MTA Számítástechnikai és Automatizálási Kutató Intézete, **Közlemények** 12. (1974)

A FIRST PASSAGE PROBLEM FOR AN M/M/1 QUEUE

By Ahmed S. Mashour

INTRODUCTION

Consider the queue size process $\{Q(t),\ 0 \leqslant t \leqslant T\}$ associated with the queueing system M/M/1. Let $L(t)$ be a given integral valued non-increasing step function with $L(0) > 0$, $L(T) = 0$. One purpose of this paper is to investigate the random variable $\tau = \inf_{0<U<T} \{u : Q(u) \geqslant L(u)\}$ and its characteristics. An algorithm is presented in Sec.1 in order to obtain an explicit formula for the distribution function of τ.

In Sec.2 it is shown that such a problem arise when we deal with an M/M/1 system which offers service for the arriving customers during a finite interval $(0,T)$. Every served customer provides a revenue $r > 0$. After the closing time T, no new customers are admitted and the present customers, if any, are to be served in an overtime, at a runing cost C/unit time. The system has to be operated in order to make the expected net revenue as high as possible. The influence of the overtime costs on the net revenue, necessitates choosing a policy to control the input process.

A rejection time policy, closing the input earlier than T, is considered. A deterministic rejection time has been discussed in [2]. The random variable τ introduced represents a stopping time the optimal choice of which is discussed in sec.2.

A formula for the expected net revenue associated with a policy $L(t)$ is given in Sec.3. Then the deterministic and the described random rejection policies are compared.

1. THE DISTRIBUTION OF τ

We are concerned with an M/M/1 queueing system where the customers arrive in a Poisson stream at mean rate λ and the service times are independently, identically and exponentially distributed with mean $1/\mu$. We assume that the system starts with no customers. Let $L(t)$ be an integral valued non-increasing step function given by:

a) $L(0) = N$,

b) $L(t_i^*) = N - i, \qquad i = 1, 2, \ldots, N$

where $t_N^* \leqslant T$.

Beérkezett: 1973. december 20.

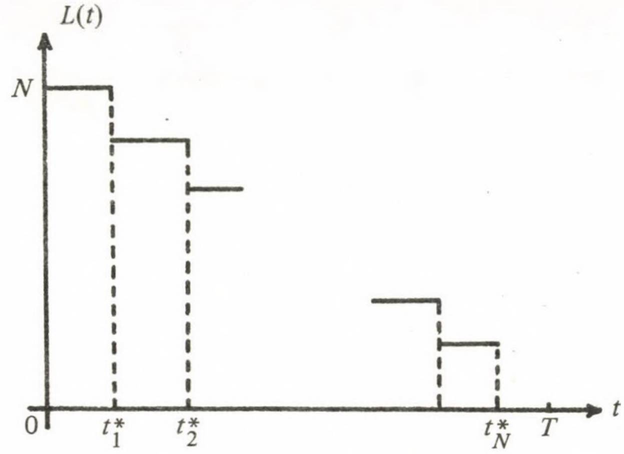

Figure 1.

In this section we show that the random variable τ given by $\tau = \inf_{0<u<T} \{u : Q(u) \geqslant \geqslant L(u)\}$ is of mixed type i.e the distribution function of τ is a mixture of an atomic and continous distribution. An algorithm is presented to obtain a formula for that distribution function and its expectation and variance.

Define the event

$$E_t = \{Q(u) \geqslant L(u) \text{ for some } 0 \leqslant u \leqslant t\}, \quad \text{and}$$

(1)

$$B_k(t) = Pr\{\overline{E}_t \cap (Q(t) = k)\}, \qquad k = 0, 1, \ldots, L(t) - 1$$

where $\overline{E}_t = \{Q(u) < L(u) \text{ for all } 0 \leqslant u \leqslant t\}$, is the complement event of E_t.

Form (1) and Fig. 1, it can be easily seen that τ may assume the discrete values t_i^*' s with probabilities

(2) $$P(\tau = t_i^*) = B_{L(t_i^*-0)-1}(t_i^* - 0), \qquad i = 1, 2, \ldots, N;$$

the queue size process touches the function $L(t)$ at t_i^* from the left. Also τ may assume any value lies between the t_i^*'s since the queue size process may touch $L(t)$ from below as a result of an arrival. It follows that

(3) $$P(\tau \leqslant u) = \lambda \int_{R(u)} B_{L(t)-1}(t)dt + \sum_{t_i^* \leqslant u} B_{L(t_i^*-0)-1}(t_i^* - 0),$$

where $R(u) = \{t : t \in (0,u), t \neq t_i^*, \quad i = 1, \ldots, N\}$.

Our problem now is to give an explicit formula for the functions $B_k(t)$. We show that $B_k(t)$, $k = 0, 1, \ldots,$ satisfies different systems of linear differential equations on different intervals.

From equation (1) it follows that for small interval h

$$B_0(t+h) = B_0(t)(1-\lambda h) + \sigma(h) \qquad t^*_{N-1} \leqslant t < t^*_N$$

then $\quad B'_0(t) = -\lambda B_0(t)$.

Also $\quad B_0(t+h) = B_0(t)(1-\lambda h) + B_1(t)\mu h + \sigma(h)\,, \qquad 0 < t \leqslant t^*_{N-1}\,,$

then $B'_0(t) = -\lambda B_0(t) + \mu B_1(t)$.

Similarly it can be shown that

$$B'_1(t) = -(\lambda+\mu)B_1(t) + \lambda B_0(t)\,, \qquad t^*_{N-2} < t < t^*_{N-1}\,,$$

$$B'_1(t) = -(\lambda+\mu)B_1(t) + \lambda B_0(t) + \mu B_2(t)\,, \qquad 0 < t \leqslant t^*_{N-2}\,,$$

$$B'_2(t) = -(\lambda+\mu)B_2(t) + \lambda B_1(t)\,, \qquad t^*_{N-3} < t \leqslant t^*_{N-2}\,,$$

$$B'_2(t) = -(\lambda+\mu)B_2(t) + \lambda B_1(t) + \mu B_3(t)\,, \qquad 0 < t \leqslant t^*_{N-3}\,,$$

. . . .

$$B'_{N-1}(t) = -(\lambda+\mu)B_{N-1}(t) + \lambda B_{N-2}(t)\,, \qquad 0 < t \leqslant t^*_1\,.$$

It is more suitable to rewrite these equations in the following form

if $\quad t^*_{N-1} < t \leqslant t^*_N\,, \quad$ then

$$B'_0(t) = -\lambda B_0(t)\,,$$

if $\quad t^*_{N-2} < t \leqslant t^*_{N-1}\,, \quad$ then

$$B'_0(t) = -\lambda B_0(t) + \mu B_1(t)\,,$$

$$B'_1(t) = -(\lambda+\mu)B_1(t) + \lambda B_0(t)\,,$$

if $\quad t^*_k < t \leqslant t^*_{k+1}\,, \qquad k = 0, 1, \ldots, N-2\,,$

$$B'_0(t) = -\lambda B_0(t) + \mu B_1(t)\,,$$

$$B'_1(t) = -(\lambda+\mu)B_1(t) + \lambda B_0(t) + \mu B_2(t)\,,$$

. . . .

$$B'_{N-(k+2)}(t) = -(\lambda+\mu)B_{N-(k+2)}(t) + \lambda B_{N-(k+3)}(t) + \mu B_{N-(k+1)}(t)\,,$$

$$B'_{N-(k+1)}(t) = -(\lambda+\mu)B_{N-(k+1)}(t) + \lambda B_{N-(k+2)}(t)\,.$$

Since the system starts with no customers at $t = 0$, then

$$B_0(0) = 1\,,$$

(4) $\qquad B_k(0) = 0\,, \qquad k = 1, 2, \ldots, N-1\,.$

We start by solving the last system of N linear differential equations on the interval $(0, t_1^*)$ using the initial values given by (4). Then we can determine the values of $B_0(t), B_1(t), \ldots, B_{N-1}(t)$ at $t = t_1^*$ and use these values as the initial condition of the next system of $N-1$ linear differential equations on the interval (t_1^*, t_2^*). On repeating this procedure we can get the forms of $B_k(t)$ on different intervals.

THE SOLUTION OF THE FIRST SYSTEM OF LINEAR EQUATIONS

In matrix form this system can be written in the form

$$\frac{d}{dt} B(t) = AB(t) \tag{5}$$

where

$$B(t) = \begin{bmatrix} B_0(t) \\ B_1(t) \\ \cdot \\ \cdot \\ \cdot \\ B_{N-1}(t) \end{bmatrix} \qquad A = \begin{bmatrix} -\lambda & \mu & 0 \ldots & 0 \\ \lambda & -(\lambda+\mu) & \mu \ldots & 0 \\ 0 & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & 0 \\ \cdot & & & \mu \\ 0 & \ldots & \ldots \lambda & -(\lambda+\mu) \end{bmatrix} \quad (N \times N)$$

with

$$B(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad (N \times 1)$$

We seek the solution in the form

$$B(t) = h\, e^{wt}$$

where

$$h = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_N \end{bmatrix} \quad (N \times 1).$$

Then from equation (5), it follows that

(6) $$(A - wI)\,h = 0\,.$$

In order to obtain a non-trivial solution, it is necessary and sufficient that

(7) $$|A - wI| = 0\,.$$

Concerning the roots of this characteristic equation we prove.

Lemma 1. *All the roots of the characteristic equation (7) are negative and distincit.*

Proof. Denote

$$D_n(w) = \begin{vmatrix} \lambda + w & -\mu & 0 & \dots & 0 \\ -\lambda & \lambda+\mu+w & -\mu & \dots & 0 \\ 0 & & & & . \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ . & & & & 0 \\ . & & & & -\mu \\ 0 & \dots & & -\lambda & \lambda+\mu+w \end{vmatrix} \qquad (n \times n)$$

It is easy to see that

(8) $$D_n(w) = (\lambda + \mu + w)D_{n-1}(w) - \lambda\mu D_{n-2}(w)\,, \qquad n \geqslant 2$$

where $D_0(w) = 1$, $D_1(w) = \lambda + w$ also $D_n(0) = \lambda^n > 0$ for all $n \geqslant 1$.
We prove now by using the recurrence relation (8) that all the roots of $D_N(w) = 0$ are distincit and negative. $D_1(w) = 0$ has only one root $w_1^{(1)} = -\lambda$
but $D_2(w) = (\lambda + \mu + w)D_1(w) - \lambda\mu$.
Since $D_2(0) = \lambda^2 > 0$, $D_2(-\lambda) < 0$ and $D_2(-\infty) > 0$ it follows that $D_2(w) = 0$ has two distincit negative roots $w_1^{(2)}$, $w_2^{(2)}$ such that

$$0 > w_1^{(2)} > -\lambda > w_2^{(2)}$$

generally if the roots of $D_{n-2}(w) = 0$ are

$$w_1^{(n-2)}, \quad w_2^{(n-2)}, \dots, w_{n-2}^{(n-2)}\,,$$

and the roots of $D_{n-1}(w) = 0$ are $w_1^{(n-1)}$, $w_2^{(n-1)}, \dots, w_{n-1}^{(n-1)}$ such that

$$0 > w_1^{(n-1)} > w_1^{(n-2)} > w_2^{(n-1)} > w_2^{(n-2)} >, \dots, > w_{n-2}^{(n-2)} > w_{n-1}^{(n-1)}$$

then $D_{n-2}(w_1^{(n-1)}) > 0$, $D_{n-2}(w_2^{(n-1)}) < 0$, $D_{n-2}(w_3^{(n-1)}) > 0, \ldots$
i.e. sign $D_{n-2}(w_k^{(n-1)}) = (-1)^{k-1}$, $k = 1, 2, \ldots, n-1$.
Then from the recurrence relation (8) it follows that the sign of $D_n(w_k^{(n-1)}) = = -\lambda\mu D_{n-2}(w_k^{(n-1)})$ alternates i.e the roots of $D_n(w) = 0$ are separated by the roots of $D_{n-1}(w) = 0$.
Consequently all the roots are negative and distincit.

We get a system of N solutions

$$B_{(1)}(t) = h^{(1)} e^{w_1^{(N)} t}, \quad B_{(2)}(t) = h^{(2)} e^{w_2^{(N)} t}, \ldots, B_{(N)}(t) = h^{(N)} e^{w_N^{(N)} t},$$

and the general solution of the system (5) is

(9) $$B(t) = \sum_{i=1}^{N} d_i B_{(i)}(t) = \sum_{i=1}^{N} d_i h^{(i)} e^{w_i^{(N)} t}$$

where d_i, $i = 1, 2, \ldots, N$ are arbitrary constants to be determined from the initial condition (4) at $t = 0$.
The general solution (9) can be written in the form

$$B_0(t) = \sum_{i=1}^{N} d_i \alpha_1^{(i)} e^{w_i^{(N)} t},$$

(9') $$B_1(t) = \sum_{i=1}^{N} d_i \alpha_2^{(i)} e^{w_i^{(N)} t}$$

. . . .

$$B_{N-1}(t) = \sum_{i=1}^{N} d_i \alpha_N^{(i)} e^{w_i^{(N)} t}.$$

The constants d_i's are determined from

$$1 = \sum_{i=1}^{N} d_i \alpha_1^{(i)},$$

$$0 = \sum_{i=1}^{N} d_i \alpha_2^{(i)},$$

. . . .

$$0 = \sum_{i=1}^{N} d_i \alpha_N^{(i)}.$$

The Column vector h^* associated with a root w^* according to the equation

(10) $$Ah^* = w^* h^*$$

is given by Lemma 2.

Lemma 2. *The componants of the eigen vector h^* corresponding to w^* are*

$$\alpha_k^* = \frac{D_{k-1}(w^*)}{\mu^{k-1}} \alpha_1^*, \qquad k = 2, \ldots, N$$

where $D_k(w)$ is defined by the recurrence relation (8).

Proof: The coefficient matrix of the equation (10) is

$$\begin{bmatrix} -(\lambda + w^*) & \mu & 0 \ldots & 0 \\ \lambda & -(\lambda+\mu+w^*) & \mu \ldots & 0 \\ 0 & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & 0 \\ \cdot & & & \mu \\ 0 & & & \lambda - (\lambda+\mu+w^*) \end{bmatrix} \quad (N \times N)$$

since the determinant of this matrix is $D_N(w^*) = 0$ but $D_{N-1}(w^*) \neq 0$, it follows that the rank of this matrix is $N-1$ and the first $N-1$ rows are linearly independent while the last row is a linear combination of the others. It follows from the first equation in the system (10), that

$$\alpha_2^* = \frac{(\lambda + w^*)}{\mu} \alpha_1^* = \frac{D_1(w^*)}{\mu} \alpha_1^*$$

from the second equation

$$\alpha_3^* = \frac{(\lambda+\mu+w^*)}{\mu^2} \alpha_2^* - \frac{\lambda}{\mu} \alpha_1^* = \frac{(\lambda+\mu+w^*)D_1(w^*) - \lambda\mu}{\mu^2} \alpha_1^* = \frac{D_2(w^*)}{\mu^2} \alpha_1^*$$

similarly

$$\alpha_4^* = \frac{D_3(w^*)}{\mu^3} \alpha_1^*,$$

$$\alpha_N = \frac{D_{N-1}(w^*)}{\mu^{N-1}} \alpha_1^*,$$

where α_1^* can be arbitrarly chosen, say $\alpha_1^* = 1$.

Now the solution of the system (5) is well defined and the values of the functions $B_0(t), B_1(t), \ldots, B_{N-1}(t)$ can be obtained at the end t_1^* of the first interval.

Repeating the same steps for the smaller system of $N-1$ linear differential equations on the interval (t_1^*, t_2^*) and so on untill the last system, which contains one differential equation on the interval (t_{N-1}^*, t_N^*), is reached.
The distribution function of τ given by (3) can be written in the form

if $\quad 0 < u < t_1^*$

$$P(\tau \leqslant u) = \lambda \int_0^u B_{N-1}(t)dt = \lambda \sum_{i=1}^{N} \frac{d_i \alpha_N^{(i)}}{w_i^{(N)}} (e^{w_i^{(N)} u} - 1) ,$$

if $\quad t_1^* < u < t_2^*$

$$P(\tau \leqslant u) = \lambda \int_0^{t_1^*} B_{N-1}(t)dt + B_{N-1}(t_1^*) + \lambda \int_{t_1^*}^{u} B_{N-2}(t)dt = \tag{11}$$

$$= \lambda \sum_{i=1}^{N} \frac{d_i \alpha_N^{(i)}}{w_i^{(N)}} (e^{w_i^{(N)} t_i^*} - 1) + \sum_{i=1}^{N} d_i \alpha_N^{(i)} e^{w_i^{(N)} t_1^*} +$$

$$+ \lambda \sum_{i=1}^{N-1} \frac{\bar{d}_i \bar{\alpha}_{N-1}^{(i)}}{w_i^{(N-1)}} (e^{w_i^{(N-1)} u} - e^{w_i^{(N-1)} t_1^*})$$

where $\bar{d}_i$'s are the arbitrary constants on the interval (t_1^*, t_2^*), $\bar{\alpha}_{N-1}^{(i)}$ represents the componants of the eigen vectors corresponding to the eigen values on the interval (t_1^*, t_2^*).
$P(\tau \leqslant u)$ where $t_{k-1}^* < u < t_k^*$, $\quad k = 3, \ldots, N$ can be easily obtained in the same way.

2. AN M/M/1 ASSOCIATED WITH REWARDS

We consider an M/M/1 which offers service for the arriving customers during a finite operating time $(0, T)$. Every served customer provides a revenue $r > 0$. At the closing time T no new customers are admitted and the present customers, if any, are to be served in an overtime, at a runing cost C/unit time with $r < c/\mu$. The system has to be operated in order to make the expected net revenue as high as possible. The influence of the overtime costs on the net revenue, necessitates choosing a policy to control the input process. Such a policy may take the form of a decreasing function $L(t)$ that give the upper limit of the number of customers present in the system at time t. Because of the steplikeness of the process $Q(t)$, one can reduce the function space $\{L(t), \quad 0 \leqslant t \leqslant T\}$ to the space of integer valued non-increasing step functions. According to such a policy the input is closed at the moment

$$\tau = \inf_{0 < u < T} \{u : Q(u) \geqslant L(u)\}$$

(i.e no customer is admitted after τ). Then a question arises how to choose $L(t)$ from the space of integer-valued non-increasing step functions in order to make the net revenue as high as possible.

THE OPTIMAL POLICY $L(t)$

Let $f(k, t)$ denotes the total expected overtime costs when there are k customers in the system at time t (t is measured in the negative direction where the origin represents the closing time T) ignoring later arrivals, then

$$(12) \qquad f(k, t) = c \int_t^\infty (x - t)\, \gamma_k(x)\, dx \,, \qquad k \geqslant 1$$

where $\gamma_i(x) = \mu^i x^{i-1} e^{-\mu x}/(i-1)! \qquad x \geqslant 0$.

At any moment $0 < u < T$, the net revenue of the queue operator according to a policy $\hat{L}(u)$ is $r\hat{L}(u) - f(\hat{L}(u), u)$.
The optimal policy $L(u)$ is that policy which maximizes the difference $r\hat{L}(u) - f(\hat{L}(u), u)$ for all $0 < u < T$. The points of jumps for the optimal $L(t)$ function can be determined as follows:

Define.

$$g_t(n) = rn - f(n, t)\,,$$

$$\Delta g_t(n) = g_t(n+1) - g_t(n) = r - \Delta f(n, t)$$

where

$$(13) \qquad \Delta f(n, t) = f(n+1, t) - f(n, t) = \frac{c}{\mu} \sum_{j=0}^{n} \frac{(\mu t)^j}{j!}\, e^{-\mu t}$$

since $\Delta f(n, t)$ is increasing in n and decreasing in t, then $\Delta g_t(n)$ is decreasing in n and increasing in t.
Thus $L(t)$ is the first integer for which $\Delta g_t(n) < 0$, i.e $L(t) = \min\{n \geqslant 0;\ g_t(n) < 0\}$.
At $t = 0$ we have

$$\Delta g_0(n) = r - \Delta f(n),\ 0 = r - c/\mu < 0$$

for any $n \geqslant 0$.

Therefore $L(0) = 0$. $L(t)$ remains zero untill the point $t_0 = \{t : \Delta g_t(0) = 0\}$ is reached, since $\Delta g_t(0)$ is increasing in t. Now $\Delta g_{t_0}(0) = 0$, but $\Delta g_{t_0}(1) < 0$ since $\Delta g_t(n)$ is decreasing in n. It follows that $L(t_0) = 1$.
Similarly $L(t) = 1$ for $t \geqslant t_0$ untill the point $t_1 = \{t : \Delta g_t(1) = 0\}$ is reached, at t_1, $L(t)$ must jump from 1 to 2 in-order to keep $\Delta g_t(.)$ negative.
With the same argument, we see that $L(t)$ jumps from i to $i+1$ at the point $t_i = \{t : \Delta g_t(i) = 0\}$. If the maximum value of $L(t)$ on the interval $(0, T)$ is N, then it is convenient to use the transform

$$t_i^* = T - t_{N-1}\,, \qquad i = 1, 2, \ldots, N\,,$$

so that all the times are measured from the opening time $t = 0$. The individual customers optimal balking strategy for the system M/M/1 was given in [1] and has a similar nature as $L(t)$.

3. THE EXPECTED NET REVENUE ASSOCIATED WITH $L(t)$

Let $\{N_t, t \geq 0\}$ be the poisson process denoting the number of the arrivals during $(0, t)$. It follows from [3] that for the stopping time τ

$$EN_\tau = \lambda E\tau .$$

If we denote

R_A : the reward associated with the admitted customers during $(0, \tau)$,

V: the overtime cost associated with the policy $L(t)$, then we have noting that our assumptions imply that $\tau > 0$,

$$E(R_A \mid \tau > 0) = \lambda r E N_\tau = \lambda r E\tau \tag{14}$$

$$\begin{aligned} E(V \mid \tau > 0) &= cE \sup \left(\sum_{i=1}^{L(\tau)} \xi_i - (T - \tau), 0 \right) = \\ &= c \int_0^{t_N^*} E\left(\left[\sum_{i=1}^{L(\tau)} \xi_i - (T - \tau) \right]^+ \mid \tau = t \right) dP_\tau(t) \end{aligned} \tag{15}$$

where ξ_i's are independently and identically exponentially distributed random variables with mean $1/\mu$ which represent the service time, and

$$P_\tau(t) = P(\tau \leq t) .$$

The integral in (15) can be written as the sum of integrals according to the nature of the distribution function of τ.

Therefore from (2) and (9) we have

$$\begin{aligned} E(V \mid \tau > 0) &= \lambda \sum_{j=1}^{N} \int_{t_{j-1}^*}^{t_j^*} f(N - j + 1, T - t) B_{N-j}(t) dt + \\ &+ \sum_{j=1}^{N} f(N - j, T - t_j^*) \cdot P(\tau = t_j^*) . \end{aligned} \tag{15'}$$

The expected net revenue given that $\tau > 0$ is

$$E(R_A \mid \tau > 0) - E(V \mid \tau > 0) . \tag{16}$$

We compare now the random stopping time policy and the deterministic optimal rejection time policy from the point of view of the expected net revenue associated with each. First we consider the queuing system M/M/1 for which a) $\rho = \lambda/\mu < 1$, b) the initial distribution of the queue size is stacionary one:

$$p_0^* = P\{Q(0) = 0\} = 1 - \rho ,$$

$$p_k^* = P\{Q(0) = k\} = (1 - \rho)\rho^k , \qquad k \geq 1 .$$

The random stopping time policy states that $\tau = 0$ whenever the system starts with N or more customers and $\tau > 0$ when the system starts with $N - 1$ or less customers. Then we get

$$P(\tau = 0) = \sum_{i=N}^{\infty} p_i^* = \rho^N ,$$

$$P(\tau > 0) = 1 - \rho^N .$$

It follows that the distribution function of τ will take the form

(17) $$P(\tau \leq t) = P(\tau = 0) + P(\tau \leq t, \tau > 0) = \rho^N + (1 - \rho^N)P(\tau \leq t \mid \tau > 0)$$

where $P(\tau \leq t \mid \tau > 0)$ is given by (3) with the initial condition

$$B_k(0) = \frac{p_k^*}{\sum_{i=0}^{N-1} p_i^*}, \qquad k = 0, 1, \ldots, N-1 .$$

If we denote

R_I : the reward associated with the initial number of customers,

R : the total reward, then $R = R_I + R_A$.

From the law of total expectation:

(18) $$ER = \rho^N E(R \mid \tau = 0) + (1 - \rho^N)E(R \mid \tau > 0) =$$

$$= \rho^N \frac{r \sum_{k=N}^{\infty} kp_k^*}{\rho^N} + (1 - \rho^N)\left\{\frac{r \sum_{k=1}^{N-1} kp_k^*}{1 - \rho^N} + E(R_A \mid \tau > 0)\right\} =$$

$$= \frac{\rho r}{1 - \rho} + (1 - \rho^N)\lambda r E\tau$$

also by (12)

(19) $$EV = \rho^N E(V \mid \tau = 0) + (1 - \rho^N)E(V \mid \tau > 0) =$$

$$= \sum_{k=N}^{\infty} p_k^* f(k, T) + (1 - \rho^N)E(V \mid \tau > 0) .$$

Thus the expected net revenue associated with the random stopping time policy is

(20) $ER - EV$.

On the other hand when the case of the deterministic optimal rejection time policy is considered for the same system., from [2] we can deduce easily that the expected net revenue $C(t')$ associated with a rejection time t' is given by

$$C(t') = \left(\lambda t' + \frac{\rho}{1-\rho}\right) r + c\rho \frac{e^{(\lambda-\mu)(T-t')}}{\lambda-\mu}$$

taking the derivative with respect to t', and equating with zero, then

$$\frac{d}{dt'} c(t') = \lambda\left[r - \frac{c}{\mu} e^{(\lambda-\mu)(T-t')}\right] = 0 ,$$

or $$T - t' = \frac{1}{\lambda-\mu} \ln\left(\frac{r\mu}{c}\right)$$

(21) $$t' = T - \frac{1}{\lambda-\mu} \ln\left(\frac{r\mu}{c}\right) .$$

Noting that

$$\frac{d^2}{dt'^2} c(t') = \rho c(\lambda-\mu) e^{(\lambda-\mu)(T-t')} < 0$$

for all t' it follows that $C(t')$ attains its maximum at t' given by (21).

The optimal rejection time = $t^* = \max(0, t')$ i.e

(22) $$t^* = \max\left(0, T + \frac{1}{\mu-\lambda} \ln\left(\frac{r\mu}{c}\right)\right) .$$

The associated Expected net revenue = $C(t^*)$ =

(23) $$= \left(\lambda t^* + \frac{\rho}{1-\rho}\right) r + c\rho \frac{e^{(\lambda-\mu)(T-t^*)}}{\lambda-\mu} .$$

4. DESCRIPTION OF THE PROGRAM USED FOR THE COMPUTATIONS

A computer program is written to fulfil the calculations necessary for obtaining numerical results. Firstly we should give the parameters λ, μ, r, C, T. Then the control function $L(t)$ should be described. This is made by the means of the points $t_1^*, t_2^*, \ldots, t_N^*$. If one would like to use the optimal control policy then the t_i^*'s have to be determined in accordance with sec.2. To do this the subroutine DELFNT calculates the value of $\Delta f(n, t)$ for $n \geqslant 1, t > 0$. Then the subroutine POINT determines the values $t_i = \{t : r = \Delta f(i, t)\}$ where the t_i's are measured from the closing time T in the negative direction.

After this we turn to find the roots of $D_n(w) = 0,\ n = 1, 2, \ldots, N$. This is done by the means of the recurrence relation (8) and the property of the sequence of roots proved. Lemma 2. is applied in calculating the eigen vectors corresponding to the intervals. After this it remains only to determine the arbitrary constants. Starting from the first interval we use the initial condition (4). The $B_k(t_1^*),\ k = 0, 1, \ldots, N-1$ values serve for the initial condition on the next interval and so on.

These quantities are quite enough to determine the mean value $E\tau$ and the expected net revenue.

As a way of checking the results obtained, the system was simulated and all the quantities of interest were estimated and compared with the corresponding calculated quantities. The results obtained by both methods showed a good agreement. Point out the effectiveness of the exast procedure, we mention that for getting the numerical results presented in the foregoing tables 1 minute 47 seconds was needed for the exact values while for those of simulated 17 minutes, 19 seconds.

In order to make the comparison mentioned in sec.3, we should start the system from the stationary state. This is equivalent to use the initial condition

$$B_k(0) = p_k^* \Big/ \sum_{i=0}^{N-1} p_i^* , \qquad k = 0, 1, \ldots, N-1$$

for the first interval.

The expected net revenue associated with the deterministic optimal rejection time is easily obtained by (23). The results obtained given that the system has been started with a stationary queue size are summarized in the following tables:

For $r = 2,\ C = 50$,

	λ	μ	T	N	cal. $E\ \tau$	sim. $E\ \tau$	cal. $E\ V$	sim. $E\ V$
1	0.5	0.75	14	5	4.570	4.573	0.0947	0.0972
2	0.6	0.80	13	5	3.386	3.391	0.2919	0.3043
3	0.6	0.80	15	6	4.829	4.832	0.2025	0.2155
4	0.5	0.75	18	7	8.013	7.986	0.0469	0.0477

This table compare between the calculated and the simulated values for $E\tau$ and the expected overtime $E\ V$.

	λ	μ	T	N	E_D	cal. E_R	sim. E_R
1	0.5	0.75	14	5	– 0.0263	3.8320	3.719
2	0.6	0.80	13	5	– 7.9263	– 4.5331	– 5.119
3	0.6	0.80	15	6	– 3.3350	1.6702	1.093
4	0.5	0.75	18	7	3.9737	9.6650	9.588

where E_D : the expected net revenue associated with deterministic optimal rejection time policy.

E_R : the expected net revenue associated with the random stopping time policy.

From the second table it follows that the random stopping time policy seems to be more better than the deterministic one.

Acknowledgement

I would like to thank my supervisor Dr. J. Tomko for suggesting the problem and the help he has given by way of many discussions.

References

[1] Emmons, H., "The optimal admission policy to a multiserver queue with finite horizon" J. Appl. Prob. 9 (1972) 103-116.

[2] Mine, H. and Ohno, K., "An optimal rejection time for an M/G/1 queueing system" Op. Res. 19 (1971) 194-208.

[3] Hall, W.J., "On Wald's equations in continuous time" J. Appl. Prob. 7 (1970) 59-68.

Summary

An M/M/1 queueing system with a simple cost structure is considered, assuming that the system operates during a finite interval after which any remaining customers will require extra overtime service costs. For controlling the input a random rejection time, which is the first passage time that the queue size hints a given non-increasing positive $L(t)$ function, is discussed. Its distribution function is obtained by solving successive systems of differential equations. A computational procedure has been written and the numerical results obtained are presented showing that the effectiveness of the random rejection policy is higher than that of a deterministic one.

Резюме

Рассматривается система обслуживания M/M/m , функционирующая в конечном интервале времени. Требования, оставшиеся в системе по окончании периода функционирования, дообслуживаются сверхурочно, что приводит к дополнительной оплате. Чтобы уменьшить сверхурочную работу, рассматривается управление входящим потоком, по которому вход системы закрывается, когда длина очереди пересекает данную невозрастающую функцию $L(t)$. Такой момент закрытия входа представляет собой случайную величину, независящую от будущего. Для нахождения её функции распределения необходимо решить последовательность систем дифференциальных уравнений. Написана вычислительная процедура и приведены численные результаты, показывающие, что рассматренное правило закрытия входа более эффективно, чем правило детерминированного типа.

TARTALOMJEGYZÉK