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ON ABSOLUTE CONTINUITY OF MEASURES DEFINED BY MULTIDIMENSIONAL DIFFUSION PROCESSES WITH RESPECT TO THE WIENER MEASURE

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In [5] Lipcer and Shiryaev dealt with the absolute continuity of measures generated by a diffusion type process and the Wiener process for one dimensional case. In this paper an attempt is made to carry out their result for multidimensional case. We summarize the result in tree theorems based on each other and we discuss; in details, a lemma the proof of which, for multidimensional case, needs considerations different from those used in [5]. Before this we give a concise list of preliminaries.

Let (Ω, F, P) be the basic probability space, $\{F_t \subset F,\ 0 \leq t \leq 1\}$ a monotonically nondecreasing family of σ-algebras, $w = (w_t, F_t, P)$ an n-dimensional standard Wiener process, i.e. it is an n-dimensional continuous martingal with respect to the family F_t, such that $w_0 = 0$ a.s. and

$$E[(w_t^i - w_s^i)(w_t^j - w_s^j) \mid F_s] = \delta^{ij}(t-s) \qquad t \geq s \text{ a.e. } i, j = 1, 2, \ldots, n$$

Let C_1 denote the space of the n-dimensional vector valued continuous functions x_t on $[0, 1]$ and B_t the σ-algebra generated by cylindric sets on $[0, t]$. Further, let $\alpha_t(x)$ be a $B_{[0,1]} x B_1$, $B_{[0,1]}$ is the σ-algebra of Borel sets of interval $[0, 1]$, measurable n dimensional nonanticipating functional, i.e. $\alpha_t(x)$ B_t measurable for every $0 \leq t \leq 1$.

Let

$$|\alpha_t(x)| = \sqrt{\sum_{i=1}^{n} (\alpha_t^i(x))^2}\ .$$

The n dimensional (ξ_t, F_t) process is called a process of diffusion type if there exists a nonanticipating measurable functional such that

$$P\left(\int_0^1 |\alpha_t(\xi)|\, dt < \infty\right) = 1$$

$(\alpha_t(\xi) = \alpha_t(\xi(\omega)),\ \ \xi(\omega) = \{\xi_t,\ 0 \leq t \leq 1\ )$ and $\xi_t = \int_0^t \alpha_s(\xi) ds + w_t$ a.s. for any $0 \leq t \leq 1$.

Denote by $\mu_\xi(\mu_w)$ the measure on the space (C_1, B_1) generated by the process

$$\xi(\omega) = \{\xi_t,\ 0 \leq t \leq 1\} \qquad (w(\omega) = \{w_t,\ 0 \leq t \leq 1\})$$

i.e. for $B \in B_1$

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$$\mu_\xi(B) = P\{\omega : \xi(\omega) \in B\} \qquad (\mu_w(B) = P\{\omega : w(\omega) \in B\}).$$

Let (γ_t, F_t) be an n-dimensional stochastic process satisfying the condition

(1) $$P\left(\int_0^1 |\gamma_t|^2 dt < \infty\right) = 1$$

and such that $\zeta_t = 1 + \int_0^t \gamma_s dw_s = 1 + \sum_{i=1}^n \int_0^t \gamma_s^i dw_s^i$ is a nonnegative martingal with respect to (F_t, P). Introduce now a new measure $\widetilde{P}$ on the measurable space (Ω, F) by the formula

(2) $$\widetilde{P}(d\omega) = \zeta_1 P(d\omega)$$

Theorem 1. *Let* $\xi_t = -\int_0^t \frac{\gamma_s}{\zeta_s} ds + w_t$, $0 \leq t \leq 1$. *Under conditions (1) and (2)* $(\xi_t, F_t, \widetilde{P})$ *is a* n*-dimensional standard Wiener process.*

Girsanov proved, see [4], this statement for a more particular case considering

$$\zeta_t = \exp\left\{\int_0^t \gamma_n dw_n - \frac{1}{2}\int_0^t |\gamma_n|^2 dn\right\}$$

Lipcer and Shiryaev, in [5], dealt with general ζ_t but for one–dimensional case. Their proof is essentially simpler than that of Girsanov. Concerning the multidimensional case one can quite easily observe that, upon replacing the ordinary scalar products by scalar products of vectors, Lipcer's and Shiryaev's arguments remain valid.

The next two theorems deal with the absolute continuity of measures generated by the process ζ_t and w_t.

Theorem 2. *Let* ξ_t *be a diffusion type process satisfying the equation*

$$\xi_t = \int_0^t \alpha_s(\xi) ds + w_t, \qquad \xi_0 = 0$$

and suppose that

(3) $$P\left(\int_0^1 |\alpha_t(w)|^2 dt < \infty\right) = 1$$

The measure μ_w *is absolutely continuous with respect to* μ_ξ $(\mu_w \ll \mu_\xi)$ *and*

$$\frac{d\mu_w}{d\mu_\xi}(w) = \exp\left\{-\int_0^1 \alpha_t(w) dw_t + \frac{1}{2}\int_0^1 |\alpha_t(w)|^2 dt\right\}, P \text{ a.s.}$$

Under the condition $F_t^\xi = F_t^w$ *(3) is necessary for the absolute continuity.*
If x_t is not a Wiener trajectory, i.e. the stochastic integral in the exponent has no meaning, the Randon–Nikodym derivative equals to 0. This remark can be correctly explained by the notion of generalized to integral introduced by Lipcer and Shiryaev in one–dimensional case.

Their considerations remains valid without any change for multidimensional case as well. An analogous theorem holds for $\frac{d\mu_\xi}{d\mu_w}$:

Theorem 3. *Let* ξ_t *be the same process as in theorem 2. The condition*

$$P\left(\int_0^1 |\alpha_t(\xi)|^2 dt < \infty\right) = 1$$

is necessary and sufficient for the absolute continuity of μ_ξ *with respect to* μ_w, *and*

$$\frac{d\mu_\xi}{d\mu_w}(\xi) = \exp\left\{\int_0^1 \alpha_t(\xi)d\xi_t - \frac{1}{2}\int_0^1 |\alpha_t(\xi)|^2 dt\right\}.$$

Concerning the meaning of this formula we should make again an analogous remark. The proofs of the sufficiency of condition in theorem 2. and 3. do not require any changes in Lipcer's and Shiryaev's proof but for proving its necessity we have to generalize a lemma used by them.

Lemma 1. *Let* (Ω, F, P) *be a probability space, and* $w = (w_t, F_t, P)$ *be a standard n-dimensional Wiener process. If* $\zeta(\omega)$ *is a* F_1^w *measurable random variable (one–dimensional) with* $E|\zeta(\omega)| < \infty$ *then there exists a measurable nonanticipating* $\gamma_t(w)$ *vector functional, such that* $\int_0^1 |\gamma_t(w)|^2 dt < \infty$ *with probability 1 and for the martingale* $\zeta_t = E(\zeta(\omega)|F_t^w)$ *for every* $t \geqslant s$ *P almost everywhere*

$$\zeta_t - \zeta_s = \sum_{i=1}^{n} \int_s^t \gamma_n^i(w)dw_n^i \,. \tag{4}$$

This result is due to Clark [1] for one–dimensional case and to Kunita and Watanabe [2] for multidimensional case, but under a bit stronger condition.

Proof. As the σ–algebra F_t^w is continuous, the martingale ζ_t is P a.e. continuous (see [3]).
Set $\tau_N = \inf_{0 \leqslant t \leqslant 1} \{t : |\zeta_t| = N\}$, $\tau_N \cap t = \min(t, \tau_N)$ and put $\zeta_N(t) = \zeta_{\tau_N \cap t}$. τ_N is obviously a Markov–point and so $\zeta_N(t)$ is a martingale

$$\sup_{0 \leqslant t \leqslant 1} |\zeta_N(t)| \leqslant N, \quad P \text{ a.e.}$$

The process $\zeta_N(t)$ is continuous and square integrable, therefore according to [2] it can be represented in the form

$$\zeta_N(t) = \sum_{i=1}^{n} \int_0^t \gamma_N^i(s, w)dw_s^i$$

where $\gamma_N^i(s, w)$, $i = 1, \ldots, n$ a square integrable F_t^w measurable for every t.

On the set

$$\chi_N(t) = \{\omega : \sup_{0 \leq s \leq t} |\zeta_s| \leq N\} \quad \text{for} \quad M > N$$

we have

$$\zeta_N(s) = \zeta_M(s) \qquad 0 \leq s \leq t \ \text{ a.e., i.e.}$$

$$\int_0^t \chi_N(s)(\zeta_N(s) - \zeta_M(s))^2 ds = \int_0^{\tau_N} (\zeta_N(s) - \zeta_M(s))^2 ds = \int_0^{\tau_N} (\zeta_s - \zeta_s)^2 ds = 0.$$

Define for every $1 \leq i \leq n$ the functional $\gamma^i(t, w)$ by $\gamma_1^i(t, w)$ on the set $\{\omega : 0 \leq \sup_{0 \leq s \leq t} |\zeta_s| \leq 1\}$ and by $\gamma_2^i(t, w)$ on the set $\{\omega : 1 \leq \sup_{0 \leq s \leq t} |\zeta_s| \leq 2\}, \ldots$ and so on. $\gamma^i(t, w)$ is, for every i obviously measurable process and for any fixed t is F_t^w measurable. Moreover

$$\left\{\omega : \sum_{i=1}^n \int_0^1 (\gamma^i(t, w))^2 dt = \infty\right\} \subset \left\{\omega : \int_0^1 \sum_{i=1}^n (\gamma^i(t, \omega) - \gamma_N^i(t, w))^2 dt > 0\right\}$$

$$\subset \{\omega : \sup_{0 \leq s \leq t} |\zeta_s| > N\}$$

The probability of the last set tends to zero as $N \to \infty$, so $\sum_{i=1}^n \int_0^1 (\gamma^i(t, w))^2 dt < \infty$ P a.e.

Thus the to integral $\int_0^t \gamma^i(s, w) dw_s$ can be correctly defined for every t. By the virtue of a well known property of to integral

$$P\left\{\left|\int_0^t (\gamma_N(s, w) - \gamma(s, w)) dw_s\right| > 0\right\} \leq P\left\{\int_0^t |\gamma_N(s, w) - \gamma(s, w)| ds > 0\right\} \to 0$$

as $N \to \infty$.

From this it follows that $\zeta_N(t)$ stochastically converges to $\int_0^t \gamma(s, w) dw_s$. Since $\lim_{N \to \infty} \zeta_N(t) = \zeta_t$ in probability so that

$$\zeta_t = \int_0^t \gamma(s, w) dw_s.$$

The uniqueness of the representation can be easily proved:

Let $\gamma_1(t, w)$, $\gamma_2(t, w)$ be functionals for which the representation (4) holds. Then for any $0 \leq t \leq 1$ applying to formula to

$$\eta_t^2 = \left(\int_0^t (\gamma_1(s, w) - \gamma_2(s, w)) dw_s\right)^2$$

we have

$$0 = \eta_t^2 = \sum_{i=1}^n \int_0^t 2 \cdot \eta_s dw_s^i + \sum_{i=1}^n \int_0^t (\gamma_1^i(s, w) - \gamma_2^i(s, w))^2 ds.$$

Therefore $\gamma_1(s, w) = \gamma_2(s, w)$ for every $0 \leq t \leq 1$ with probability 1.

References

[1] Clark, J. M. C., "The representation of functionals of Brownian motion by stochastic integrals" AMS 41(1970) 4, 1282–1295.

[2] Kunita, H. and Watanabe, S., "On square integrable martingales" Nagoya Math. J. 30(1967) 209–245.

[3] Neveu, J., Bases mathématiques du calcul des probabilités (Masson et Cie, Paris, 1964).

[4] Гирсанов, И.В., "О преобразовании одного класса случайных процессов с помощью абсолютно-непрерывной замены меры" Теория вероятностей и ее применен. 5 (1960) 314-330.

[5] Липцер, Р.Ш. и Ширяев, А.Н., "Об абсолютной непрерывности мер, соответствующих процессам диффузионного типа, относительно винеровской" Изд. АН СССР Сер. матем. 36 (1972) 847-889.

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Об абсолютной непрерывности мер, соответствующих n мерным процессам диффузионного типа, относительно винеровской

В настоящей работе рассматривается абсолютная непрерывность мер (относительно винеровской меры) соответствующих многомерным процессам диффузионного типа. Обобщаются результаты Липцера и Ширяева на многомерный случай.