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ON THE LOCAL LIMIT THEOREM FOR GENERAL LATTICE DISTRIBUTION

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ABSTRACT. The scheme of series of lattice casual vectors analogical to the probable is considered in the paper. A local limit theorem have been received, and the indicated directions of its use in differential methods theory of mathematical physics problems solving were received.

A local limit theorem for lattice distributions, and their quasiprobable analogues are successfully used in the approximate solution of some kinds of mathematical physics problems [1-3].

Quasiprobable distribution on the plain (according to Y. P. Studnyev [4]) on the whole is the complex number set $\{p(k,l)\}$, for which

$$\sum_{k,l} p(k,l) = 1; \qquad \sum_{k,l} |p(k,l)| < +\infty.$$

Let $\{(\xi_{nn}, \eta_{nn})\}$ be sequence of a mutually independent series of probable vectors (first index - series number, the second is changed from 1 to n), which are equally distributed on $\{kh_1, lh_2\}$ lattice, and let $\{p\,(n,k,l)\}$ be a quasiprobable vector distribution in series frames in the meaning of correspondence $p\,(n,k,l) = P\,\{(\xi_{nj}, \eta_{nj}) = (kh_1, lh_2)\}$, where $k,l \in Z; h_1, h_2 > 0; j = \overline{1,n}$. Then

$$w(t,s) = \sum_{k,l} e^{i(tkh_1 + slh_2)} p(n,k,l)$$

is a Fourier-Stieltjes distribution transformation $\{p(n,k,l)\}$.

As in the theory of probabilities, the formula of reverse is being proved:

$$p(n,k,l) = \frac{h_1 h_2}{4\pi^2} \iint\limits_D e^{-i(tkh_1 + slh_2)} w(t,s)\, dt\, ds,$$

where D rectangle is in the form of $\left[-\frac{\pi}{h_1}, \frac{\pi}{h_1}, -\frac{\pi}{h_2}, \frac{\pi}{h_2}\right]$.

If $\{p_n\,(k,l)\}$ is $\left(\sum\limits_{j=1}^{n} \xi_{nj}, \sum\limits_{j=1}^{n} \eta_{nj}\right)$ sum vector distribution, then the reverse formula is in

$$P_n(n,k,l) = \frac{h_1 h_2}{4\pi^2} \iint\limits_D e^{-i(tkh_1 + slh_2)} w(t,s)\, dt\, ds, \tag{1}$$

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where $\{P_n(k,l)\}$ distribution is $n-$divisible fold of the set $\{p_n(k,l)\}$ with itself. Let, further

$$\alpha_{r,q}=\sum_{k,l}(kh_1)^r(lh_2)^q p(n,k,l)$$

be an initial moment with (r,q) order of $\{p(n,k,l)\}$ distribution.

Let us consider two auxiliary lemmas.

Lemma 1. *Let for the casual vector with distribution conditions exist :*

1) $\alpha_{1,0}=\alpha_{0,1}=\alpha_{3,0}=\alpha_{2,1}=\alpha_{1,2}+\alpha_{0,3}=\alpha_{3,1}=\alpha_{2,2}=\alpha_{1,3}=0$;

2) $\alpha_{2,0}=-\frac{2a}{\sqrt{n}},\alpha_{1,1}=-\frac{2b}{\sqrt{n}},\alpha_{0,2}=-\frac{2c}{\sqrt{n}}, at^2+bts+cs^2$ *is a positively determined square form;*

3) $\alpha_{4,0}=\alpha_{0,4}=4!A, A>0$;

4) $a_{r,q}$ *exist, when* $r=q=5$.

Then

$$\left|w^n\left(\frac{t}{\sqrt[4]{n}},\frac{s}{\sqrt[4]{n}}\right)-e^{at^2+bts+cs^2-A\left(t^4+s^4\right)}dtds\right|=O\left(\frac{|t|^5+|s|^5}{\sqrt[4]{n}}\right),n\to\infty \tag{2}$$

correlation comes true.

Proof. We should note, that the role of moments in apportion is the same, as for characteristic functions in the analogical situation. That is why

$$w(t,s)=1+\frac{1}{\sqrt{n}}\left(at^2+bts+cs^2\right)-A\left(t^4+s^4\right)+O\left(|t|^5+|s|^5\right), t,s\to 0.$$

But then

$$w\left(\frac{t}{\sqrt[4]{n}},\frac{s}{\sqrt[4]{n}}\right)=1+\frac{at^2+bts+cs^2}{n}-\frac{A\left(t^4+s^4\right)}{n}+O\left(\frac{|t|^5+|s|^5}{n^4\sqrt{n}}\right),\quad n\to\infty;$$

or

$$w\left(\frac{t}{\sqrt[4]{n}},\frac{s}{\sqrt[4]{n}}\right)=e^{\frac{1}{n}(at^2+bts+cs^2)-\frac{A\left(t^4+s^4\right)}{n}}+O\left(\frac{|t|^5+|s|^5}{\sqrt[4]{n}}\right),\quad n\to\infty$$

for any fix couple $(t,s)\in R_2$. Hence, it follows, after the $n-$th power production of both parts of the last equality we receive (2). Then Lemma 1 is proved.

Let's present$w(t,s)$ into

$$w(t,s)=w_0(t,s)+\frac{1}{\sqrt{n}}\left(at^2+bts+cs^2\right)$$

form. □

Lemma 2. *Let in* $\{\xi_{nn},\eta_{nn}\}$ *series sequence, which are mutually independent and equally distributed within the limits of lattice casual vector series for every vector in series Lemma 1 conditions and condition :* $|w_0(t,s)|<1$, *when* $(t,s)\in D\backslash\{0,0\}$, *are executed. Then* $A_0>0$ *exists that in* $\sqrt[4]{n}D=\left[-\frac{\sqrt[4]{n}\pi}{h_1},\frac{\sqrt[4]{n}\pi}{h_1},-\frac{\sqrt[4]{n}\pi}{h_2},\frac{\sqrt[4]{n}\pi}{h_2}\right]$ *rectangle execute*

$$\left|w^n\left(\frac{t}{\sqrt[4]{n}},\frac{s}{\sqrt[4]{n}}\right)\right|\le e^{at^2+bts+cs^2-A_0\left(t^4+s^4\right)}. \tag{3}$$

Proof. In the work [3] it has been shown that by Lemma 2 conditions $A_0>0$ would exist that $|w_0(t,s)|\le e^{-A_0\left(t^4+s^4\right)}, (t,s)\in D$.

Then in D executes

$$|w(t,s)|\le|w_0(t,s)|+\frac{at^2+bts+cs^2}{\sqrt{n}}\le e^{\frac{at^2+bts+cs^2}{\sqrt{n}}-A_0\left(t^4+s^4\right)},$$

whence directly (3) comes out. □

These lemmas permit to prove such a theorem.

Theorem. *Let in $\{(\xi_{nn}, \eta_{nn})\}$ series sequences, which are mutually independent and equally distributed within the limits of lattice casual vector series for every vector in series conditions :*

1) $\alpha_{1,0} = \alpha_{0,1} = \alpha_{3,0} = \alpha_{2,1} = \alpha_{1,2} = \alpha_{0,3} = \alpha_{3,1} = \alpha_{2,2} = \alpha_{1,3} = 0$;

2) $\alpha_{2,0} = -\frac{2a}{\sqrt{n}}, \alpha_{1,1} = -\frac{2b}{\sqrt{n}}, at^2 + bts + cs^2$ *is a positively determined square form;*

3) $\alpha_{4,0} = \alpha_{0,4} = 4!A, A > 0$ *;*

4) $\alpha_{r,q}$ *exist, when* $r + q = 5$ *;*

5) $|w_0(t, s)| < 1$*, when* $(t, s) \in D \backslash \{0, 0\}$.

Then evenly on $k \in Z$ *by* $n \to \infty$ *:*

$$\sqrt{n}\left(\frac{p_n(k,l)}{h_1 h_2} - \frac{1}{4\pi^2}\iint e^{-i(tkh_1 + slh_2) + n\left(\frac{at^2+bts+cs^2}{\sqrt{n}} - A\left(t^4+s^4\right)\right)}dtds\right) \to 0. \tag{4}$$

Proof. Use the reverse formula (1) and in integrals of the left part of (4) correlation defined through R_n, execute the substitute: $t \to \frac{t}{\sqrt[4]{n}}, s \to \frac{s}{\sqrt[4]{n}}$.

We receive

$$4\pi^2 R_n = \iint\limits_{\sqrt[4]{n}D} e^{-i\left(\frac{lkh_1}{\sqrt[4]{n}} + \frac{slh_2}{\sqrt[4]{n}}\right)} w^n\left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}}\right) dt\, ds -$$

$$-\iint e^{-i\left(\frac{lkh_1}{\sqrt[4]{n}} + \frac{slh_2}{\sqrt[4]{n}}\right) + at^2+bts+cs^2 - A\left(t^4+s^4\right)} dt\, ds == I_1 + I_2 - I_3,$$

where

$I_1 = \iint\limits_{\Delta} e^{-i\left(\frac{lkh_1}{\sqrt[4]{n}} + \frac{slh_2}{\sqrt[4]{n}}\right)}\left(w^n\left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}}\right) - e^{at^2+bts+cs^2-A\left(t^4+s^4\right)}\right) dt\, ds,$

$I_2 = \iint\limits_{\sqrt[4]{n}D\backslash\Delta} e^{-i\left(\frac{lkh_1}{\sqrt[4]{n}} + \frac{slh_2}{\sqrt[4]{n}}\right)} w^n\left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}}\right) dt\, ds,$

$I_3 = \iint\limits_{R_2\backslash\Delta} e^{-i\left(\frac{lkh_1}{\sqrt[4]{n}} + \frac{slh_2}{\sqrt[4]{n}}\right) + at^2+bts+cs^2-A\left(t^4+s^4\right)};$

$\Delta = \left\{(t, s) : |t| \le n^{\lambda}, 0 < \lambda < \frac{1}{28}\right\}.$

Integral I_1 estimation comes out of Lemma 1. Really, according to (2) $B > 0$ exists, that

$$\left|w^k\left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}}\right) - e^{at^2+bts+cs^2-A\left(t^4+s^4\right)}\right| \le B\left(\frac{|t|^5 + |s|^5}{\sqrt[4]{n}}\right), (t, s) \in \Delta.$$

Then

$$|I_1| \le \frac{B}{\sqrt[4]{n}}\iint\limits_{\Delta}\left(|t|^5 + |s|^5\right) dtds = \frac{4B}{3}n^{7\lambda - \frac{1}{4}} = \frac{4B}{3}n^{-7\left(\frac{1}{28} - \lambda\right)}.$$

So as $\lambda < \frac{1}{28}$, then $I_1 \to 0$ by $n \to \infty$.

Out of the Lemma 2 the I_2 estimation comes out. Using (3), we receive

$$|I_2| \le \iint\limits_{\sqrt[4]{n}D\backslash\Delta} e^{at^2+bts+cs^2-A\left(t^4+s^4\right)}dtds.$$

The positively determined square form permits an upper estimation:

$$at^2 + bts + cs^2 \le a_0\left(t^2 + s^2\right), \quad a_0 > 0.$$

Then

$$|I_2| \leq \iint\limits_{\sqrt[4]{n}D\setminus\Delta} e^{a_0\left(t^2+s^2\right)-A\left(t^4+s^4\right)}\,dt\,ds \leq 4\left(\int\limits_{n^\lambda}^{\infty} e^{a_0t^2-A_0t^4}dt\right)^2.$$

We should take into consideration that by sufficiently large n for $t \geq n^\lambda$

$$4A_0t^3 - 2a_0t > 4A_0n^{3\lambda} - 2a_0n^\lambda,$$

I_2 integral permits the further estimation :

$|I_2| \leq \frac{1}{(2A_0n^{3\lambda}-a_0n^\lambda)^2}\int\limits_{n_\lambda}^{\infty}\left(\left(4A_0t^3 - 2a_0t\right)e^{a_0t^2-A_0t^4}dt\right)^2 = \left(\frac{e^{a_0n^{2\lambda}-A_0n^{4\lambda}}}{2A_0n^{3\lambda}-a_0n^\lambda}\right)^2,$

therefore, by $n \to \infty$ $I_2 \to 0$.

I_3 integral estimation is analogical to the second stage of I_2 integral estimation. The Theorem is proved. □

The received theorem could easy be generalized in case of quasiprobable lattice distributions with Fourier-Stieltjes transformation in

$$w\left(t,s\right) = e^{\Psi_2(t,s)+\Psi_4(t,s)+\ldots+\Psi_{2q-2}(t,s)-\Psi_{2q}(t,s)}$$

form,where $\Psi_2\left(t,s\right), \Psi_4\left(t,s\right), \Psi_{2q}\left(t,s\right)$ are positively-determined forms with orders indicated by indexes.

Such generalization could be used to the approximate problems solving linked with the evolutional equation in the form of:

$$\frac{\partial u(x,y,\tau)}{\partial\tau} = \left((-1)^{q+1}\Psi_{2q}\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y}\right) - (-1)^q\Psi_{2q-2}\left(i\frac{\partial}{dx}, i\frac{\partial}{\partial y}\right) - \right.$$
$$\left. - \ldots - \Psi_2\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y}\right)\right)u\left(x,y,\tau\right)$$

according to the scheme, for example, in [3].

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