set. In the case $|A|=3$ we shall prove the existence of many functionally complete algebras. In the case $|A|=3$ we cannot prove any equality, but we can easily construct a nonfinite set of functionally complete algebras.

Proofs of our theorems are based on the well-known constructions of Janov and Mucnik, and on the fact that for every k there is a function $s(x,y)$ with $[s]=O_{E_k}$.
Let us define for $n>3$ $g_n(x_1,\dots x_n)$ as follows:

$$g_n(x_1,\dots,x_n)=\begin{cases} 1 \text{ if, } x_j=1 \text{ and for } i\neq y\ x_i=z \\ s(x_1,x_2) \text{ if for } i>2\ x_i=3 \\ 0 \text{ otherwise.}\end{cases}$$

Since $g_n(x_1,x_2,3,3,\ldots,3)=s(x_1,x_2)$ and $[s]=0_{E_k}$, all the g_n-s are functionally complete. Let $\mathcal{G} = \bigcup_{n>3} g_n$. We shall prove that $g_n \notin [G \setminus g_n]$:

Let $\mathfrak{A}(x_1,\ldots x_n)$ be a polynomial of $\langle E_k, G\setminus g_n\rangle$.

Let $g_m(x_{i1},\ldots,x_{im})$ be any subformula of $\mathfrak{A}$ so, that all the s_{ij}-s are variables. Since $m\neq n$ we have two cases:

1. $|\{i_1,\ldots,i_m\}|<n$. Then let $g_e=1$ for a fixed $\ell\notin\{i_1,\ldots i_m\}$ and $y_i=z$ for $\ell\neq i$. Then we have $\mathfrak{A}(y_1,\ldots,y_m)=0$ and - by definition - $g_n(g_1,\ldots y_n)=1$.
2. $|\{i_1,\ldots i_m\}|=n$. Then there are at least two indices i_{ℓ_1}, i_{ℓ_2} so, that $i_{\ell_1}=i_{\ell_2}=t$; we can choose $y_t=1$ and $y_j=1$ for $j\neq t$, and so we also obtain $\mathfrak{A}(y_1,\ldots y_n)=0$ and $g_n(y_1,\ldots,y_n)=1$.

Theorem 1: Let $|A|>3$. Then the cardinality of non-equivalent functionally complete algebras is $\mathfrak{c}$. In the case $k=3$ let us define g_n as follows:

$$g_n(x_1,\ldots,x_n)=\begin{cases} 2 \text{ if } |\{i/k_i=2\}|\geq n-1 \text{ and } |\{i/k_i=0\}|=0. \\ s(x_1,x_2) \text{ if for } j\geq 3\ x_j=0, \\ 1 \text{ otherwise.}\end{cases}$$

In this case all g_n are also functionally complete. Let ${}^n= \bigcup_{i>n} g_i$. We shall prove $g_n\notin[\mathcal{G}^n]$, and so we obtain, that

$$\mathcal{G}^m \supsetneq \mathcal{G}^{m+1} \supsetneq \cdots \supsetneq \mathcal{G}^{m+k} \supsetneq \cdots$$

If $g_n\in[\mathcal{G}^n]$, then there is a polynomial of $\langle E_k,\mathcal{G}^n\rangle$, $\mathfrak{A}(x_1,\ldots,x_n)$, so that $\mathfrak{A}(x_1,\ldots,x_n)=g_n(x_1,\ldots,x_n)$. Let us consider the set $Y\ (E_k)^n$, where

$$Y=\{(y_1,\ldots,y_n)|(y_1,\ldots,y_n)\in\{1,2\}^n \text{ and } |\{y|y_j=1\}|=1\}.$$

Since $\mathfrak{A}(x_1,\ldots x_n)=g_n(x_1,\ldots,x_n)$ holds, we can choose a "minimal" polynomial $\mathfrak{A}^*\in[\mathcal{G}^n]$ so that

$$\mathfrak{A}^*|_Y = g_n|_Y \quad \text{and,}$$

if $\mathfrak{A}^*$ is written in the form $\mathfrak{A}^*=g_q(\mathcal{L}_1,\ldots,\mathcal{L}_q)$ then there is no $\mathcal{L}_i$ with $\mathcal{L}_i|_Y=g_n|_Y$.

By $\mathfrak{A}^*\in[\mathcal{G}^n]$ we have $q>n$. The set Y has exactly n elements and so $\mathcal{L}_i|_Y\neq g_n|_Y$ implies that there is at least one $y\in Y$ and j_1,j_2 such, that $\mathcal{L}_{j_1}(y)=\mathcal{L}_{j_2}(y)=1$, and hence $\mathfrak{A}^*|_Y\neq g_n|_Y$.

This contradiction proves our

<u>Theorem 2.</u> For all $k>2$ there are at least countable many non-equivalent functionally complete algebras.

REFERENCES

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Összefoglaló

Hogyan konstruáljunk sok nem-ekvivalens funkcionálisan teljes algebrát

Jelen dolgozatban a szerzők a következő kérdésre keresik a választ: mi a nem-ekvivalens funkcionálisan teljes algebrák száma egy rögzített véges alaphalmazon. Az <A, F> és <A,B> algebrákat ekvivalensnek nevezzük, ha az általuk generált klonok egyenlőek ([F] = [B]). Konstruálnak kontinuum sok nem ekvivalens funkcionálisan teljes algebrát legalább 4 elemű alaphalmaz fölött.

Р е з ю м е

Как надо построить много неэквивалентных функционально полных алгебр

В настоящей работе авторы изучают следующий вопрос: сколько неэквивалентных функционально-полных алгебр существует. Алгебры <A,F> и <A,B> эквивалентные, если F = B . В настоящей работе авторы построят континуум неэквивалентных функционально полных алгебр, если $|A| \geq 4$.