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ON THE ESTIMATION OF REGRESSION COEFFICIENTS IN CASE OF AN AUTOREGRESSIVE NOISE PROCESS

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INTRODUCTION

In statistical time series analysis one of the most frequently discussed problem has the following formulation: a time series on the form

$$y(t) = m(t) + x(t), \quad t = 1, 2, \dots, N$$

is observed, where $m(t)$ is an unknown deterministic function and $x(t)$ is a stochastic process with 0 mean and known spectrum. The purpose is to draw some conclusions for $m(t)$ from the observed process $y(t)$. In the practice we seek the function $m(t)$ in the form

$$m(t) = \sum_{\nu=1}^{k} a_\nu \varphi^{(\nu)}(t),$$

where a_ν are unknown coefficients and $\varphi^{(\nu)}(t)$ are known functions (usually polinomials or trigonometric polinomials). We have to estimate the coefficients a_ν. The most natural way is the method of least squares.

With the following notations

$$\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_k \end{pmatrix}, \qquad y = \begin{pmatrix} y(1) \\ y(2) \\ \cdot \\ \cdot \\ \cdot \\ y(N) \end{pmatrix}, \qquad \varphi^{(j)} = \begin{pmatrix} \varphi^{(j)}(1) \\ \varphi^{(j)}(2) \\ \cdot \\ \cdot \\ \cdot \\ \varphi^{(j)}(N) \end{pmatrix}$$

and

$$\Phi = (\varphi^{(1)}, \varphi^{(2)}, \dots, \varphi^{(k)})$$

the least square estimator $\hat{\alpha}$ of the vector α takes the form

$$\hat{\alpha} = (\Phi^* \Phi)^{-1} \Phi^* y.$$

In the case of normal white noise estimator $\hat{\alpha}$ coincides with the maximum likelihood estimator of the vector α. If we suppose, that the noise process $x(t)$ is normal, but not white and it has known correlation matrix R, we have the maximum likelihood estimator α_0 of the vector α in the form

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$$\alpha_0 = (\Phi^* R^{-1}\Phi)^{-1}\Phi^* R^{-1} y.$$

It is well known, that α_0 has minimal dispersion among the linear unbiased estimators of α. From the point of wiew of computational technics the inversion of the matrix R for enormously large N is a difficult problem. Both the estimators $\hat{\alpha}$ and α_0 are normally distributed (as linear combinations of Gaussian variables) with expectations and variances

$$E\hat{\alpha} = (\Phi^* \Phi)^{-1}\Phi^* Ey = (\Phi^* \Phi)^{-1}\Phi^* \Phi\alpha = \alpha$$

$$E(\hat{\alpha} - \alpha)(\hat{\alpha} - \alpha)^* = (\Phi^* \Phi)^{-1}\Phi^* R\Phi(\Phi^* \Phi)^{-1}$$

$$E\alpha_0 = (\Phi^* R^{-1}\Phi)^{-1}\Phi^* R^{-1}\Phi\alpha = \alpha$$

$$E(\alpha_0 - \alpha)(\alpha_0 - \alpha)^* = (\Phi^* R^{-1}\Phi)^{-1}.$$

In this work we investigate the problem of the distribution of the estimators by the method of computer simulation. The question is, how they depend on simple parameters as damping and hidden periodicity.

1. Let us regard the process

$$y(t) = a \cos \omega t + x(t)$$

where the frequency ω is a given constant, a is the unknown parameter and $x(t)$ is a discrete time parameter second order autoregressive process i.e. $x(t)$ satisfies the difference equation

$$x(t) = \alpha x(t-1) + \beta x(t-2) + \epsilon(t).$$

The coefficients α and β are known real numbers satisfying the condition $\alpha^2 + 4\beta < 0$, the process $\epsilon(t)$ is a standard discrete time parameter white noise. The "period" of this scheme is $2\pi/\omega_1$, where

$$\omega_1 = \arccos \frac{|\alpha_1|}{2\sqrt{-\alpha_2}}.$$

On this example we can investigate another curious problem of the time series analysis, namely the distinction of a process with periodic mean value function from a process with hidden periodicity. We summarise the results of our computer simulation experiments about the statistical behaviour of the least square estimator $\hat{\alpha}$ and the maximum likelihood estimator a_M of the unknown parameter in tabular form, when the damping parameter λ and the hidden frequency ω_1 of the process $x(t)$ where varied.

The least square estimator $\hat{a}$ has the form

$$\hat{a} = \frac{\sum_{t=1}^{N} y(t) \cos \omega t}{\sum_{t=1}^{N} \cos^2 \omega t}$$

(The estimator $\hat{a}$ is the maximum likelihood estimator of a under the false hypothesis that the noise is white.)

The maximum likelihood estimation a_M can be calculated from the conditional density function

$$f = \frac{1}{(2\pi)^{N/2}} \exp\left\{-\frac{1}{2}\sum (x(t) - \alpha x(t-1) - \beta x(t-2))^2\right\}$$

of the process $x(t) = y(t) - a\cos\omega t,\ t = 1, 2, \ldots, N,$ under the condition that $x(0) = x_0$. The solution of the likelihood equation

$$\frac{d\ln f}{da} = 0$$

can be written in the form

$$a_M = \frac{B}{A}$$

where

$$A = \sum_t \{\cos^2\omega t + \alpha^2\cos^2\omega(t-1) + \beta^2\cos^2\omega(t-2)$$
$$- 2\alpha\cos\omega t\cos\omega(t-1) - 2\beta\cos\omega t\cos\omega(t-2)$$
$$+ 2\alpha\beta\cos\omega(t-1)\cos\omega(t-2)\}$$

and

$$B = \sum_t \{y(t)\cos\omega t + \alpha^2 y(t-1)\cos\omega(t-1) +$$
$$+ \beta y(t-2)\cos\omega(t-2) - \alpha y(t-1)\cos\omega t -$$
$$- \alpha y(t)\cos\omega(t-1) - \beta y(t-2)\cos\omega t -$$
$$- \beta y(t)\cos\omega(t-2) + \alpha\beta y(t-2)\cos\omega(t-1)$$
$$+ \alpha\beta y(t-1)\cos\omega(t-2)\}\ .$$

2. In our concrete example the parameters were chosen as follows:

$$a = 6.8,\quad \alpha = 1.83,\quad \beta = -0.98,\quad \omega = \frac{2\pi}{10}.$$

So the period of the noise is 16.04 and the damping parameter is small ($\sqrt{0.98}$). Table 1. shows the dependence of estimators on the number of observations. In the first column we can find the numbers of observations, in the second column the type of the estimator, in the 2.nd–6.th columns the 0.05, 0.1, 0.2, 0.8, 0.9 and 0.95 respectively, quantiles of estimators (calculated from 200–500 samples), and the last two columns contain the mean value and the dispersion of the estimators.

We get similar results in every case, when the period of $m(t)$ and the hidden period of $x(t)$ are far from each other (e.g. $\omega \leq 2\pi/40$ or $\omega \geq 2\pi/12$). The two estimators differ essentially in the case of small number of observations, while for $N > 300$ they almost coincide.

Table 2. shows the dependence of estimator on the frequency ω – in this case $N = 40$. The construction of table 2. is similar to the first one.
When the frequency of $m(t)$ is equal to the hidden frequency of $x(t)$ $(2\pi/\omega = 16)$ the signal and the noise cannot be separated. In this case the least square estimator is better than the maximum likelihood one – in the sequel we return to this phenomenon. For large ω both estimators are better: there are more waves on the interval of observations. To avoid this effect we investigated the behaviour of estimators on the intervals the length of which is 2 or 1 waves.

These results are contained in tables 3. and 4.

The least square estimation gives very bad results for $T \leq 5$, while the maximum likelihood estimator becomes continuously better as the distance between the frequencies of the noise and the signal grows.

Figure 1. shows the dependence of dispersions of the two variant of estimators on ω in the neighbourhood of the frequency of the noise, observing 2 waves.

Experiments were made to determine, how the damping influences the statistical behaviour of the estimates. In a natural way, when the damping grows, the distance between the two estimates decreases. If $\alpha = 1.488$ and $\beta = -0.64$ (then the frequency of the noise coincides with the previous, and the damping equals 0.8) the two estimates are not essentially different.

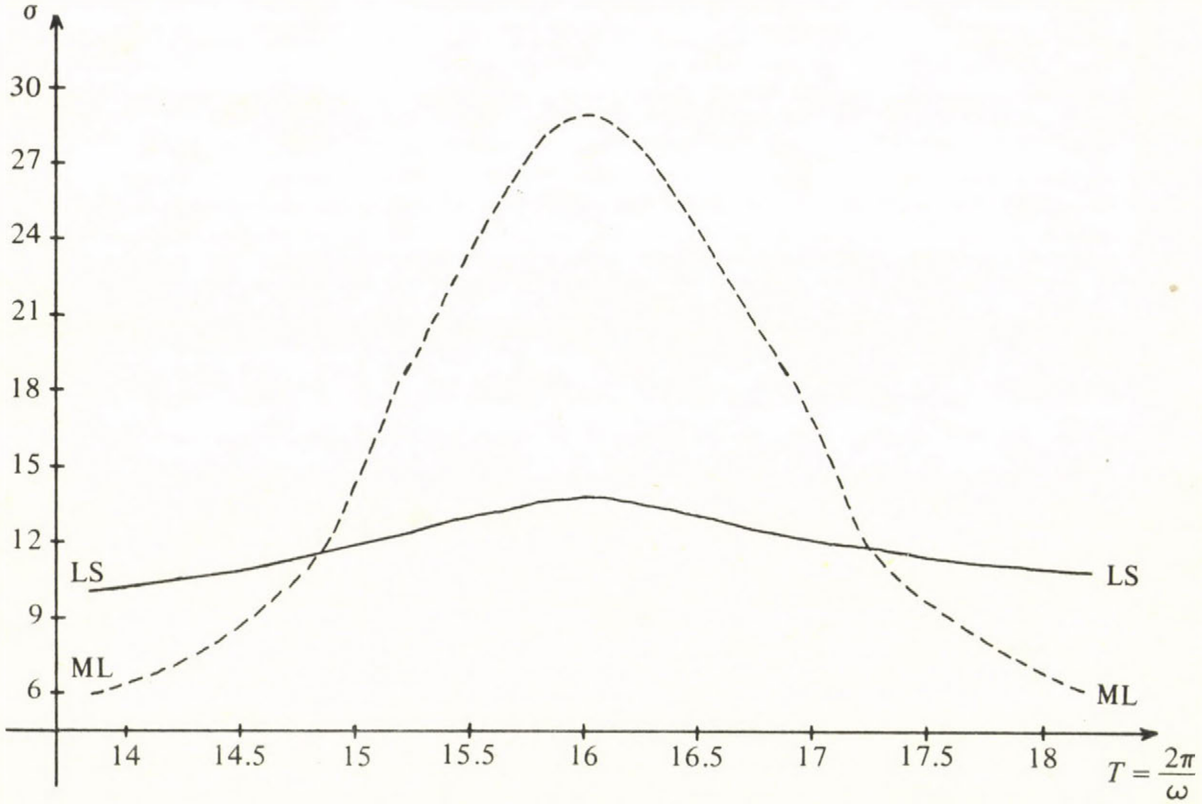
So far we have supposed that the noise was a second order autoregressive process with known parameters. By simulation we examined the behaviour of the estimators in the case if the noise is a higher 4–5 order autoregressive process and we use a second order approximation for the maximum likelihood estimation, the so called R-estimators (see Holevo [2]).

N	Type of estim.	0.05	0.1	0.2	0.8	0.9	0.95	m	σ
5	ML LS	– 0.82 – 8.73	– 0.07 – 4.12	1.97 0.35	9.99 13.76	11.81 16.67	13.61 19.94	6.12 6.82	4.47 8.86
10	ML LS	3.15 – 5.80	3.84 – 2.49	4.84 – 0.06	8.77 13.84	9.95 16.10	10.81 18.48	6.85 6.78	2.34 7.65
15	ML LS	3.89 – 2.71	4.43 – 0.87	5.17 2.02	8.20 12.07	8.96 14.57	9.73 17.14	6.78 7.09	1.75 5.80
20	ML LS	3.99 0.68	5.03 2.02	5.70 3.94	7.87 9.60	8.38 11.14	8.99 12.10	6.71 6.79	1.42 3.54
30	ML LS	5.03 3.31	5.33 4.27	5.77 5.20	7.64 8.86	8.25 9.98	8.62 10.40	6.73 6.73	1.17 2.22
40	ML LS	5.04 3.90	5.41 4.26	5.92 5.19	7.64 8.16	8.20 9.10	8.59 9.82	6.89 6.90	1.10 1.75
50	ML LS	5.09 4.38	5.44 4.96	5.94 5.58	7.54 7.93	7.86 8.47	8.13 9.02	6.72 6.72	0.97 1.38
60	ML LS	5.38 4.95	5.68 5.23	6.11 5.60	7.49 7.86	7.83 8.43	8.01 8.84	6.78 6.80	0.86 1.28
80	ML LS	5.62 5.83	5.92 5.72	6.20 6.05	7.48 7.59	7.83 8.06	7.91 8.28	6.87 6.86	0.73 0.92
100	ML LS	5.66 5.36	5.89 5.61	6.19 6.01	7.37 7.48	7.59 7.83	7.88 8.22	6.74 6.73	0.65 0.88
200	ML LS	6.05 5.86	6.19 6.07	6.42 6.29	6.97 7.12	7.19 7.54	7.39 7.81	6.76 6.74	0.40 0.58
300	ML LS	6.15 6.06	6.22 6.18	6.44 6.36	7.00 7.10	7.20 7.26	7.35 7.44	6.76 6.75	0.35 0.41

Table 1.

$T = \frac{2\pi}{\omega}$	Type of estim.	0.01	0.1	0.2	0.8	0.9	0.95	m	σ
40	ML LS	2.94 – 0.09	4.14 2.04	5.25 3.97	8.20 9.59	9.56 11.60	9.79 13.04	6.76 6.74	1.90 3.84
30	ML LS	2.45 0.66	3.54 2.58	4.40 3.74	8.99 9.07	9.95 10.84	10.33 11.65	6.62 6.63	2.52 3.33
20	ML LS	0.13 – 9.32	1.56 – 5.81	2.98 – 1.13	10.15 14.39	11.73 18.03	13.28 22.30	6.47 6.53	4.22 9.15
18	ML LS	– 4.18 – 11.43	– 2.71 – 7.08	0.36 – 1.46	12.32 15.17	16.50 21.04	18.46 24.45	6.78 6.93	7.49 10.66
16	ML LS	– 42.39 – 14.62	– 35.53 – 9.44	– 18.34 – 4.23	29.71 19.11	39.62 24.52	52.11 27.16	4.26 6.76	28.28 13.01
14	ML LS	– 1.59 – 7.07	0.09 – 4.24	1.85 – 0.17	11.26 13.58	13.64 17.60	15.28 21.79	6.98 7.18	5.35 8.35
12	ML LS	3.13 – 0.73	3.77 2.10	4.70 4.00	8.55 9.79	9.51 11.35	10.31 12.86	6.73 6.57	2.16 3.95
10	ML LS	5.00 2.97	5.48 3.99	5.82 4.95	7.71 8.93	8.07 9.78	8.47 10.29	6.81 6.83	1.10 2.25
8	ML LS	5.87 4.52	6.06 5.13	6.37 5.70	7.30 7.85	7.48 8.50	7.77 8.85	6.81 6.82	0.57 1.29
5	ML LS	6.48 5.76	6.53 5.93	6.65 6.17	6.94 7.35	7.06 7.61	7.14 7.86	6.80 6.79	0.20 0.65

Table 2.


σ
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T = 2π/ω

Figure 1.

$T = \frac{2\pi}{\omega}$	Type of estim.	0.05	0.1	0.2	0.8	0.9	0.95	m	σ
50	ML	5.16	5.54	5.90	7.67	8.08	8.61	6.76	1.06
	LS	4.19	4.65	5.34	8.09	8.77	9.28	6.74	1.56
40	ML	4.74	5.10	5.67	7.86	8.58	8.93	6.83	1.29
	LS	4.06	4.73	5.44	8.25	9.10	9.54	6.88	1.67
30	ML	4.09	4.58	5.31	8.35	9.00	9.73	6.75	1.77
	LS	1.73	2.77	4.14	9.13	10.41	11.66	6.73	2.91
20	ML	0.13	1.56	2.98	10.15	11.73	13.28	6.73	4.22
	LS	– 9.32	– 5.81	– 1.13	14.39	18.09	22.30	6.62	9.15
18	ML	– 9.74	– 3.58	0.29	13.24	16.76	19.32	6.25	8.21
	LS	– 10.92	– 7.27	– 2.10	14.37	20.56	26.40	6.18	10.56
16	ML	– 48.55	– 37.01	– 22.16	35.52	45.05	57.99	5.31	32.45
	LS	– 14.78	– 11.68	– 3.87	15.71	19.60	24.26	5.22	11.95
14	ML	– 4.25	– 1.98	0.59	12.18	15.25	17.14	6.46	6.48
	LS	– 12.09	– 7.98	– 2.91	15.69	20.59	23.46	6.47	10.87
12	ML	1.85	3.41	4.78	8.91	9.82	10.41	6.69	2.54
	LS	– 5.13	– 2.32	1.05	12.56	15.56	17.54	6.72	6.98
10	ML	3.99	5.03	5.70	7.87	8.38	8.99	6.71	1.42
	LS	0.68	2.02	3.94	9.60	11.14	12.10	6.79	3.54
8	ML	5.14	5.40	5.90	7.50	7.76	8.05	6.70	0.89
	LS	4.59	5.22	5.55	7.84	8.31	8.74	6.76	1.25
6	ML	5.93	6.20	6.39	7.26	7.46	7.59	6.83	0.51
	LS	3.55	4.14	5.08	8.55	9.52	9.94	6.80	2.06
5	ML	6.11	6.22	6.35	7.12	7.26	7.47	6.78	0.44
	LS	2.74	3.52	4.48	18.75	9.96	10.70	6.78	2.39
4	ML	6.25	6.36	6.49	7.05	7.18	7.29	6.79	0.32
	LS	1.51	2.10	3.84	9.62	11.25	12.10	6.77	3.22
3	ML	6.38	6.52	6.61	7.02	7.14	7.22	6.82	0.24
	LS	– 0.62	1.26	3.72	9.69	12.11	13.53	6.77	4.24

Table 3.

$T = \frac{2\pi}{\omega}$	Type of estim.	0.05	0.1	0.2	0.8	0.9	0.95	m	σ
50	ML LS	4.35 3.54	5.04 4.25	5.55 5.49	8.16 8.37	8.72 9.24	9.38 10.04	6.85 6.84	1.49 1.94
40	ML LS	2.94 – 0.09	4.14 2.04	5.25 3.97	8.20 9.59	9.56 11.60	9.79 13.04	6.76 6.74	1.90 3.84
30	ML LS	2.39 0.93	3.27 2.22	4.86 3.29	9.06 10.32	10.46 11.95	11.44 13.64	7.00 7.08	2.75 3.88
20	ML LS	– 3.79 – 14.39	– 1.63 – 11.13	0.67 – 5.25	13.52 18.63	15.28 24.08	16.42 26.77	7.00 6.76	6.44 13.29
18	ML LS	– 11.57 – 12.28	– 8.80 – 9.70	– 4.30 – 3.96	15.48 16.01	20.34 22.80	22.65 29.85	6.34 6.22	11.23 12.24
16	ML LS	– 73.55 – 17.87	– 59.34 – 12.91	– 37.95 – 8.60	42.88 13.37	62.07 17.23	87.86 19.96	2.89 3.12	48.04 12.15
14	ML LS	– 9.21 – 11.99	– 5.14 – 9.56	– 1.95 – 4.22	14.00 17.79	17.86 21.75	20.81 25.25	6.22 6.32	9.00 11.57
12	ML LS	– 1.03 – 10.99	0.83 – 6.88	3.01 – 2.75	10.27 15.04	12.54 20.12	13.65 25.65	6.70 6.68	4.39 10.78
10	ML LS	3.15 – 5.8	3.84 – 2.49	4.84 – 0.06	8.77 13.84	9.95 16.10	10.81 18.48	6.85 6.78	2.34 7.65
8	ML LS	4.29 – 3.36	4.64 – 1.66	5.44 1.46	7.74 11.92	8.52 13.93	9.25 15.57	6.70 6.64	1.47 5.66
6	ML LS	5.33 – 1.90	5.68 – 0.27	5.95 2.28	7.63 10.55	8.06 13.28	8.44 15.99	6.81 6.82	0.96 5.46
5	ML LS	5.72 – 4.13	5.98 – 1.28	6.26 1.23	7.42 12.46	7.84 14.99	8.03 16.93	6.84 6.83	0.71 6.37
4	ML LS	5.74 – 0.77	6.04 1.34	6.29 2.95	7.27 10.44	7.49 11.75	7.71 14.16	6.77 6.77	0.58 4.37
3	ML LS	5.67 0.95	5.83 2.63	6.24 4.08	7.34 9.29	7.71 11.03	7.88 11.96	6.80 6.79	0.69 3.35

Table 4.

References

[1] Grenander, U. and Rosenblatt, M., Statistical analysis of stacionary time series (John Wiley, New York, 1957).

[2] Холево, А.С., "Об оценках коэффициентов регрессии" Теория вероятностей XIУ (I969) I.

Р е з ю м е

Об оценке параметра регрессии, когда процесс шума является процессом авто-регрессии

В настоящей работе рассматривается процесс $y(t) = a\cos\omega t + x(t)$, где ω-данная константа, a - неизвестный параметр и $x(t)$ удовлетворяет стохастическому разностному уравнению

$$x(t) = \alpha x(t-1) + \beta x(t-2) + \epsilon(t).$$

Постоянные α и β удовлетворяют условию $\alpha^2 + 4\beta < 0$, $x(t)$ предполагается стационарным и $\epsilon(t)$ является стандартным белым шумом с дискретным временем.

Методом статических испытаний исследуется поведение разных типов оценок.

Результаты показывают, что при малом числе наблюдений зачитывая специальную форму получаются лучшие оценки параметра a (доверительное множество уже). В случае большого выбора оценки таким образом не станут лучшими.

Если период сигнала и скрытый период шума близки друг к другу, то тогда нельзя отделить сигнал от шума.