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THE CONNECTION BETWEEN GAUSSIAN MARKOV PROCESSES AND AUTOREGRESSIVE–MOVING AVERAGE PROCESSES

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In this paper we examine the connection between stochastic difference (differential) equations and multidimensional Gaussian Markov processes. We are using the definitions and notations of [1].

Definition 1. We call a stationary Gaussian process $\xi(n)$ an autoregressive moving average (ARMA) process if it satisfies the equation

$$\xi(n) = \sum_{i=1}^{\alpha} a_i \xi(n-1) + \sum_{i=1}^{\beta} b_i \epsilon(n-1) + \epsilon(n) \tag{1}$$

where $\{\epsilon(n)\}$ is a sequence of independent, identifically, distributed (i.i.d.) Gaussian random variables, and $\epsilon(n)$ is independent of $\mathfrak{A}_{-\infty}^{n-1}(\xi)$.

Theorem 1. *The equation (1) has a unique stationary solution if and only if all zeros of the characteristic polynom of the autoregressive part* $p_1(\rho) = \rho^\alpha - \sum_{i=1}^{\alpha} a_i \rho^{\alpha-i}$ *are inside the unite circle. In this case* $\xi(u)$ *is the first component of a* $k = \max\{\alpha, \beta+1\}$ *dimensional stationary Gaussian Markov process.*

$$\xi(t) = \{\xi^{(1)}(t), \ldots, \xi^{(k)}(t)\}$$

Proof. Let us assume that $\xi^{(1)}(t) = \xi(t)$ and consider the system of equations

$$\xi^{(i)}(n) = \xi^{(i+1)}(n-1) + C_{i-1}\,\epsilon(n) \quad \text{if} \quad i \leqslant \alpha - 1 \tag{2}$$

$$\xi^{(\alpha)}(n) = \sum_{i=1}^{\alpha} a_{\alpha+1-i} \xi^{(i)}(n-1) + \sum_{i=\alpha+1}^{\beta+1} b_{i-1} \xi^{(i)}(n-1) + C_{\alpha-1}\epsilon(n)$$

$$\xi^{(\alpha+1)}(n) = \epsilon(n)$$

$$\xi^{(\alpha+i)}(n) = \xi^{(\alpha+i-1)}(n-1) \qquad \text{if} \quad 1+\alpha < i \leqslant \beta+1$$

(Naturally in the case $\alpha < \beta$ the suitable terms and equations are omitted.)
If the constants c_j $(j = c_1 \ldots (\alpha-1))$ satisfy the equations

$$c_0 = 1 \tag{1}$$

$$c_1 - a_1 \qquad c_0 = b_1$$

$$\vdots$$

$$c_{\alpha-1} - a_1 c_{\alpha-2} \ldots - a_{\alpha-1}\, c_0 = b_{\alpha-1}$$

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then the system (2) is equivalent to the equation (1). It is easy to see that the characteristic polynom $p_2(\rho)$ of (2) is equal to $p_1(\rho)$ if $\beta < \alpha$, and $\rho^{\beta+1} p_1(\rho)$ otherwise. So the system (2) of stochastic difference equations has a unique stationary solution, which is a k-dimensional Gaussian Markov process and its first component will be the unique stationary solution of the equation (1).
Q.E.D.

Remark 1. The solution of the equation (1) can be obtained in a constructive way similarly to the first order autoregressive process

(4) $$\xi(n) = \sum_{k=0}^{\infty} c_k \epsilon(n-k)$$

Proof. Indeed, if the coefficients c_k satisfy the infinite recursive system of equations

(5) $$\begin{aligned} &c_0 = 1 \\ &c_1 - a_1 c_0 = b_1 \\ &\cdot \\ &\cdot \\ &\cdot \\ &c_k - \sum_{i=1}^{\alpha} a_i c_{k-i} = b_n, \quad \text{if} \quad k \geqslant \alpha, \end{aligned}$$

(notice that the first α equations concide with system (3)), and $\sum_{k=1}^{\infty} |c_k|^2 < \infty$, then the process (4) is a correctly defined stationary Gaussian process satisfying (1).

As $b_k = 0$ for $k > \beta$, and the roots of characteristic polynom $p_1(\rho)$ are inside the unite circle, system (5) has a unique solution with the desired property.

A multidimensional Gaussian Markov process $\underline{\xi}(n)$ has the representation $\underline{\xi}(n) = \sum_{k=0}^{\infty} \underline{\underline{Q}}^k \underline{\epsilon}(n-k)$. As the matrix Q satisfies its own characteristic equation:

$$Q^{\alpha} - \sum_{i=1}^{\alpha} a_i Q^{\alpha - i} = 0,$$

all the elements of $\{Q^n\}$ satisfy a recursive system of equations similar to (5), therefore the components of $\underline{\xi}(n)$ are sums of ARMA processes. Notice that if $\xi(n) = \sum_{k=1}^{l} d_k \xi^{(k)}(n)$, where $\xi^{(k)}(n) = \sum_{i=1}^{\alpha} a_i \xi(n-i) + \sum_{i=0}^{\beta} b_{(i)}^{(k)} \epsilon^{(k)}(n-i)$ and $\{\epsilon^{(k)}(n)\}$ is a sequence of i.i.d. Gaussian vectors, then $\xi(n)$ is ARMA process. So we get the converse of theorem 1.

Theorem 2. *Any component of a multidimensional stationary Gaussian Markov process is ARMA process.*

In the continuous time case the equation

(1') $$\xi^{(\alpha)}(t) = \sum_{i=1}^{\alpha} a_i \xi^{(\alpha-1)}(t) + \sum_{i=1}^{\beta} b_i w^{(\beta+2+i)}(t) + w'(t)$$

would correspond to equation (1). Before giving an exact meaning to (1') we try to solve it formally. For this purpose we need the following.

Lemma 1. *If the function* $f(t)$ *is differentiable and*

$$\int_0^\infty (|f(t)|^2 + |f'(t)|^2)\,dt < \infty,$$

then

(6) $$\int_{-\infty}^{t+h} f(t+h-s)\,dw(s) - \int_{-\infty}^{t} f(t-s)\,dw(s) =$$

$$\int_t^{t+h} \int_{-\infty}^{\tau} f'(\tau-s)\,dw(s) + f(0)(w(t+h)-w(t)).$$

The proof can be carried out by changing the order of integration. The relation (6) formally can be considered as a "rule of differentiation":

(7) $$\Big(\int_{-\infty}^{t} f(t-s)\,dw(s)\Big)' = \int_{-\infty}^{t} f'(t-s)\,dw(s) + f(0)w'(t).$$

We are looking for a solution of (1') in the form $\xi(t) = \int_{-\infty}^{t} f(t-s)\,dw(s)$, suggested by the representation of the first order autoregressive process. If $\beta < \alpha$, then there exists a unique function $f(t)$ satisfying the homogeneous differential equation

(8) $$f^{(\alpha)}(t) - \sum_{i=1}^{\alpha} a_i f^{(\alpha-i)}(t) = 0$$

and the initial conditions

(9) $$f(0) = 1$$

$$f'(0) - a_1 \cdot f(0) = b_1$$

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$$f^{(\alpha-1)}(0) - \sum_{i=1}^{\alpha-1} a_i f^{(\alpha-1-i)} = b_{\alpha-1}$$

(if $i > \beta$, $b_i = 0$).
Using the formal differentiation rule (7) we may convince that

(10) $$\xi(t) = \int_{-\infty}^{t} f(t-s)\,dw(s)$$

is a formal solution of (1').

. If the roots of the characteristic polynom $p_1(\lambda) = \lambda^\alpha - \sum_{i=1}^{\alpha} a_i \lambda^{\alpha-i}$ have negative parts, then $\int_0^\infty |f^{(i)}(t)|^2 dt < \infty$ for every $i = 0, 1, \dots$. In this case the process $\xi(t)$ given by (10) is a correctly defined stationary Gaussian process. We may assume (10) as the definition of continuous time ARMA process. (We notice that for $\beta \geqslant \alpha$ (1') has only generalized solution.) For continuous time ARMA processes theorems corresponding to theorems 1 and 2 are valid too:

Theorem 3. *A continuous time Gaussian process* $\xi(t)$ *is ARMA if and only if it is a component of a multidimensional stationary Gaussian process* $\xi(t)$.

Proof. The first part of the proof is obvious. The α-dimensional process $\{\xi^{(i)}\} = \{\int_{-\infty}^{t} f^{(i)}(t-s)dw(s)\}$ $(i = 0, \dots \alpha - 1)$ satisfies system of equations:

$$(11) \qquad d\xi^{(i)} = \xi^{(i+1)}(t) + c_i dw(t), \qquad i = 0, \dots, \alpha - 1,$$

$$d\xi^{(\alpha-1)} = \sum_{i=0}^{\alpha-1} a_{\alpha-i}\xi^{(i)} + c_{\alpha-1}dw(t),$$

where $c_i = f^{(i)}(0)$.

The converse assertion can be obtained similarly to the discrete time case, using the integral representation of a multidimensional Gaussian Markov process, and the fact that the matrix function e^{Ht} satisfies the differential equation $(e^{At})^{(\alpha)} = \sum_{i=0}^{\alpha-1} a_{i-e}(At)^{(i)}$, where the coefficients a_i coincide the coefficients of the characteristic polynom of A.

Remark 1. If we suppose that $\beta \geqslant \alpha$ we would have to add further equations to system (11) among them the equation $d\xi^{(\alpha+1)}(t) = dw(t)$ which has no stationary solution. This is the reason of the additional condition $\beta < \alpha$.

Remark 2. The system of equation (11) has the following visual meaning: an ARMA process $\xi(t)$ is not differentiable in general – but by the addition of a suitable Wiener process it becomes differentiable. This procedure can be continued up to the $(\alpha - 1)$-th derivative of $\xi(t)$.

Remark 3. Combining theorems 1., 2. and 3. with Doob's theorem (see [2]) we get that the discrete time sample process $\xi(n\delta)$ of a continuous time ARMA process $\xi(t)$ is also ARMA. But, the sample process $\xi(n\delta)$ of a pure autoregressive process isn't generally a discrete time pure autoregressive process, because if a matrix A has the form

$$\begin{pmatrix} 0 & 1 & & 0 \\ . & 0 & 1 & \\ . & 0 & 0 & 1 \\ . & & & \\ a_1 & & & a_\alpha \end{pmatrix}$$

its exponent $e^{A\delta}$ has not the same one.

In this work we have avoided the spectral approach to stationary processes because of the necessity of deep analytic tools. But in some technical applications the spectral density function has a simple visual meaning and it can be easily measured. For this reason we breefly summarize (without proofs) the basic facts concerning to the ARMA processes. A regular discrete (continuous) time stationary process has the representation (see [3])

$$\xi(n) = \int_0^{2\pi} e^{in\varphi} g(\varphi) dw(\varphi) \tag{12}$$

$$\xi(t) = \int_{-\infty}^{\infty} e^{it\lambda} h(\lambda) dw(\lambda) \tag{13}$$

where $w(\varphi)$, $w(\lambda)$ are standard Wiener processes ("random measures"), and functions $g(\varphi)$ resp. $h(\lambda)$ can be analitically continued to the open unit circle resp. upper halfplane. The sequence of i.i.d. Gaussian random variables (resp. the white noise process) corresponds to the identically constant function on the interval $(0, 2\pi)$ (resp. $(-\infty, \infty)$). Using this fact we can easily find the connection between the "moving–average" representations (4) and (10) and the spectral representations (12) and (13):

$$g(\varphi) = \sum_{n=0}^{\infty} c_n e^{in\varphi},$$

$$h(\lambda) = \int_{-\infty}^{0} f(-s) e^{i\lambda s} ds.$$

Using the formal correspondences

$$\xi(n) \sim g(\varphi) e^{in\varphi}, \qquad \xi(t) \sim h(\lambda) e^{i\lambda t}$$
$$\xi'(t) \sim h(\lambda) i\lambda e^{i\lambda t},$$

$w'(t) \sim e^{t\lambda t}$ we get for ARMA process the correspondencees

$$g(\varphi) = \frac{\sum_{n=0}^{\beta} b_n e^{-in\varphi}}{\sum_{n=0}^{\alpha} a_n e^{-in\varphi}}, \qquad h(\lambda) = \frac{\sum_{n=0}^{\beta} b_n (i\lambda)^n}{\sum_{n=0}^{\alpha} a_n (i\lambda)^n}$$

In continuous time case we can see from the form of $h(\lambda)$ that in the case $\beta \geqslant \alpha$ the integral of the spectral density function $|h(\lambda)|^2$ would be infinite. By physical reasons such a system can't exist.

References

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Р е з ю м е

Связь между процессами гауссовского марковского типа и типа авторегресси с конечным скользящим суммированием

В статье элементарными методами доказывается, что гауссовский процесс удовлетворяет стохастическому разностному (дифференциальному уравнению типа I (I') тогда и только тогда, когда он является компонентом многомерного стационарного гауссовского марковского процесса. Процесс являющийся решением уравнения I (I') в случае дискретного (непрерывного) временного параметра называется процессом типа авторегрессии с конечным сколзящим суммированием.