MTA Számítástechnikai és Automatizálási Kutató Intézete, Közlemények 25/1982

ON FUNCTIONALLY COMPLETENESS OF PRIME-ELEMENT ALGEBRAS

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Several authors have discussed the question of giving conditions for the functional completeness of a finite algebra. A major part of these theorems is concerned with the fact that if the automorphism-group of the algebra is a sufficiently large subgroup in the permutations of the base set, then the algebra is functionally complete, if we disregard some exceptions. For example for every finite algebra $<A,t>$ or $<A,d>$, $|A| > 2$ where t,d are the discriminator function and dual discriminator function respectively, it was known, that they are functionally complete. (See H. Werner [12] and E. Fried, A. Pixley [4].) In this case Aut ($\mathfrak{A}$) is the full symmetric group on . A generalization of this result was given by B. Csákány [2] who proved that all but six homogeneous algebras (algebras for which Aut ($\mathfrak{A}$)$=S_A$ where A is the base set of $\mathfrak{A}$) are functionally complete. Analogous results were showed by Á. Szendrei - L. Szabó [10] and P.P. Pálfy - Á. Szendrei - L. Szabó [6]. These theorems are the following:

[10]: A finite algebra $\mathfrak{A} = <A,F>$ with triply transitive automorphism group is either functionally complete or equivalent to an affine space over the $GF(2)$.

[6]: A finite algebra $\mathfrak{A} =<A,F>$, with doubly transitive automorphism group is either functionally complete or equivalent to an affine space over a finite field.

The structure of affine spaces (or in terms of k-valued logics: linear closed classes) is well known. (See J. Bagyinszki - - J. Demetrovics [1] and Á. Szendrei [11].) According to the above results, these exceptional algebras are sufficiently described. The authors give a similar result for special algebras with transitive automorphism groups. Other notions and notations are to be found in G. Grätzer [5] and R. Pöschel, L.A. Kaluznin [7].

THEOREM.

Let $\mathfrak{A} = \langle A, F \rangle$ be an algebra where $|A| = p$ and $p \geqq 3$ is a prime number. Suppose that Aut $(\mathfrak{A})$ is a transitive subgroup of S_A. Then $\mathfrak{A}$ is either functionally complete or polynomially equivalent to an affine algebra over the $GF(p)$.

In the two element lattice the dual discriminator is an algebraic function (this is the median) but this structure is not functionally complete - since all algebraic functions are isotone. On the other hand d is a homogeneous function, hence the restriction $p \geqq 3$ is essential.

It is easily seen that in our case the transitivity of Aut $(\mathfrak{A})$ means that Aut $(\mathfrak{A})$ contains a cycle of length p, which we shall denote by φ. Of course, the subgroup $\langle\varphi\rangle$ itself acts transitively on the set A. To simplify notations, we can suppose $A = \{0, 1, \ldots, p-1\}$ and $\varphi : x \to x+1$ for all $x \in A$ where + denotes the addition mod p.

The proof uses Rosenberg's completeness theorem. (I.G. Rosenberg [9], R.W. Quackenbush [8].) The authors show that whenever F preserves none of the linear relations of A, then it will preserve none of the six types in Rosenberg's classification. The main problem was to show that in the above case $\mathfrak{A}$ is simple.

It is well known that for an arbitrary algebra $\mathrm{Con}(\mathfrak{A}) = \mathrm{Con}(\langle A, F_1 \rangle)$ with F_1 denoting the set of all unary algebraic functions of $\mathfrak{A}$. This simple fact suggests to investigate the unary algebraic functions of $\mathfrak{A}$. A basic tool in the proof was the following

LEMMA 1.

Let $\mathfrak{A} = \langle A, F\rangle$ be a non-trivial algebra with $|A| = p, p \geq 3$ prime with its operations being not all projections and with $\varphi \in \mathrm{Aut}(\mathfrak{A})$.

Then at least one of the following cases holds

a) $F_1 \cap S_A \neq \langle id_A\rangle\ T$

b) There is a $k \not\equiv 0 \pmod p$ and there are f, g elements of F_1 for which $1 < |imf| < p$ and for all $y \in imf$ $g(y) = y + k$ hold.

An easy consequence of Lemma 1 is

LEMMA 2.

In the case b.) in Lemma 1 there is a $h \in F_1$ for an arbitrary $k \in A$ with $h(y) = y + k$ for all $y \in imf$.

For all $f \in F_1$ $\varphi^{-1} f \varphi \in F_1$ holds, and in the case a.) of Lemma 1 it means that $F_1 \cap S_A$ is a transitive subgroup of S_A. So in each of the above two cases we have a "large" set of unary algebraic functions.
The following two lemmas facilitate handling central and k-regular relations. (See [7], [8]). Let O_A denote the set of all operations on A with finite arites.

LEMMA 3.

Let $\rho \subseteq A^k$ be a k-ary totally reflexive relation and $k \geq 3$. If $H \subseteq O_A$ such that all operations in H are surjective (or constant) then $\rho \in InvH$ implies $\rho' \in InvH$ where

$$\rho' = \{(x,y) \mid (x,y,a_1,\ldots,a_{k-2}) \in \rho \quad \text{for all} \quad a_1,\ldots,a_{k-2} \in A\}.$$

LEMMA 4.

Let H, ρ, ρ' be as defined in Lemma 3.

i) If ρ is a non-trivial central relation then ρ' is also central with the same center.

ii) If ρ is a k-regular relation defined by the equivalence relations $\Theta_1, \ldots, \Theta_m$ then

$$\rho' = \bigcap_{i=1}^{m} \Theta_i$$

In the possession of the above media Theorem is easy to prove. Its statement underlines the technical character of Lemma 1. J. Bagyinszki and J. Demetrovics proved that every linear closed class properly containing constant functions and projections will contain a permutation different from the identity. So, if $\mathfrak{A} = \langle A, F \rangle$ is a non-trivial affine algebra then $S_A \cap F_1 \neq \langle id_A \rangle$. This fact and the Theorem show that in all cases the statement a.) of Lemma 1 holds. At last we mention two consequences of the Theorem.
A subset H of O_A is called basic if for all Slupecki functions $f, [\{f\} \cup H] = O_A$ holds. (Then for arbitrary $X \subseteq O_A$ $[X]$ denotes the closed class generated by X.) A group $G \leq S_A$ is a basic group if G is a basic subset in O_A. L. Szabó has conjectured (personal communication) the following:

Let $\mathfrak{A} = \langle A, F \rangle$ be a nontrivial finite algebra and assume that Aut($\mathfrak{A}$) is a basic group. Then $\mathfrak{A}$ is functionally complete. From our theorem follows:

COROLLARY 1.

If $\mathfrak{A} = \langle A, F \rangle$, $|A| = p$ and p is a prime number then Szabó's conjecture holds.

Let A be a nonempty set and $\pi \in S_A$. The graph ρ_π of π can be defined as follows:

$$\rho_\pi = \{(x_1 \pi(x)) \mid x \in A\}$$

That is, $\rho_\pi \subseteq A^2$ is binary relation on A. J. Demetrovics and L. Hannák in [3] have proved the following: if $|A| \geq 5$ then for arbitrary $\pi \in S_A$, $Pol\ \rho_\pi$ contains a continuum cardinality set of closed classes. In other words in this case there is a continuum cardinality set of polynomially non-equivalent algebras $\mathfrak{A}_\beta = \langle A, F_\beta \rangle$ $(\beta < \mathfrak{c})$ for which $\pi \in \mathrm{Aut}\ (\mathfrak{A}_\beta)$ $(\beta < \mathfrak{c})$. This result and our Theorem imply:

CORALLY 2:

If A is a prime element set and $|A| \geq 5$ then there is a continuum cardinality set of pairwise polynomially non-equivalent functionally complete algebras $\mathfrak{A}_\beta = \langle A, F_\beta \rangle (\beta < \mathfrak{c})$.

In the case $|A| = 3$ the authors cannot tell even the number of different algebras with $\varphi \in \mathrm{Aut}\ (\mathfrak{A})$.

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Összefoglaló

Prímszám elemű algebrák függvény-teljességéről

A jelen dolgozatban a szerzők a tranzitív automorfizmus csoporttal rendelkező algebrákkal foglalkoznak. A fő eredmény szerint ezen algebrák vagy függvényteljesek, vagy lineárisok. Az eredmények bizonyítás nélkül szerepelnek.

Резюме

О функциональной полноте алгебр с простыми числами элементов

В настоящей работе занимаемся функциональной полнотой некоторых алгебр. Главный результат настоящей работы заключается в том, что изучаемые алгебры либо функционально полные , либо линейные.