Self-organizing patterns in an evolutionary rock-paper-scissors game for stochastic synchronized strategy updates

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We study a spatial evolutionary rock-paper-scissors game with synchronized strategy updating. Players gain their payoff from games with their four neighbors on a square lattice and can update their strategies simultaneously according to the logit rule, which is the noisy version of the best-response dynamics. For the synchronized strategy update two types of global oscillations (with an ordered strategy arrangement and periods of three and six generations) can occur in this system in the zero noise limit. At low noise values, all nine oscillating phases are present in the system by forming a self-organizing spatial pattern due to the comprising invasion and speciation processes along the interfaces separating the different domains.

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I. INTRODUCTION

Game theory gives us a general mathematical framework to study interactive decision situations, when the aims and choices of the players are potentially in conflict [1]. Evolutionary games are introduced to investigate multiagent social and biological systems [2] using the concept of game theory to quantify the interactions. Players can modify their strategy following certain evolutionary rules taking into account Darwinian selection in biological systems and/or individual rationality in social systems. In recent decades, many scientific fields approached game theory; physics proved to be exceptional in handling the spatial systems because of the direct applicability of statistical physics methods (for a survey see [3,4]).

Now our investigations are focused on a spatial evolutionary game where pair interactions are described by a cyclic three-strategy game, called the rock-paper-scissors (RPS) game. For the traditional RPS game, players choose simultaneously one of three strategies: rock, paper, or scissors. According to the rules of the game, a strategy is superior to another and inferior to the third one: rock beats scissors, scissors beat paper, and paper beats rock, i.e., the game represents the cyclic dominance of strategies. This type of cyclic dominance plays a crucial role in the maintenance of biodiversity in ecological systems [5–7] and can support the cooperative behavior in social dilemmas when a third strategy (e.g., loners) is present [3,8,9]. Similarly, population-level cyclic motions were observed in a human laboratory experiment of the discrete-time iterated RPS game [10]. Moreover, biological experiments have justified the presence of cyclic dominance in populations of bacteria [11–13] and lizards [14,15]. These results have initiated progressive activity to clarify the consequences of cyclic dominance on the macroscopic, populationwide behavior. It is found that these systems exhibit a nontrivial effect when the cyclic symmetries are disturbed [16–19].

Very recently [20] it turned out that all symmetric $3 \times 3$ games can be decomposed as the linear combination of potential and RPS games. For potential games, a potential can be derived from the payoff matrix. The random sequential application of the logit rule drives such systems into a state where the stationary strategy distribution is described by the Boltzmann distribution [21,22]. The RPS game can be considered as the force responsible for driving the system out of equilibrium by generating probability currents (entropy production) throughout the microscopic states of multiagent models [23,24].

For well-mixed populations, systems with RPS-type interactions show global oscillations in the strategy frequencies [25]. For structured populations, i.e., when the interaction network of players is defined by a complex graph, the oscillation can either evolve towards a limit cycle or its amplitude increases continuously until only one strategy (species) remains alive [26,27]. On two-dimensional lattices, due to the short-range interactions, numerical simulations have indicated the survival of all three strategies through the formation of self-organizing patterns with rotating spirals [7,28–30]. The general features of the self-organizing patterns (including spatial correlations) have been studied quantitatively in several previous papers [28,31–35]. In contrast to this behavior, in finite one-dimensional systems, investigations indicate domain coarsening [36,37] or fixation for stochastic imitation rules [33,38,39]. In three-dimensional lattice systems, on the other hand, vortices (rotating three-edged vertices where the three types of interfaces meet) form closed loops [29,40] in analogy to systems described by the complex Ginzburg-Landau model (see, e.g., [41] and references therein).

Both the complexity of the system and the richness of phenomena increase gradually if we study the effect of cyclic dominance with four [42–51], five [52–56], six [43,57–60], or nine [60–63] strategies. In particular, the studies about nine-species systems help interpret the results of the present model where one can observe nine ordered spatial arrangements of the three strategies for synchronized strategy update. It is worth mentioning that some of the mentioned three-, four-, and six-species models are subsystems (systems where the use of certain strategies are prohibited) of the nine-species one. In the present model we can explore the competitive relationships among nine strategy associations (which can be identified as complex species) that are built up from three strategies and possess an ordered spatial strategy arrangement.
It is already well known that the macroscopic behavior of a spatial evolutionary game depends mainly on the number of strategies, the payoff matrix, the connectivity structure, and the dynamical rule(s) controlling the choice of new strategies [3,4,64–66]. The first dynamical rules were introduced to take the Darwinian selection into consideration, allowing the population of the fittest strategy (or species) to grow at the expense of the weaker ones [67,68]. The first spatial evolutionary game was introduced by Nowak and May [69,70] to study the maintenance of cooperative behavior in a lattice model with synchronized strategy update in analogy to cellular automata [71]. In this model, players have modified their strategy at discrete time steps by simultaneously adopting the strategy of the neighbor who received the highest income in the previous step. Subsequently, this deterministic model was modified by the addition of stochastic noise to the imitation process [72]; these rules are used by the followers to study different versions of evolutionary games [73–77]. In the meantime, additional variants of the dynamical rules (including random sequential stochastic updates) were introduced and studied systematically (for a brief survey see [3,4,64]).

In the literature of physics and economics, a fundamentally different set of dynamical rules was also introduced to take human rationality into account: In the so-called logit rule [22,78–81], players are capable of evaluating their income by testing each available strategy and playing fictitious games against their co-players, assuming the current (quenched) strategy environment. The stochastic logit rule prefers to choose the strategy providing the highest income in an exponential manner. In the low noise limit, it becomes equivalent to the best-response rule [82–84] that may be used to find the Nash equilibria for certain types of games. This rule is intimately related to equilibrium statistical physics because in the case of multiagent potential games, the random sequential applications of the logit rule drive the system into a Boltzmann distribution [22].

In the present work we study an evolutionary RPS game where players are located on the sites of a square lattice with periodic boundary condition. Following one of the three strategies, players play RPS games with their four nearest neighbors in every discrete-time step and after collecting their income, they are allowed to modify their strategy according to the next generation. In this model, players have modified their strategy at discrete time steps by simultaneously adopting the strategy of the neighbor who received the highest income in the previous step. Subsequently, this deterministic model was modified by the addition of stochastic noise to the imitation process [72]; these rules are used by the followers to study different versions of evolutionary games [73–77]. In the meantime, additional variants of the dynamical rules (including random sequential stochastic updates) were introduced and studied systematically (for a brief survey see [3,4,64]).

Using this formalism, the income of player \( x \) can be calculated as

\[
\mathbf{u}_x(s_x, s_{\Omega_x}) = \sum_{y \in \Omega_x} s_x^T \cdot \mathbf{M} \cdot s_y,
\]

where \( s_x^T \) is the transpose of the player’s strategy vector and \( s_{\Omega_x} \) stands for the strategy profile of players in the interaction neighborhood \( \Omega_x \) of player \( x \), thus the summation runs over the four nearest neighbors of player \( x \). The matrix \( \mathbf{M} \) contains the payoff elements describing the cyclically dominant RPS interactions, namely,

\[
\mathbf{M} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.
\]

After players collected their payoff at time \( t \), they have the option to adopt a (possibly) more successful strategy according to the evolutionary aspect of the model. In this paper we use the so-called logit rule. In this strategy adoption process players check the payoff they could earn by adopting the possible strategies, provided their neighbors keep using their present strategies. Individuals keep their strategy or adopt a new strategy with probabilities depending on these expectations:

\[
\Pr(s'_{x} | s_{x}, s_{\Omega_x}) = \frac{\exp[\mathbf{u}_x(s_x, s_{\Omega_x})]}{\sum_{s'_x \in \{R,P,S\}} \exp[\mathbf{u}_x(s'_x, s_{\Omega_x})]}.
\]

Here \( \Pr(s'_{x} | s_{x}, s_{\Omega_x}) \) determines the probability for player \( x \) to adopt strategy \( s'_{x} \in \{R,P,S\} \) where \( \mathbf{u}_x(s'_x, s_{\Omega_x}) \) is the expected payoff for playing strategy \( s'_x \) with neighbors who play their present strategies. In addition, \( K \) indicates the measure of noise (or temperature) that can originate from different sources and enables the occasional appearance of less successful strategies. The probability of choosing a strategy with smaller expected payoff increases with increasing \( K \). Every player decides about the strategy to adopt for the subsequent generation and then the strategy of all players is updated at the same time, i.e., synchronized strategy updating is applied.

We studied the model by performing Monte Carlo simulations on a square lattice with periodic boundary conditions. The size of the lattice varied, depending on the actual problem, from \( N = 4 \times 10^4 \) to \( 10^6 \). Due to the dynamics and the emerged phenomena. In Sec. III we discuss the results of the model and present and explain the emerging
domain structure, these system sizes are large enough to avoid finite-size effect problems. In most of the cases, the simulation is started using random initial conditions where players have equal chance to adopt one of the three different strategies initially; however, to study the invasion phenomena we applied prepared initial states. After a suitable transient time $t_{tr}$, simulation data are obtained by averaging over a $t_s$ sampling time. These parameters are adjusted to the system behavior and are changed typically between $t_{tr} = 1000–5000$ Monte Carlo steps (MCS) and $t_s = 10^4–10^6$ MCS, where, within the time unit (MCS), one full strategy updating sequence takes place, i.e., each player modifies strategy once.

III. RESULTS AND DISCUSSION

Examining the evolution of the model from a random initial state, we can observe a fast cluster-formation process resulting in domains that exhibit a synchronously oscillating internal structure at sufficiently low noise levels. After some time the domain-growing process stops and the system evolves into a self-organizing spatiotemporal pattern. A typical snapshot is shown in Fig. 1, where nine types of ordered strategy arrangements can be distinguished. Apart from the three obvious homogeneous clusters, there are six more that play an important role during the evolutionary process. These additional clusters have a chessboardlike structure where the dark and light squares of the chessboard are occupied by two different strategies. In fact, for a given strategy pair we can distinguish chessboard and antichessboard patterns where the sublattice occupancy is reversed. Since the sites of the underlying square lattice are identical, chessboard and antichessboard structures may seem to be the same (after all, they are equivalent apart from a translation by one site), but we are going to show that they interact quite differently with other types of clusters, thus they have to be treated as different types as well. We will refer to the different cluster types also as strategy associations further on.

First we discuss what happens inside the homogeneous domains in the limit $K \to 0$ that corresponds to the synchronized best-response dynamics. Following this myopic rule, players choose the strategy providing the highest income, assuming that co-players do not modify their strategy. If initially ($t = 0$) all players use strategy $R$, then in the next generation ($t = 1$) everyone will choose $P$ and subsequently ($t = 2$) $S$. In short, the homogeneous strategy distribution changes cyclically as illustrated in the inner ring in Fig. 2. These homogeneous states are denoted by the same strategy pair (e.g., $SS$) positioned on the two sublattices. However, these pairs can be different as well. For example, state $PR$ denotes a sublattice ordered strategy arrangement where the first sublattice is uniformly occupied by $P$ players and the other one by $R$ players. Within such a domain, $P$ players are satisfied and do not wish to modify their strategy, while the dissatisfied $R$ players are enforced uniformly to apply strategy $S$, thus the system evolves into the $PS$ state. In this newly formed domain type, $P$ players are motivated to change their strategy to $R$ while their co-players keep theirs in the next step. Consequently, one can observe a cyclic behavior that is illustrated in the outer ring in Fig. 2. It is worth mentioning that similar cycles emerge when we consider only two players using the same dynamical rule.

At low noise levels, point defects can appear inside the ordered patterns. These defects, however, vanish in the next generation as the ordered neighborhood dictates the course of evolution. The frequency of isolated point defects increases
of the player to indeed adopt the suboptimal strategy is about 0.7% for $K = 0.2$, i.e.,
\[
\Pr(R) = \frac{e^{1/K}}{e^{1/K} + e^{2/K} + e^{-3/K}} \approx 0.0067.
\]
This kind of strategy adoption can happen at any place along the domain interface, therefore the overall probability for the starting step of the invasion grows substantially for longer interfaces. As soon as such a steplike irregularity forms, one of the domains can gain ground at the expense of the other. The arrows (and states) following the second panel of Fig. 4 are the result of the highest probability events: The irregularity spreads upward and downward in every generation until the whole column is filled up by the superior domain. Invasion is induced by a (noise-dependent) random event, but after that it continues basically deterministically. To give an idea about the measure of the invasion’s noise dependence, we calculated the probability of the above-described steplike irregularity’s appearance for two additional $K$ values: $\Pr(R)_{K=0.1} = 0.00005$ and $\Pr(R)_{K=0.4} = 0.076$. Consequently, for higher noise values, the domain interfaces become rough and the invasion proceeds very fast. On the other hand, in the low noise limit, the domain structure is basically frozen. Similarly to the appearance of the above-described steplike irregularity, other strategies can appear at the domain boundaries as well; however, the likelihood of these events is several magnitudes smaller than that of the event inducing the invasion. The successive panels of the figure demonstrate the six-generation-long cyclic behavior of the domains in a spectacular way.

We have to emphasize that the above invasion process only considered vertical (or horizontal) domain interfaces. For the propagation of diagonal interfaces, no noise-induced starting step is needed. The diagonal interface is essentially a series of steplike irregularities, thus the whole front moves by one site perpendicular to the interface in every generation as the result of the most likely strategy changes.

Besides the black lines in Fig. 3, there are dashed blue lines connecting three strategy associations. These dashed lines indicate cases when the encounter of two domains gives birth to a third type of cluster that consumes the first two. The arrow points towards this newly established domain type. This process can also be considered as speciation: The encounter of two species (associations) generates a third one. The actual course of speciation takes place similarly to that of invasion: The third species forms at the interface and then it consumes both original associations according to the scenario described in Fig. 4. We illustrate this process in Fig. 5 on a larger scale.
In some cases, the invasion process goes on in a slightly different way. A third type of strategy association can form at the interface, however; in this instance, this new species does not emerge victoriously; it only helps one of the domains to invade the other. It pushes forward into the territory of the inferior domain and as soon as the region of the inferior domain is consumed, this new species acts as a catalyst for invasion in a narrow strip and incorporates its territory from the other side. As a result, the inferior domain and at the same time the superior domain invade the other. It pushes forward into the territory of the other. We have determined the autocorrelation function where \( c(\cdot, \cdot) \) denotes averaging over the lattice sites \( x \) and time \( t \) in the stationary state and the occupation number \( n_s(\mathbf{R}) = 1 \) if \( s_x = \mathbf{R} \) and \( n_s(\mathbf{R}) = 0 \) otherwise. Evidently, similar correlation functions can be defined for the other two strategies, however, these quantities are equivalent due to the cyclic symmetries.

As the spatial pattern is a mixture of homogeneous and chessboardlike domains, an oscillation (with a period of 2) can be observed in the autocorrelation function where \( c(z) \) tends to zero with a different speed at the odd and even values of \( z \). At high noise values, the correlations vanish exponentially. In contrast, at low noise level, a suppressed oscillation can be observed in \( c(z) \) at even distances as illustrated in Fig. 6 for \( K = 0.3 \) and 0.4. The estimated value where \( c(z) \) becomes zero is proportional to the average linear domain size that increases when the noise is decreased. The quantitative analysis of the system’s behavior in the low noise limit is prevented by technical difficulties related to the extremely slow motion of interfaces.

### IV. SUMMARY

We have studied an evolutionary RPS game with myopic synchronized strategy updates on a square lattice. We observed peculiar self-organizing patterns including invasion and speciation processes. One of the striking features of these globally oscillating patterns is that we can distinguish domains representing ordered phases of the three- or six-step-long limit cycles. These limit cycles can be considered as oscillating solutions consisting of strategy associations with spatiotemporal structure governing the system’s behavior at small sizes and low noise levels. The self-organizing patterns emerge in a sufficiently large system if it is started from a random initial state.

The self-organizing pattern formation of these domains is maintained by invasions between them. The direction of invasion is determined by the RPS-type competitions between the components of the oscillating domains. It is found that this direction is preserved during the consecutive steps. The invasion directions reflect symmetries that are assumed in models of bacterial warfare where bacteria use two toxins (antitoxins) during their evolutionary competition. Besides the cyclic invasion phenomena we could observe speciation processes when a new phase emerges and extends along the interfaces of the original domains. All these features are summarized in an extended flow diagram characterizing the invasion and speciation processes between the strategy associations.

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