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ON THE USE OF DIFFUSION APPROXIMATIONS FOR THE CYCLIC QUEUE MODEL

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Consider the following model of a multiprogrammed computer system (see Figure 1.).

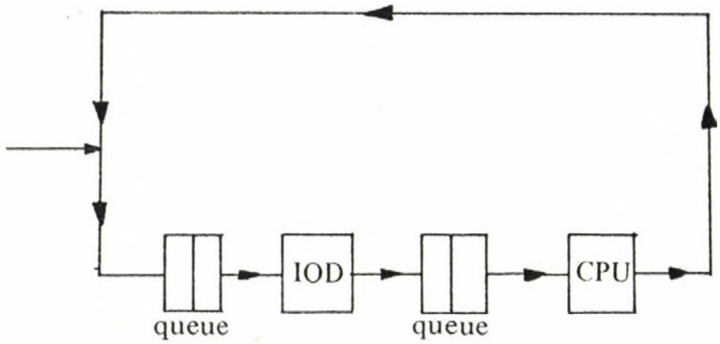

Figure 1.

This is a closed two-server system. At any instant exactly K programs are in the cycle formed by the central processor unit (CPU) and the input-output devices (IOD). Each program is either at the so-called IOD stage or at the CPU-stage (awaiting or under service). This model has been analized by many authors (see [3]-[6]) using a diffusion approximation for the number of programs in the CPU-stage.

In this note we shall give an exact proof for the validity of diffusion approximation in the case of exponentially distributed service times. We shall use the following

Notations:

$N(t)$ – the number of programs at the CPU-stage at time t,

ξ_i – the service time of the i-th demand on the IOD,

η_j – the service time of the j-th demand on the CPU,

K – degree of multiprogramming.

We assume that the following conditions are fulfilled:

1.) ξ_i and η_j, $i = 1,2,\ldots,$ $j = 1,2,\ldots$ are mutually independent random variables.

2.) ξ_i, $i = 1,2,\ldots$ are independent identically distributed (i.i.d.) random variables with distribution function $P\{\xi_i \leq t\} = 1 - e^{-\lambda t}$.

3.) η_j, $j = 1,2,\ldots$ are i.i.d. random variables with distribution function $P\{\eta_j \leq t\} = 1 - e^{-\mu t}$.

Let $\Delta = 1/K$ and $X_\Delta(t) = N(t)\Delta$. Obviously $X_\Delta(t)$ is a time-homogeneous Markov process with countable state-space. The states of $X_\Delta(t)$ are the values $x_i = i\Delta$, $i = 0,1, \ldots K$. Let us denote the interval $[0,1]$ of the real axis and the σ-algebra of their Borel-sets with $\{ X, \mathscr{B} \}$.

Let $\mathscr{D}_{[0,T]}(X)$ be the space of functions with discontinuities of the first kind, defined on the time interval $[0,T]$ and with range in the separable, complete, metrical space X.

Let $\mathcal{L}$ be the space of real, continuous and $\mathscr{B}$-measurable functions on $[0,1]$. Let the norm on $\mathcal{L}$ be $\| f \| = \sup_x |f(x)|$, if $f \in \mathcal{L}$.

We will allow Δ to tend to zero and both λ and μ to depend on Δ. Therefore we shall denote them by λ_Δ and μ_Δ, respectivelly. The following statement will be proved.

Theorem

Let $/\lambda_\Delta - \mu_\Delta / \Delta \to m$

$/\lambda_\Delta + \mu_\Delta / \Delta^2 \to \sigma^2 > 0$

if $\Delta \to 0$,

then for arbitrary constant α

$$P\{F(X_\Delta(t)) < \alpha\} \to P\{F(X_0(t)) < \alpha\},$$

F being continuous functional on $\mathscr{D}_{[0,1]}(X)$.

$X_0(t)$ – *is a diffusion process on the real axis with reflecting barriers at points* 0 *and* 1, *with the infinitesimal generator* $A_0 f = \sigma^2/2f'' + mf'$. *The domain* $\mathcal{L}'$ *of* A_0 *is the space of twice continuously differentiable functions for which* $f'(0) = f'(1) = f''(0) = f''(1) = 0$.

Proof. The proof of this theorem is based on a Gikhman-Skorohod's theorem ([2], Vol. I, page 508). This theorem states, that $P\{F(X_\Delta(t)) < \alpha\} \to P\{F(X_0(t)) < \alpha\}$ as $\Delta \to 0$ if

1.) the finite dimensional distributions (f.d.d.) of $X_\Delta(t)$ tend to corresponding f.d.d. of $X_0(t)$ and

2.) for arbitrary $\epsilon > 0$

$$\lim_{h \downarrow 0} \overline{\lim_{\Delta \to 0}} \sup_{s-t} \{P_\Delta(t,x,s,V_\epsilon(x));\ x \in X,\ 0 \leqslant s - t \leqslant h\} = 0.$$

Here $P_\Delta(t,x,s,\Gamma)$, $\Gamma \in \mathscr{B}$ is the probability of transition for the process $X_\Delta(t)$ and $V_\epsilon(x) = \{ y : |y - x| > \epsilon \}$.

As $X_\Delta(t)$ is a time-homogeneous process, it is enough to prove that

$$\lim_{h \downarrow 0} \overline{\lim_{\Delta \to 0}} \{P_\Delta(t,x,V_\epsilon(x));\ x \in X,\ 0 \leqslant t \leqslant h\} = 0.$$

Furthermore, we known that as soon as $\epsilon > \Delta$,

$$P_\Delta(t,x,V_\epsilon(x)) = o(t); \quad x\epsilon X, \quad 0 \leq t \leq h.$$

Thus, we can see that the second condition of Gikhman-Skorohod' theorem is satisfied. The fulfilment of the first condition is proved in the next lemma.

Lemma. *Under the conditions of the theorem the f.d.d.'s of* $X_\Delta(t)$ *tend to the f.d.d.'s of* $X_0(t)$.

Proof. Denote $T_\Delta(t)f(x)$ the transition operator (see [2] Vol. II., page 30.) corresponding to the Markov process $X_\Delta(t)$ and $T_0(t)f(x)$ the same for the process $X_0(t)$, where $f\epsilon\mathcal{L}$. To prove the lemma it is well enough to verify that

$$T_\Delta(t)f(x) \to T_0(t)f(x) \qquad \text{as} \qquad \Delta \to 0,$$

because in our case the convergence of these transition operators implies the convergence of the transition probabablities

$$P_\Delta(t,x\,\Gamma) \quad \to \quad P_0(t,x,\Gamma), \qquad \Gamma\epsilon\mathcal{B}$$

(where $P_0(t,x,\Gamma)$ is the transition probability of the process $X_0(t)$) and as it is known the finite dimensional distributions of the processes can be described by the help of the transition probabilities and the initial distribution.

We will use the next lemma, proved by Feller ([1], page 460.).

Approximation lemma *Let* $\{T_\Delta(t)\}$ *be a family of pseudo–Poissonian semigroups commuting with each other and generated by the endomorphisms* A_Δ. If $A_\Delta f \to A_0 f$ *for all* f *of a set* $\mathcal{L}'$ *dense in* $\mathcal{L}$, *then*

$$T_\Delta(t) \to T_0(t) \qquad as\ \Delta \to 0,$$

where $\{T_0(t)\}$ *is a semi-group of contractions whose infinitesimal generator agrees with* A_0 *for all* $f\epsilon\mathcal{L}'$. $\mathcal{L}'$ *is the domain of the operator* A_0.

Continue the function $f(x)$ outside the interval $[0,1]$ so that for $\Delta > 0$ $f(-\Delta) = f(0)$ and $f(1+\Delta) = f(1)$. In this case the transition operator of the process $X_\Delta(t)$ has a form of

$$T_\Delta(t)f(x) = \lambda_\Delta t f(x+\Delta) + \mu_\Delta t f(x-\Delta) + (1-(\lambda_\Delta+\mu_\Delta))f(x) + o(t),$$

and for the infinitesimal generator after some rearrangements we get a form of

$$A_\Delta f(x) = (\lambda_\Delta + \mu_\Delta)(S_\Delta f(x) - f(x)),$$

where

$$S_\Delta f(x) = \frac{\lambda_\Delta}{\lambda_\Delta + \mu_\Delta} f(x+\Delta) + \frac{\mu_\Delta}{\lambda_\Delta + \mu_\Delta} f(x-\Delta)$$

is a transition operator, $(\lambda_\Delta + \mu_\Delta) \to \infty$ and $S_\Delta \to \mathbb{1}$ as $\Delta \to 0$. From the form of A_Δ can be seen that $\{T_\Delta(t)\}$ is really a family of pseudo-Poissonian semi-groups (see [1], page 353.) generated by A_Δ.

The elements of $\{T_\Delta(t)\}$ commute with each other too if A_Δ-s for different $\Delta > 0$ do.

$$A_{\Delta_1}A_{\Delta_2}f(x) = \lambda_{\Delta_1}(\lambda_{\Delta_2}f(x + \Delta_1 + \Delta_2) + \mu_{\Delta_2}f(x + \Delta_1 - \Delta_2) - (\lambda_{\Delta_2} + \mu_{\Delta_2})f(x + \Delta_1))$$
$$+ \mu_{\Delta_1}(\lambda_{\Delta_2}f(x - \Delta_1 + \Delta_2) + \mu_{\Delta_2}f(x - \Delta_1 - \Delta_2) - (\lambda_{\Delta_2} + \mu_{\Delta_2})f(x - \Delta_1))$$
$$- (\lambda_{\Delta_1} + \mu_{\Delta_1})(\lambda_{\Delta_2}f(x + \Delta_2) + \mu_{\Delta_2}f(x - \Delta_2) - (\lambda_{\Delta_2} + \mu_{\Delta_2})f(x)) =$$
$$= A_{\Delta_2}A_{\Delta_1}f(x).$$

Let $\mathcal{L}'$ be the space of the twice continuously differentiable functions defined on $[0,1]$, for which $f'(0) = f'(1) = f''(0) = f''(1) = 0$. Using a Taylor-formula for $f(x) \epsilon \mathcal{L}'$ the infinitesimal generator has a form of

$$A_\Delta f(x) = (\lambda_\Delta - \mu_\Delta)\Delta f'(x) + (\lambda_\Delta + \mu_\Delta)\Delta^2 \frac{f''(x)}{2} + (c_1\lambda_\Delta + c_2\mu_\Delta)\frac{f''(x)}{2},$$

where the values of the constants c_1 and c_2 are

$$|c_1| \leq \max_{x < \zeta < x + \Delta} |f''(\zeta) - f''(x)| \text{ and } |c_2| \leq \max_{x - \Delta < \zeta < x} |f''(\zeta) - f''(x)|.$$

Under the conditions of the theorem for every $f \epsilon \mathcal{L}'$ $\sup\limits_{0 \leq x \leq 1} |A_\Delta f(x) - A_0 f(x)| \to 0$

as $\Delta \to 0$.

Using the statement of Feller's approximation lemma our theorem is proved.

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Összefoglaló

Diffuziós közelités jogossága multiprogramozású számitógépek vizsgálatában

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A multiprogramozásu számitógépek vizsgálatára felállitott u.n. ciklikus modellben a központi egység előtti sor eloszlására diffuziós közelitést lehet alkalmazni. A cikkben exponenciális tartásidők esetére bebizonyitjuk egyfajta diffuziós közelités jogosságát.

Р Е З Ю М Е

ОБОСНОВАННОСТЬ ДИФФУЗИОННОГО ПРИБЛИЖЕНИЯ В ОДНОЙ МОДЕЛИ МУЛЬТИПРОГРАММ ВЫЧИСЛИТЕЛЬНЫХ МАШИН

Мария Рет

В т.н. циклической модели мультипрограммной вычислительной машины применяется диффузионное приближение для вычисления распределения очереди перед центральным процессором.

В докладе доказывается справедливость одного диффузионного приближения в случае показательного распределения времен обслуживания.