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ON AN ESTIMATE FOR THE PARAMETER OF A MULTIDIMENSIONAL STATIONARY GAUSSIAN MARKOV PROCESS, AND AN APPLICATION

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INTRODUCTION

In the paper estimates and confidence limits for the parameter of a multidimensional stationary Gaussian Markov process are considered. The estimate of the coefficient–matrix of a multidimensional stationary Gaussian Markov process, under certain weak conditions, may be reduced to the estimate for the parameter of a onedimensional real (respectively complex) stationary Gaussian Markov process.

These latter estimates and the distribution of the estimates are given in [4] and in [5].

Through the research of a geophysical problem (the axis of instantaneous rotation of the Earth) the authors compare the effeciency of the methods of the various estimates for the parameter.

Computer realizations are given for the various parameter estimation procedures, and their application to the above mentioned geophysical problem.

The estimation of the parameters and the determination of the confidence limits of one–dimensional continuous Gaussian Markov processes can be found in the papers of Arató [4] and Arató–Benczúr [5].

Employing the results of these papers we give results for similar problems in the multi-dimensional case as well for continuous as for discrete processes.

The results are valid under certain conditions; the case, when the conditions are not satisfied, requires further investigations. In the final part of the paper the efficiencies of the different parameter estimation methods are compared in connection with a geophysical problem, concerning variations of the axis of rotation of the Earth.

1. The continuous case

Let $\xi(t)$ be a multidimensional continuous stationary Gaussian Markov process:

$$d\underline{\xi}(t) = A\,\underline{\xi}(t)dt + d\underline{w}(t),$$

where $\underline{w}(t)$ is a Wiener process and we assume that the real component of the eigenvalues of matrix A is negative.

By Baxter's theorem we have: if

$$E(d\underline{w} \cdot d\underline{w}^*) = B\,dt,$$

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then

$$\lim_{\max(t_k - t_{k-1})} [\underline{\xi}(t_k) - \underline{\xi}(t_{k-1})][\underline{\xi}(t_k) - \underline{\xi}(t_{k-1})]^* = B \cdot T$$

(with probability 1), where $0 = t_0 < t_1 < \ldots < t_n$ is a partition of the interval $[0, T]$, and B is the covariancy matrix of $\underline{w}(t)$. As it is well–known, all matrices can be brought to the Jordan form.

Denote with B' the Jordan form of B:

$$B' = SBS^{-1}.$$

Since B is symmetrical, B' is a diagonal matrix.

The elements of the principal diagonal of B' are just the eigenvalues of B and if $\underline{s}_i(s_{1i}, s_{2i}, \ldots, s_{ni})$ is the eigenvector belonging to the i-th eigenvalue, then $S = (s_{ij})$. This makes S unambiguous.

We suppose also, that $A' = SAS^{-1}$ is in Jordan form.

Consider now the transform of the process $\underline{\xi}(t)$, that is the stationary Gaussian Markov process:

(1) $$d\underline{\xi}'(t) = A'\underline{\xi}'(t)dt + d\underline{w}'(t),$$

where $\underline{\xi}'(t) = S\underline{\xi}(t)$, $\underline{w}'(t) = S\underline{w}(t)$. It is easy to verify, that SBS^* gives the covariancy matrix of $\underline{w}'(t)$. Because S in unitery, the latter equals SBS^{-1}. This means, that B' is precisely the covariancy matrix of $\underline{w}'(t)$. The eigenvalues of A are all different (simple) with probability 1.

Then A' also is of diagonal form, therefore the equation (1) is decomposed in the following n equations:

(2) $$d\xi_k'(t) = -\lambda_k \xi_k'(t)dt + dw_k'(t), \qquad k = 1, 2, \ldots, n,$$

where $-\lambda_k$ is an eigenvalue of A and $\lambda_i \neq \lambda_j$ $(i \neq j)$. As S can be determined using $B' = SBS^{-1}$, we can deduce confidence limits for the elements of A from the confidence limits for the parameters of the transformed process.

a) λ_k is real

The process (we leave the index and the prime) is:

$$d\xi(t) = -\lambda\xi(t)dt + dw(t).$$

We can state the following on the basis of [4]: if the observation happens in the interval $[0, T]$, then

$$\hat{\lambda} = \frac{-(s_1^2 - \frac{1}{2}T) + \sqrt{(s_1^2 - \frac{1}{2})^2 + 2Ts_2^2}}{2Ts_2^2}$$

provides and estimation, where

$$s_1^2 = \frac{1}{2}(\xi^2(0) + \xi^2(T)), \quad s_2^2 = \frac{1}{T}\int_0^T \xi^2(t)dt.$$

In the case of the given realization $\hat{\lambda}$ can calculated from the above equation. Similarly on the basis of [4] confidence limits can be given for λ.

b) λ_k is complex

(We leave the index k and the prime.)

The process is given by

$$d\xi(t) = -\lambda\xi(t)dt + dw(t).$$

Now, if $\xi(t) = \eta(t) + i\zeta(t)$, $w(t) = \varphi(t) + i\psi(t)$, $\lambda = \alpha - \beta i$ (thus $\beta > 0$), from the relation:

$$d[\eta(t) + i\zeta(t)] = (-\alpha + i\beta)[\eta(t) + i\zeta(t)]dt + d[\varphi(t) + i\psi(t)]$$

we deduce the processes:

$$d\eta(t) = -\alpha\eta(t)dt - \beta\zeta(t) + d\varphi(t)$$
$$d\zeta(t) = \beta\eta(t)dt - \alpha\zeta(t)dt + d\psi(t).$$

The estimations for β and for α can be found in [5]:

$$\hat{\alpha} = \frac{-\left(\frac{s_1^2}{a} - T\right) + \sqrt{\left(\frac{s_1^2}{a} - T\right)^2 + 4T\frac{s_2^2}{a}}}{\frac{2Ts_2^2}{a}} \quad \text{and} \quad \hat{\beta} = \frac{r}{s_2^2}$$

where

$$s_1^2 = \frac{1}{2}[|\xi(0)|^2 + |\xi(T)|^2],$$

$$s_2^2 = \frac{1}{T}\int_0^T |\xi(t)|^2 dt,$$

$$r = \frac{1}{T}\,{}^{T}_{0}\,(\eta d\zeta - \zeta d\eta)$$

and $a = b'_{kk}$ (if we consider the process ξ_k), i.e. $\underline{a}$ is the k-th element of the principal diagonal of B'.

In the case of a given realization s_1^2, s_2^2 a and r – therefore also $\hat{\alpha}$ and $\hat{\beta}$ – can be calculated.

With the help of the table in [5] we can give confidence limits for $\hat{\alpha}$; $\left(\frac{r}{s_2} - \beta s_2\right)\sqrt{\frac{2a}{T}}$ has $N(0, 1)$ distribution.

Finally we determine confidence limits for the elements of A.

Let be $S = P + iQ$, $A' = A_1 + iA_2$. Then $S^{-1} = P' + iQ'$, where

$$P' = (P + QP^{-1}Q)^{-1}, \quad Q = -(Q + PQ^{-1}P)^{-1}$$

(the existence of the inverses can be proved).

$$A = S^{-1}A'S = P'A_1P - P'A_2Q - Q'A_1Q - Q'A_2P,$$

because A is a real matrix.

We suppose, that confidence limits can be determined for the elements of A' (precisely for the elements of A_1 and of A_2) at the confidence level $1 - \epsilon$ (let all eigenvalues of A have non–zero imaginary components).

By a simple calculation we obtain for the element α_{ij} of A:

$$P(\alpha_{ij}^{(1)} \leqslant \alpha_{ij} \leqslant \alpha_{ij}^{(2)}) \geqslant (1 - 2\epsilon)^n = 1 - \epsilon^*,$$

where

$$\epsilon^* = \sum_1^n \binom{n}{k} (-2\epsilon)^k,$$

$$(\alpha_{ij}^{(1)}) = P'A_1^{(1)}P - P'A_2^{(2)}Q - Q'A_1^{(2)}Q - Q'A_2^{(2)}P,$$

$$(\alpha_{ij}^{(2)}) = P'A_1^{(2)}P - P'A_2^{(1)}Q - Q'A_1^{(1)}Q - Q'A_2^{(1)}P,$$

where we denote with $A_1^{(1)}$ $(A_1^{(2)})$ respectively with $A_2^{(1)}$ $(A_2^{(2)})$ the matrices formed by the left (right) endpoints of the intervals at the confidence level $1 - \epsilon$.

In the case where of the eigenvalues of A precisely l are real, the "sharper" inequality

$$P(\alpha_{ij}^{(1)} \leqslant \alpha_{ij} \leqslant \alpha_{ij}^{(2)}) \geqslant (1 - \epsilon)^l \cdot (1 - 2\epsilon)^{n-l}$$

is valid.

2. The discrete case

Let $\underline{\xi}(k)$ be a multidimensional discrete stationary Gaussian Markov process:

$$\underline{\xi}(k) = Q\underline{\xi}(k - 1) + \underline{w}(k)$$

where $\underline{w}(k)$ is a Wiener process and the modules of the eigenvalues of Q are less than 1.

We approximate this process by a continuous process:

$$d\underline{\xi}(t) = A\underline{\xi}(t)dt + d\underline{w}(t),$$

where $A = \ln Q$ and the random variables $\underline{\xi}(1), \underline{\xi}(2), \dots$ are the realizations of $\underline{\xi}(t)$ at the moments $\delta, 2\delta, \dots$.

We can estimate A on the basis of the previous paragraph.

It is easy to verify that from the equation $Q = e^A$ and from the monotonity of the function e^x follows that the confidence interval $(q_{ij}^{(1)}, q_{ij}^{(2)}) = (e^{\alpha_{ij}^{(1)}}, e^{\alpha_{ij}^{(2)}})$ has at least the confidence level $1 - \epsilon$.

3. Variation of the instantaneous Earth rotation axis

The instantaneous rotation axis of the Earth constantly changes its position relatively to the Earth itself.

Several authors deal with the investigation of these variations. We can mention e.g. the paper [3] of A. M. WALKER and A. YOUNG or the paper [2] of D. R. BRILLINGER. A model of a solution of the problem can be found also in the paper [1] of M. ARATÓ. As it is known, this change – as a two–dimensional process – consists of two components: of a component varying regularly every year and of a component varying with a period, of about 14 months. The two components are clearly indicated in the periodograms of fig. 1/a and fig. 1/b.

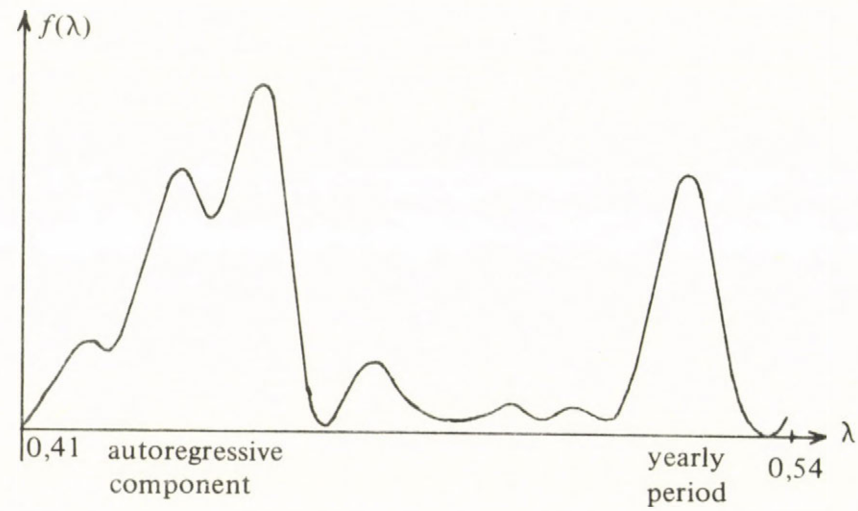

Figure 1/a

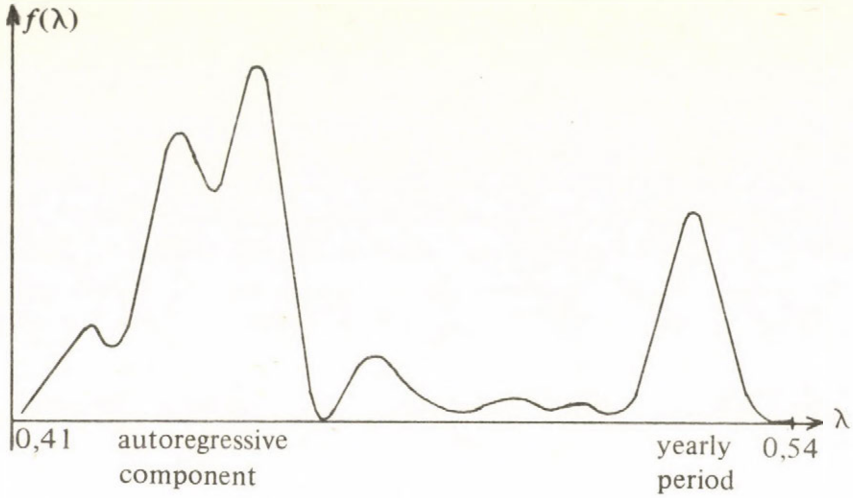

Figure 1/b

We apply the following model to the description of the phenomenon (see [1]). The data (600–600 observations) were taken from the paper [3] of Walker and Young. For a comparison we made also calculations on the ground of data published by Orlov (see [6]).

The Component with the yearly period we consider as a deterministic process, as a sinus function with given amplitude and phase. The estimation of the parameters for this component is a regression problem. We note, that Brillinger [2] subtracted simply the monthly averages from the original process, as the values of the component with the 12-months period.

In the regression problem we assume that both components of the process is in the form

$$x_t = A + B \sin \omega t + C \cos \omega t + \epsilon_t$$

where A, B, C are constans, ϵ_t is a white noise process and $\omega = 2\pi/12$ we have one observation per month.

Applied the estimated coefficients A, B, C we can subtract from the original process the yearly component.

Now we regard the residue process as a two dimensional first order autoregressive process (see [1] and §. 2. in this work).

In our case the matrix Q, in 2. paragraph, is in form

$$Q = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and $w(t)$ is a white noise process with independent components.

Let as denote

$$\hat{Q} = \begin{pmatrix} \hat{a}_1 & -\hat{b}_1 \\ \hat{b}_2 & \hat{a}_2 \end{pmatrix}$$

the estimate of the matrix Q, where $\hat{a}_1$ and $\hat{a}_2$ respectively $\hat{b}_1$ and $\hat{b}_2$ are in general not equal. Let

$$\hat{a} = \frac{1}{2}(\hat{a}_1 + \hat{a}_2),$$

$$\hat{b} = \frac{1}{2}(\hat{b}_1 + \hat{b}_2)$$

be the estimate of a and b.

The estimated values for 600–600 observations: $\hat{a}_1 = 0.87, \hat{b}_1 = 0.37, \hat{a}_2 = 0.39,$ $\hat{b}_2 = 0.89$.

We can characterize the accuracy of the model fittin with the components of the $w(t)$ residue noise process. In this case the two residue components $\epsilon_{1,t}$ and $\epsilon_{2,t}$ have variances 0"0.34 resp. 0".035, similarly to [2]. The autocovariancie functions and the estimate of periodograms see on the Figures 2/a, 2/b resp. 3/a, 3/b.

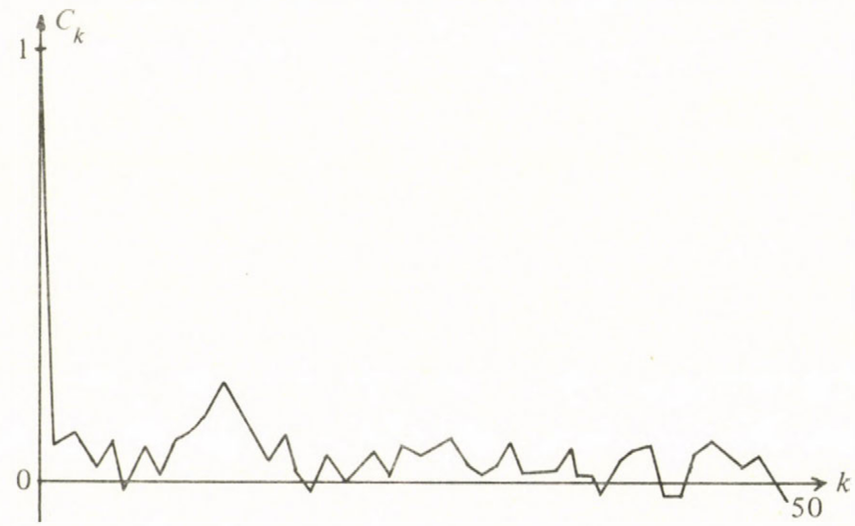

Figure 2/a

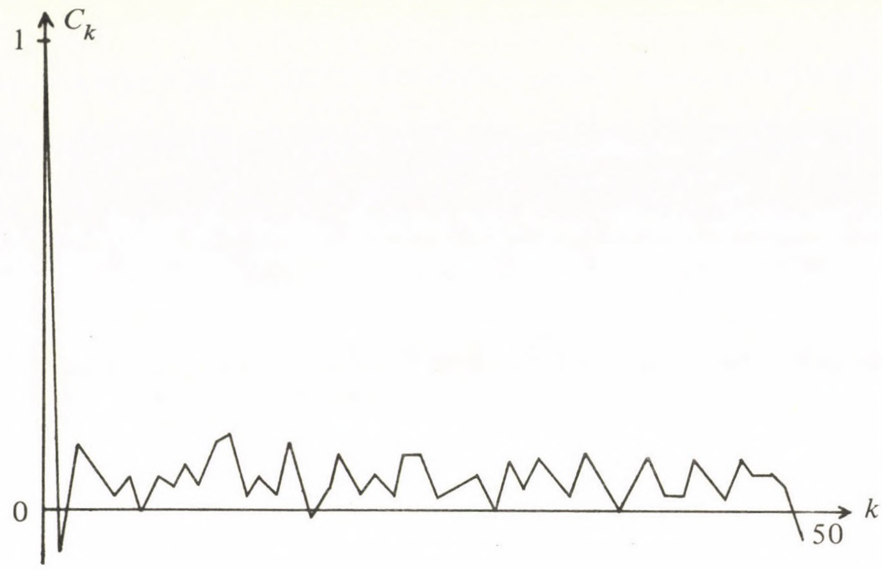

Figure 2/b

Figure 3/a

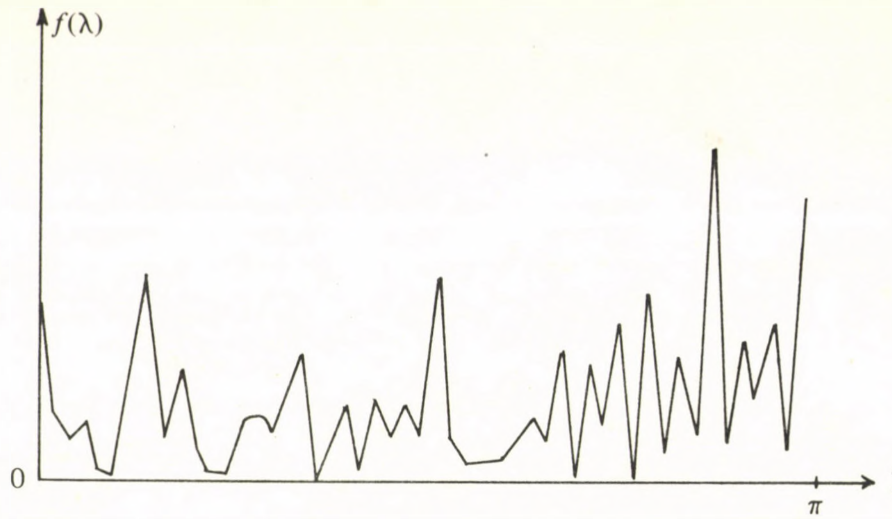

Figure 3/b

We can examine the properties of the $\epsilon_{1,t}$ and $\epsilon_{2,t}$ processes by the means of various statistical tests. On the basis of the number of the local maximum and minimum sets we can accept that $\epsilon_{1,t}$ and $\epsilon_{2,t}$ are independent noise processes. However considering the serial test can be saw that the sign of both components are changing very often that is the length of the series is very short–although the case is similar for example at the usual library random number generators.

The following question arize in connection with the later problem: What is the cause of the bad fitting of the serial test, the incomplete model or the inacurate parameter estimate?

An other question: Does the above mentioned model present any reason for the change of the original process, which is not connected with the 12 and 14 monthly periodes?

Note: We made the various parameter estimates using the time series program package of the institut.

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Р е з ю м е

Об оценке параметров многомерного стационарного гауссовского марковского процесса и её применение

В настоящей работе даются оценки параметра гауссовского марковского процесса, и определяются доверительные границы. Примером из практики решается геофизическая задача.