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AN ESTIMATION PROBLEM IN THE PROCESS OF SERVECING MACHINES

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Introduction

Consider the repairman problem described in [3], pp. 462, in which a set of m machines are attended by one repairman. If a machine breaks down, it is served immediately, unless the repairman, is already at work on other machine, in which case it joins a waiting line. We say that the system in state i at time t if i machines are not working. Thus the state space $X = \{0, 1, \ldots, m\}$ contains $m + 1$ elements. Let us assume that the intervals between the breakdown of machines are independent identically distributed random variables with exponential distribution with parameter λ, and the service time of a machine is exponentially distributed random variable with parameter μ.

Thus, the number of machines failed is a birth–and–death process with transition intensities:

$$\lambda_i = (m - i)\lambda \qquad 0 \leqslant i < m,$$

$$\mu_i = \mu \qquad 0 < i \leqslant m,$$

the other transition intensities being zero.

Billingsley [2], had investigated the estimation of the parameters λ and μ assuming that m is known.

In this paper we assume that λ and μ are known, and find estimators for the discrete parameter m. We discuss two methods for estimating m. The first method, we call it, the direct method, and the second is the Maximum Likelihood method, Numerical results based on simulation are also given to illustrate and Compare the two methods.

The direct method for estimating m:

Assuming that we observe the system for a period of time of length T, let:

$\nu(t)$ = denotes the number of failed machines at time t,

τ_i = denotes the first passage time to the state i,

$p_i(t)$ = denotes the probability that the system is in the state i at time t, assuming that the state m is an absorbing state.

Since, the number of failed machines is always less than or equal m, then we may consider the statistic.

$$\hat{m}_T = \max_{0 \leqslant t \leqslant T} \nu(t)$$

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as an estimator of the true number m of machines.

To investigate the properties of this estimate, we first calculate the probability:

$$Q_m(t) = p(\hat{m}_t = m)$$

For this purpose we have by [6], that:

$$Q_m(t) = p(\hat{m}_t = m) = p(\tau_m \leqslant t) = p_m(t),$$

and it may be found by solving the sequence of forward equations assuming that m is an absorbing state.

This sequence of forward equations is:

$$(1)\quad \begin{cases} p_0'(t) = -\lambda_0 p_0(t) + \mu p_1(t) \\ p_i'(t) = -(\lambda_i + \mu)p_i(t) + \lambda_{i-1}p_{i-1}(t) + \mu p_{i+1}(t), & 1 \leqslant i \leqslant m-2 \\ p_{m-1}'(t) = -(\lambda_{m-1} + \mu)p_{m-1}(t) \\ p_m'(t) = \lambda_{m-1}p_{m-1}(t) \end{cases}$$

which may be written in the form:

$$\frac{d}{dt} P(t) = AP(t),$$

where

$$A = \begin{bmatrix} -\lambda_0 & \mu & & & & & & \\ \lambda_0 & -(\lambda_1 + \mu) & \mu & & & & & \\ & \lambda_1 & -(\lambda_2 + \mu) & \mu & & & & \\ & & \cdots\cdots\cdots & & & & & \\ & & & \cdots\cdots\cdots & & & & \\ & & & & \lambda_{m-3} & -(\lambda_{m-2} + \mu) & \mu & \\ & & & & & \lambda_{m-2} & -(\lambda_{m-1} + \mu) & 0 \\ & & & & & & \lambda_{m-1} & 0 \end{bmatrix} \quad (m+1) \text{ x } \text{ x } (m+1)$$

$$P(t) = \begin{bmatrix} p_0(t) \\ p_1(t) \\ \cdot \\ \cdot \\ \cdot \\ p_m(t) \end{bmatrix}, \qquad (m+1) \text{ x } 1.$$

From (1), we get the general solution:

(2) $$p_m(t) = \sum_{i=1}^{m+1} C_i \alpha_{m+1}^{(i)} e^{w_i t}$$

where C_i, $i = 1, 2, \ldots, m+1$ are arbitrary constants, and the w_i 's are the eigenvalues of the matrix A, that is, the solution of the characteristic equation:

(3) $$|A - wI| = 0,$$

and $\alpha_j^{(i)}$ is the j th component of the eigenvector corresponding to the i th eigenvalue w_i.

Since we assume that all the machines are in working state at time $t = 0$, then:

$$P(0) = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad (m+1) \times 1.$$

Thus the arbitrary constants C_i's are determined by:

$$\sum_{i=1}^{m+1} C_i \alpha_1^{(i)} = 1,$$

$$\sum_{i=1}^{m+1} C_i \alpha_2^{(i)} = 0,$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$\sum_{i=1}^{m+1} C_i \alpha_{m+1}^{(i)} = 0.$$

By a similar way as in [1] we can prove that, one of the roots of the characteristic equation (3) is $w_1 = 0$ and the other roots $w_2, w_3, \ldots, w_{m+1}$ are all negative and distinct.

Thus (2) may be written as:

(2') $$p_m(t) = C_1 \alpha_{m+1}^{(1)} + \sum_{i=2}^{m+1} C_i \alpha_{m+1}^{(i)} e^{w_i t}$$

where, the second term on the right hand side tends to zero as t tends to ∞.

It is well known that $p_m(t)$ is a proper distribution, since it represents the distribution of the first passage time to the state m of a finite Markov chain, which is irreducible and recurrent. Thus

(4) $\quad p_m(\infty) = 1$

(See [3], pp. 392.) We can prove (4) also by simple calculations using the laplace transform of the system (1).

Note that, this fact is true for any $p_k(t)$, $k \leqslant m$, if we assume that k is an absorbing state.

Relation (4) means that $\hat{m}_t$ is a consistent estimator for m.

Now, from (2') and (4) we have

$$C_1 \alpha^{(1)}_{m+1} = 1,$$

i.e.

$$(5) \qquad \begin{aligned} Q_m(t) = p_m(t) &= 1 + \sum_{i=2}^{m+1} C_i \alpha^{(i)}_{m+1} e^{w_i t} \\ &= 1 + A(t) \end{aligned}$$

where the constants C_i's are determined from the relations:

$$\sum_{i=1}^{m+1} C_i \alpha^{(i)}_1 = 1,$$

$$\sum_{i=1}^{m+1} C_i \alpha^{(i)}_j = 0, \qquad 2 \leqslant j \leqslant m,$$

$$\sum_{i=2}^{m+1} C_i \alpha^{(i)}_{m+1} = -1.$$

Since $p_m(t)$ is the distribution function of the first passage time to the state m, then it is an increasing function in t, this fact is clear also from table (4) of the last section. This means that:

$$A(t) = \sum_{i=2}^{m+1} C_i \alpha^{(i)}_{m+1} e^{w_i t}$$

is an increasing function. In fact we have:

$$A(0) = -1, \qquad \text{and} \qquad A(\infty) = 0.$$

Thus, for given λ, μ, m and a small positive value ϵ, we can find a value $\hat{t}$ for which

$$p_m(\hat{t}) \geqslant 1 - \epsilon.$$

This means that, if we observe the system for a time interval of length $\hat{t}$, then the probability that we get the true value of the number of machines is, at least, equal to $(1 - \epsilon)$.

To determine $\hat{t}$, we have to find the minimum value of t for which

$$-\epsilon \leqslant A(t) \leqslant 0.$$

The distribution of $\hat{m}_t$:

To investigate the distribution of $\hat{m}_t$ we note that the event $\hat{m}_t \geqslant k$ occurs if and only if, the event $\tau_k \leqslant t$ also occurs, $k = 1, 2, \ldots, m$.

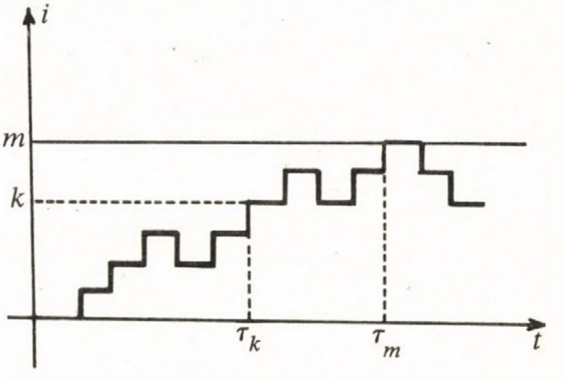

Figure 1.

Then:

$$p_k(t) = P(\tau_k \leqslant t) = P(\hat{m}_t \geqslant k),$$
$$p_{k+1}(t) = P(\tau_{k+1} \leqslant t) = P(\hat{m}_t \geqslant k+1).$$

So:

$$Q_k(t) = P(\hat{m}_t = k) = p_k(t) - p_{k+1}(t), \qquad 1 \leqslant k \leqslant m-1$$

while:

$$Q_m(t) = p_m(t) = p(\hat{m}_t = m)$$

as discussed before.

The quantities $p_k(t)$, $k = 1, 2, \ldots, m-1$ are again obtained by solving the system (1), assuming that k is an absorbing state. Thus:

$$p_k(t) = \sum_{i=1}^{k+1} C_i \alpha_{k+1}^{(i)} e^{w_i t}.$$

By the same argument as for $p_m(t)$, we have:

$$p_k(t) = 1 + \sum_{i=2}^{k+1} C_i \alpha_{k+1}^{(i)} e^{w_i t}$$

when $t \to \infty$, each $Q_k(t)$, $k = 1, 2, \ldots, m-1$ tends to zero, while:

$$Q_m(t) \to 1.$$

Also, we have:

$$\alpha_m(t) = E(\hat{m}_t) = \sum_{i=1}^{m} p_i(t)$$

and when $t \to \infty$, each $p_i(t) \to 1$, i.e.

(6) $$\lim_{t \to \infty} E(m_t) = m.$$

So, $\hat{m}_t$ is an asymptotically unbiased estimate for m.

For the variance of $\hat{m}_t$ we have:

$$\operatorname{Var}(\hat{m}_t) = E(\hat{m}_t - \alpha_m(t))^2 = \sum_{i=0}^{m-1} (i - \alpha_m(t))^2 [p_i(t) - p_{i+1}(t)] + (m - \alpha_m(t))^2 p_m(t)$$

i.e.

(7) $$\operatorname{Var}(\hat{m}_t) \to 0 \quad \text{as} \quad t \to \infty.$$

From (6) and (7), it follows again that the estimator m_t is consistent estimator for m (cf. [5], pp. 281).

Maximum likelihood estimate for m:

Assuming that we observe the breakdown times and repairtimes of the machines in a time interval of length T, let:

a_i = denotes the total number of transitions from the state i to the state $i+1$,

b_i = denotes the total number of transitions from the state $i+1$ to the state i,

γ_i = denotes the total time spent in the state i during the time of observation.

The log– likelihood function of this sample, by [2] pp. 50, is

(8) $$L_T(m) = \sum_{i=0}^{\hat{m}_T - 1} [a_i \ln((m-i)\lambda) - \gamma_i(m-i)\lambda] + \sum_{i=0}^{\hat{m}_T - 1} (b_i \ln \mu - \gamma_{i+1}\mu).$$

The maximum likelihood estimate for m is that value $\bar{m}$ which maximizes the function $L_T(m)$.

Thus we try to find an integer $\bar{m} > 0$ such that:

$$\Delta L_T(\bar{m}) < 0 < \Delta L_T(\bar{m} - 1)$$

where $\Delta L_T(m)$ is the first difference of the function $L_T(m)$ given by:

(9) $$\Delta L_T(m) = L_T(m+1) - L_T(m) = \sum_{i=0}^{\hat{m}_T - 1} [a_i \ln \frac{m+1-i}{m-i} - \gamma_i \lambda].$$

In other words, the maximum likelihood estimator for m is the first value $\bar{m}$ which satisfies:

$$\Delta L_T(\bar{m}) < 0,$$

i.e.

$$\sum_{i=0}^{\hat{m}_T - 1} [a_i \ln \frac{\bar{m} + 1 - i}{\bar{m} - i} - \gamma_i \lambda] < 0.$$

Since

$$\frac{a_i}{\gamma_i} \to \lambda_i = (m - i)\lambda \quad \text{as} \quad T \to \infty,$$

then, for large values of T we have:

$$\text{(9')} \qquad \Delta L_T(k) \simeq \sum_{i=0}^{\hat{m}_T - 1} \gamma_i \lambda [(m - i) \ln \frac{k + 1 - i}{k - i} - 1],$$

Now, it is quite easy to see that the right hand side takes its maximum at $k = m$ which is the true value of the number of machines, and that this maximum is unique.

To see this, one can show by elementary calculations that, for any $0 \leqslant i \leqslant k$,

$$(m - i) \ln \frac{k - i + 1}{k - i} - 1 \begin{cases} > 0 & \text{for} \quad k < m \\ < 0 & \text{for} \quad k \geqslant m. \end{cases}$$

From (9) and (9') we see that for $k \geqslant \hat{m}_T$ we have:

$$D_T^k = \frac{1}{T} \Delta L_T(k) - \sum_{i=1}^{\hat{m}_T - 1} \frac{\lambda \gamma_i}{T} [(m - i) \ln \frac{k + 1 - i}{k - i} - 1]$$

$$= \sum_{i=0}^{\hat{m}_T - 1} \frac{\gamma_i}{T} \left(\frac{a_i}{\gamma_i} - (m - i)\lambda \right) \ln \frac{k + 1 - i}{k - i}$$

$$\to 0, \quad \text{as} \quad T \to \infty,$$

because $\frac{a_i}{\gamma_i} \to (m - i)\lambda$ and $\frac{\gamma_i}{T} \to p_i^*$ as $T \to \infty$ where p_i^*'s are the stationary probabilities of finding the system in the state i.

This proves the consistency of the maximum likelihood estimation.

Numerical examples and illustrations:

With some modifications of the algorithm given in [4] to simulate the system M/M/m, we can get a simulating algorithm for our present case. Hundred samples are simulated and estimates for the number m of machines are founded. The following tables summarize some of these results for different values of the parameters.

Let $\hat{N}^*$ denotes the number of times we get the true value of the number of machines out from the 100 samples using the direct method, and $\bar{N}^*$ represents the same but using the maximum likelihood method of estimation.

In the tables we present both the values $\hat{N}^*$ and $\bar{N}^*$ in addition to the probability $Q_m(t)$ calculated from formula (5).

Table (1) gives the results for fixed λ, μ and m and different values of t.

T	$\bar{N}^*$	$\hat{N}^*$	$Q_m(t)$
20	47	38	0.41
50	77	76	0.77
100	91	94	0.95
200	98	100	0.99
500	100	100	1.00

$\lambda = 0.3, \ \mu = 1.0, \ m = 5$

Table 1.

Table 2. shows the effect of increasing the value of $\rho = \lambda/\mu$ on the results.

λ	μ	ρ	$\bar{N}*$	$\hat{N}*$	$Q_m(t)$
0.1	1.0	0.1	54	5	0.05
0.2	1.0	0.2	78	56	0.57
0.3	1.0	0.3	93	97	0.95
0.5	1.0	0.5	98	100	0.99
0.7	1.0	0.7	97	100	1.00
0.1	1.0	0.1	54	5	0.05
0.1	0.7	1/7	70	12	0.15
0.1	0.5	0.2	67	29	0.33
0.1	0.3	1/3	70	66	0.68
0.1	0.1	1.0	52	99	0.98

$T = 100,\ M = 5$

Table 2.

From this table, we see that, by increasing ρ we get better results. But, in fact, the accuracy of the estimators does not effect noticeably by increasing the value of m. To show this, we present in table 3., the values of $\bar{N}*$, $\hat{N}*$ and $Q_m(t)$ for the same values of λ, μ and T, but for $m = 8$, and also we give in table 4. the values of $\bar{N}*$ and $\hat{N}*$ for fixed λ, μ and T and increasing m.

λ	μ	ρ	$\bar{N}^*$	$\hat{N}^*$	$Q_m(t)$
0.1	1.0	0.1	42	0	0.01
0.2	1.0	0.2	77	38	0.40
0.3	1.0	0.3	86	97	0.92
0.5	1.0	0.5	95	100	0.99
0.7	1.0	0.7	99	100	1.00
0.1	1.0	0.1	42	0	0.01
0.1	0.7	1/7	50	8	0.06
0.1	0.5	0.2	57	24	0.21
0.1	0.3	1/3	71	56	0.61
0.1	0.1	1.0	59	100	0.97

$T = 100, \ M = 8$

Table 3.

m	$\bar{N}^*$	$\hat{N}^*$
3	96	100
5	90	92
7	91	98
10	81	89
15	90	89

$\lambda = 0.3, \ \mu = 1.0, \ T = 100$

Table 4.

From tables 2. and 3. we note that, although, the accuracy of the maximum likelihood estimator increases with increasing ρ, but when λ and μ have small values, ($\lambda = 0.1$, $\mu = 0.1$), the accuracy of the method is not so good as in the preceding cases with smaller values of ρ. This may be because of the decreasing of the number of transition for such small values of the parameters, which effect the accuracy of the method.

Table 5 shows that, when we increase λ and μ by the same ratio keeping ρ fixed, both the maximum likelihood method and the direct method give better results.

λ	μ	$\bar{N}^*$	$\hat{N}^*$	$Q_m(t)$
0.1	1.0	54	5	0.05
0.2	2.0	68	14	0.11
0.3	3.0	78	15	0.16
0.5	5.0	94	19	0.25
0.7	7.0	97	29	0.34
1.0	10.0	100	49	0.45
2.0	20.0	100	74	0.70
3.0	30.0	100	81	0.83
5.0	50.0	100	97	0.95
10.0	100.0	100	100	0.99

$\rho = 0.1$, $M = 5$, $T = 100$

Table 5.

To explain the effect of increasing λ and μ, by the same ratio keeping ρ fixed, on the accuracy of the direct method, we replace λ and μ by $R\lambda$ and $R\mu$ respectively. Thus we can easily prove that the eigenvalues are replaced by $Rw_1, Rw_2, \ldots, Rw_{m+1}$ while the components of the corresponding eigenvectors and also the values of the constants $C_1, C_2, \ldots, C_{m+1}$ do not effect. (See the proof in the appendix.)

This means that the probability $Q_m(t)$, in this case, takes the form

(10) $$Q_m(t) = 1 + \sum_{i=2}^{m+1} C_i \alpha_{m+1}^{(i)} e^{R w_i t}$$

and we have

$$Q_m(t) \to 1 \quad \text{as} \quad R \to \infty.$$

In fact, formula (10) shows that, increasing λ and μ by a ratio R is equivalent to increasing T by the same ratio R. We can insure this fact numerically, by comparing tables 1. and 6.

λ	μ	$\bar{N}*$	$\hat{N}*$	$Q_m(t)$
0.06	0.2	47	38	0.41
0.15	0.5	77	76	0.77
0.3	1.0	91	94	0.95
0.6	2.0	98	100	0.99
1.5	5.0	100	100	1.00

$\rho = 0.3$, $M = 5$, $T = 100$

Table 6.

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Appendix

In this appendix we prove that if we increase the arrival rate λ and the service rate μ by the same ratio R, keeping ρ fixed, then the eigenvalues of the matrix A increases by the same ratio R while the components of the corresponding eigenvectors and the constants $C_1, \ldots, C_{m+1}$ do not change.

To see this, let us write the characteristic equation in the form:

$$|A - wI| = wf_m(w) = 0$$

where generally:

$$f_i(w) = \begin{bmatrix} \lambda_0 + w & -\mu & & & & \\ -\lambda_0 & \lambda_1 + \mu + w & -\mu & & & \\ & -\lambda_1 & \lambda_2 + \mu + w & -\mu & & \\ & & \cdots\cdots\cdots & & & \\ & & & \cdots\cdots\cdots & & \\ & & & -\lambda_{i-3} & \lambda_{i-2} + \mu + w & -\mu \\ & & & & -\lambda_{i-2} & \lambda_{i-1} + \mu + w \end{bmatrix}.$$

Then it is easily seen that, the sequence $f_0(w), \ldots, f_m(w)$ of functions satisfies the recurrence relation:

(11) $$f_i(w) = (\lambda_{i-1} + \mu + w)f_{i-1}(w) - \lambda_{i-2}\mu f_{i-2}(w), \qquad 2 \leqslant i \leqslant m$$

with

$$f_0(w) = 1,$$
$$f_1(w) = w + \lambda_0, \quad \text{and}$$
$$f_{m+1}(w) = wf_m(w).$$

We know that one of the eigenvalues of the matrix A is equal to zero and the other m eigenvalues are the solution of the equation:

$$f_m(w) = 0.$$

Let us write

(12) $$f_i(w) = a_i w^i + a_{i-1} w^{i-1} + \ldots + a_1 w + a_0 = 0$$

where the coefficients $a_0, a_1, \ldots, a_i$ are functions of λ and μ.

Now we prove the following:

Lemma. *The coefficient* a_j *of* w^j *in* $f_i(w)$ *is a homogeneous function of order* $(i-j)$ *in* λ *and* μ.

$$j = 0, 1, \ldots, i, \qquad i = 0, i, 2, \ldots, m.$$

Proof. It is clear that the lemma is true for:

$$f_1(w) = w + \lambda_0, \quad \text{and}$$
$$f_2(w) = w^2 + (\lambda_0 + \lambda_1 + \mu)w + \lambda_0\lambda_1.$$

Let: $a_j^{(i)}$ denote the coefficient of w^j in $f_i(w)$. Thus using the recurrence relation (11) we have:

$$(13) \quad \begin{cases} a_j^{(i)} = (\lambda_{i-1} + \mu)a_j^{(i-1)} + a_{j-1}^{(i-1)} - \lambda_{i-2}\,\mu a_j^{(i-2)}, & 1 \leqslant j \leqslant i \quad \text{and} \\ a_0^{(i)} = f_i(0) = \lambda_0\lambda_1 \ldots \lambda_{i-1}. \end{cases}$$

Thus by mathematical induction and using (13) it is possible to see that the lemma is true.

Now, replacing λ and μ by $R\lambda$ and $R\mu$ respectively, equation (12) for $i = m$ takes the form

$$(14) \qquad a_m w^m + a_{m-1}Rw^{m-1} + a_{m-2}R^2w^{m-2} + \ldots + a_1R^{m-1}w + a_0R^m = 0.$$

Thus, if the roots of (12) are $w_1, w_2, \ldots, w_m$ then the roots of (14) are Rw_1 $Rw_1, Rw_2, \ldots, Rw_m$ as required.

Also, since the components of the eigenvector corresponding to any eigenvalue w^* are determined by:

$$\alpha_1^* = 1$$

$$\alpha_2^* = \frac{\lambda_0 + w^*}{\mu}\alpha_1^*,$$

$$\alpha_k^* = \frac{\lambda_{k-2} + \mu + w^*}{\mu}\alpha_{k-1}^* - \frac{\lambda_{k-3}}{\mu}\alpha_{k-2}^*, \qquad 3 \leqslant k \leqslant m,$$

$$\alpha_{m+1}^* = \frac{\lambda_{m-1}}{w^*}\alpha_m^*.$$

Then, it is clear that the values of these components and consequentely the values of the constants $C_1, C_2, \ldots, C_{m+1}$. are not effected by the replacement of λ and μ by $R\lambda$ and $R\mu$.

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Резюме

Рассматриваются два метода оценивания числа машин выстанавливаемых одним рабочим. Длительность работы и время ремонта предполагается иметь экспоненциальное распределение с параметрами λ и μ соответственно.

Определяется распределение максимума в интервале $(0, \tau)$ числа ломанных машин.

Приведены численные примеры полученные стохастическим моделированием для сравнения рассматриваемых двух методов.