ON THE REGULARITY OF THE SOLUTIONS OF THE VARIATIONAL INEQUALITIES

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INTRODUCTION

Let X be a real Hilbert's space, X' be the dual to X space (i.e. the space of linear functionals on X).

Let $a(u, v)$ be a bilinear, not necessarily symmetrical form on X satisfying the following conditions ($\|..\|$-norm on X):

(1) $\quad |a(u, v)| \leqslant c\|u\|\,\|v\|$; $\quad u, v$ from X, $\quad c > 0$ constant

(2) $\quad a(v, v) \geqslant \alpha\|v\|^2$ $\quad$ for every $v \in X$, $\quad \alpha > 0$ constant.

$R \subset X$ is a closed, convex set, f is an arbitrary element of X'.

Problem A. Find such an element u of R such that

(3) $\quad a(u, v - u) \geqslant (f, v - u)$ $\quad$ for every v from R, $\quad$ where $(f, v - u)$ is the value of f on the element $v - u$.

In [1] it is proved, that the problem A has a unique solution. The inequalities of type (3) are called variational inequalities. Their significance lays in the fact that with their help one can search important problems arising in physics and leading to the non-standard boundary problems of the equations with partial derivatives. For illustration this see an example (cf. [4]).

Let Ω be an open, bounded set of the space $\mathbf{R}^n$, Γ the bound of Ω.

Problem B. (cf. [5]). Find the function $u = u(x) = u(x_1, \ldots, x_n)$, where x from Ω, for which

(4) $\quad -\Delta u + u = f$ in Ω, $\quad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $\quad$ and

(5) $\quad u \geqslant 0, \quad \frac{\partial u}{\partial \nu} \geqslant 0, \quad u \cdot \frac{\partial u}{\partial \nu} = 0$ on Γ, $\quad$ where ν is the external normal to Γ.

We verify that the problem B is a simple variational problem of type (3).

Let us denote the Sobolev's space, i.e. the space of functions belonging together with their first derivatives to $L^2(\Omega)$, by $W_2^1(\Omega)$. This is a Hilbert's space, the scalar product is defined by

$$(v, u) = \int_{\Omega} (v \cdot u + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i})\, dx \, .$$

Define a functional on $W_2^1(\Omega)$ (supposing that f from $L^2(\Omega)$)

$$J(v) = \frac{1}{2} \int_{\Omega} (v^2 + \sum_{i=1}^{n} (\frac{\partial v}{\partial x_i})^2)\, dx - \int_{\Omega} fv dx \, .$$

Let $K \subset W_2^1(\Omega)$ be a set of (almost everywhere) non-negative on Γ functions. K is obviously a closed and convex set.

Let us prove that the problem B is equivalent to the problem of finding $\operatorname{Inf} J(v)$, where v from K and consequently, it allows a unique solution. Indeed, the function J(v) is continuous and strictly convex on W_2^1 and $J(v) \to +\infty$ when $\| v \| \to \infty$. There exists a unique u from K, that

(6) $\quad J(u) \leqslant J(v)$

for every v from K. (cf [7]).

Set $a(u, v) = \int_{\Omega} (u \cdot v + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i})\, dx$. It is clear, that

(7) $\quad (J'(u), v) = a(u, v) - (f, v) \, .$

The inequality (6) is equivalent to $(J'(u), v - u) \geqslant 0$ for every v from K .

From this and (7) we obtain

$$(8) \quad \begin{cases} a(u, v - u) \geqslant (f, v - u) & \text{for every } v \text{ from } K \\ u \text{ comes within } K \, . \end{cases}$$

So we came up to the problem A. We shall verify that (8) is equivalent to (4) - (5).

Let us suppose, that the function φ is infinitely differentiable and let has a compact support. (The support of a function φ (denoted $\operatorname{supp} \varphi$) is the closure of the set of the points, where φ is different from zero.) Substitute $v = u \pm \varphi$ in (8). We obtain

(9) $\quad a(u, \varphi) = (f, \varphi)$,

consequently u satisfies (4) in sense of distributions in Ω. Multiply (4) by $v - u$, and integrate the obtained equality on Ω and apply the Green's formula, we obtain

$$-\int_{\Gamma} \frac{\partial u}{\partial \nu} (v - u)\, d\Gamma + a(u, v - u) = (f, v - u) \, .$$

Thus, (8) is equivalent to the following:

$$(10) \quad \begin{cases} u \in K, \quad -\Delta u + u = f \\ \int_{\Gamma} \frac{\partial u}{\partial \nu} (v - u)\, d\Gamma \geqslant 0 \quad \text{for every } v \text{ from } K \, . \end{cases}$$

It can be easily proved that $u \geq 0$ on Γ, $\int\limits_\Gamma \frac{\partial u}{\partial \nu}(v - u)\,d\Gamma \geq 0$ for every v from K are equivalent to (5).

Other problems of physics and functional analysis leading also to variational inequalities one can find in [6]. In this paper we have proved the a priori estimations of the solution of the inequalities of type (3).

A few words on the notations:

$a \in A$ means, that the element a comes within A; $\forall a$ for every a; $\forall a \in A \equiv$ for every a from A $a = \| a_{ij} \|$ a matrix with element a_{ij}.

$(\, , \,)$ the scalar product in the number space $\mathbf{R}^n$ $|\xi|^2 = \sum\limits_{i=1}^{n} \xi_i^2$, $u_x = u_x(x) = (\frac{\partial u}{\partial x_1}, \ldots$ $\ldots, \frac{\partial u}{\partial x_n})$.

The essential maximum of a measurable in Ω function $f(x)$ is the constant M, that $f(x) \leq M$ almost everywhere and $\forall \epsilon > 0$ $\operatorname{mes}\{x \in \Omega: f(x) \geq M - \epsilon\} > 0$.

Riesz's theorem: When $u(x) \in L_p(\Omega)$ $\forall p \geq 1$ then

$$\lim_{p \to \infty} \| u \|_p = \operatorname*{vrai\,max}_{\Omega} |u|,$$

where $\| u \|_p = (\int\limits_\Omega |u|^p dx)^{1/p}$ and $\operatorname*{vrai\,max}\limits_{\Omega} |u|$ is the essential maximum of $|u|$.

THE A PRIORI ESTIMATIONS

Let Ω be a bounded domain in $\mathbf{R}^n$, $\mathring{W}_2^1 = \mathring{W}_2^1(\Omega)$ is the Sobolev's space. ($u \in \mathring{W}_2^1 \Leftrightarrow$ $\Leftrightarrow u \in W_2^1$ and $u = 0$ on $\partial\Omega$) $R = \{v \in \mathring{W}_2^1: v \geq 0 \text{ on } \Omega\}$.

Let $f \in L_2(\Omega)$, $a = \| a_{ij} \|$, $a_{ij} = a_{ij}(x)$ be bounded, measurable functions on Ω, $(|a_{ij}| \leq \Lambda)$ and $(a\xi, \xi) \geq \lambda^{-1} |\xi|^2$, $\forall \xi \in \mathbf{R}^n$, $\forall x \in \Omega$.

Let us see the following problem ("problem A"):

Find the function $u(x)$ from R, that

$$(11) \qquad \int_\Omega (au_x, (V - u)_x)\,dx \geq \int_\Omega f(V - u)\,dx \qquad \forall V \in R.$$

It is known (cf. [1]), that the problem A has a unique solution. Further the properties of this solution are studied.

Theorem 1. Let $u(x)$ be the solution of the problem A;

$$f \in L_p(\Omega), \quad p > \frac{n}{2}.$$

Then $\operatorname{vrai}\max_{\Omega} u^2 \leqslant c\|f\|_{L_p(\Omega)}$, where $c = c(n, p, \lambda, \Omega)$.

Proof. Let the function $F(u)$ satisfy the conditions

a) $F''(u) \geqslant 0$, $F''(u) = 0$, at $u \geqslant u_0$

b) $F(u)$ is defined at $u \geqslant 0$

c) $F(u) \geqslant 0$, $F'(u) \geqslant 0$, $F(0) = 0$.

The function $\frac{F(u) \cdot F'(u)}{u} \equiv \frac{g(u)}{u}$ is bounded on the u-axis, so if $\epsilon > 0$ is sufficiently small then the function $V = u - \epsilon g(u)$ comes within R.

Substitute it (11):

$$\int_\Omega (au_x, (g(u))_x)\, dx \leqslant \int_\Omega f \cdot g(u)\, dx. \tag{12}$$

As $(g(u))_x = F'^2(u)\, u_x + F(u) F''(u)\, u_x$, we obtain from (12)

$$\int_\Omega (au_x F'(u), u_x F'(u))\, dx + \int_\Omega F(u) F''(u) (au_x, u_x)\, dx \leqslant \int_\Omega f F(u) F'(u)\, dx. \tag{13}$$

The second integral in the left-hand side is non-negative, we omit it.

Set $w = F(u)$. From (13) we have

$$\int_\Omega w_x^2 dx \leqslant \lambda \left| \int_\Omega f F(u) F'(u)\, dx \right| \equiv \lambda I.$$

Estimate I in two cases: 1) $F(u) \equiv u$. Then

$$I \leqslant \int_\Omega |f|\, u\, dx \leqslant \|f\|_{L_2} \cdot \|u\|_{L_2}. \tag{14}$$

The following fact is known (Sobolev's inequality, cf. e.g. [3]): if $u \in \mathring{W}_2^1$ then

$$\left(\int_\Omega |u|^{2k} dx\right)^{1/k} \leqslant c(n) \int_\Omega u_x^2 dx, \tag{15}$$

where $1 \leqslant k \leqslant \frac{n}{n-2}$ at $n \geqslant 3$ and k is an arbitrary positive constant at $n = 2$.

From (14) and (15) we obtain the estimation

$$\left(\int_\Omega u^{2k} dx\right)^{1/k} \leqslant c\|f\|_{L_p}. \tag{16}$$

2) $F(u)$ is one of the members of the serie convergent to u^m $(m > 1)$ in norm of C^1. Set $z = u^m$. Suppose that $z \in L_{2q}(\Omega)$, $\frac{1}{q} + \frac{1}{p} = 1$. The function $w = F(u)$ can be chosen so, that $uF'(u) F(u) \leqslant mu^{2m} = mz^2$ and $F(u) F'(u) \leqslant mu^{2m-1} = mz^{2(1-\frac{1}{2m})}$.

Due to Hölder's inequality we have

$$I \leqslant m \,|\int_\Omega f z^{2(1-\frac{1}{2m})} dx\,| \leqslant m \|f\|_{L_p} \cdot (\int_\Omega z^{2q(1-\frac{1}{2m})} dx)^{\frac{1}{q}} .$$

As

$$(\int_\Omega z^{2q(1-\frac{1}{2m})} dx)^{\frac{1}{q}} \leqslant (\operatorname{mes}\Omega)^{\frac{1}{2mq}} \; (\int_\Omega z^{2q} dx)^{(1-\frac{1}{2m})\cdot\frac{1}{q}} \equiv$$

$$\equiv (\operatorname{mes}\Omega)^{\frac{1}{2mq}} \cdot N^{1-\frac{1}{2m}} .$$

(17) $$I \leqslant m \| f \|_{L_p} \cdot m^{-1} (\operatorname{mes}\Omega)^{\frac{1}{2mq}}) \cdot m \cdot N^{1-\frac{1}{2m}} .$$

Apply the inequality $|ab| \leqslant \frac{|a|^\alpha}{\alpha} + \frac{|b|^\beta}{\beta}$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $\alpha > 1$ to (17), setting

$$a = mN^{1-\frac{1}{2m}}, \quad b = m^{-1} (\operatorname{mes}\Omega)^{\frac{1}{2mq}}, \quad \alpha = \frac{2m}{2m-1} \text{ and } \beta = 2m:$$

$I \leqslant c(m^3 N + m^{-2m}) \cdot \| f \|_{L_p}$; that is

(18) $$\int_\Omega w_x^2 dx \leqslant c \| f \|_{L_p} [m^3 (\int_\Omega z^{2q} dx)^{\frac{1}{q}} + m^{-2m}] .$$

In this inequality due to Fatou's lemma one can take $w \to z$ where $w = F(u)$, $z = u^m$:

$$\int_\Omega z_x^2 dx \leqslant c \| f \|_{L_p} [m^3 (\int_\Omega z^{2q} dx)^{\frac{1}{q}} + m^{-2m}] .$$

Having combined this inequality with (15) (at $u = z$) we obtain the estimation

(19) $$(\int_\Omega z^{2k} dx)^{\frac{1}{k}} \leqslant c \| f \|_{L_p} [m^3 (\int_\Omega z^{2q} dx)^{\frac{1}{q}} + m^{-2m}] .$$

Set $k = \frac{n}{n-2}$. As $q = \frac{p}{p-1}$ and $p > \frac{n}{2}$, $k > q$ (at $n \geqslant 3$).

If $n = 2$ then let $k \in (q, 2)$.

Set $m = (\frac{k}{q})^\nu$, $\nu = 1, 2, \dots$, and

$$\Theta_\nu \equiv (\int_\Omega u^{2k^{\nu+1}q^{-\nu}} dx)^{k^{-\nu-1}q^{\nu}} = (\int_\Omega z^{2k} dx)^{\frac{1}{km}} \quad (\text{as } z = u^m) .$$

Note, that $m^3 \leqslant c^\nu$. (19) gives

$$(20)\qquad \frac{\Theta_\nu}{\|f\|_{L_p}^{\frac{1}{m}}} \leqslant (c^{\nu+1}\Theta_{\nu-1}^m + cm^{-2m})^{\frac{1}{m}} \leqslant c^{(\nu+1)(\frac{q}{k})^\nu}[\Theta_{\nu-1} + (\frac{q}{k})^{2\nu}] .$$

From the inequality (15) follows, that $\Theta_0 = (\int_\Omega u^{2k}dx)^{\frac{1}{k}} < \infty$. From (20) we obtain successively $\Theta_\nu < \infty$, $\Theta_2 < \infty$, etc. Iterate (20):

$$\Theta_\nu \leqslant c^{\sum_1^\infty (\nu+1)(\frac{q}{k})^\nu}[\Theta_0 + \sum_1^\infty (\frac{q}{k})^{2\nu}]\|f\|_{L_p}^{\frac{1}{m}} .$$

Due to Riesz's theorem

$$\operatorname{vrai}\max_\Omega u^2 \leqslant \lim_{\nu\to\infty} \Theta_\nu \leqslant c[(\int_\Omega u^{2k}dx)^{\frac{1}{k}} + 1] .$$

Having combined this with (16) we obtain the statement of the theorem.

K_ρ-ball with radius ρ ; $\operatorname{vrai}\max_{K_r} u = M$, $\operatorname{vrai}\min_{K_r} u = m$, $\operatorname{osc}(u; K_r) = M - m$;
$\operatorname{vrai}\max_\Omega u = M_0$.

$\Omega_\delta = \{x \in \Omega:\ \operatorname{dist}(x, \partial\Omega) \geqslant \delta\}$, $\delta < 1$, $\operatorname{dist}(x, \delta\Omega)$ the distance from x to $\delta\Omega$.

Theorem 2. Let $u(x)$ be the solution of the problem A. Then for each $x_1 x_2 \in \Omega_\delta$ the estimation

$$|u(x_1) - u(x_2)| \leqslant c_\delta |x_1 - x_2|^\alpha$$

is valid, where c_δ and α depends only on n, p, λ, Λ, M, f and δ .

Lemma. Let $u(x)$ be the solution of the problem A; $f \in L_p(\Omega)$, $p > \frac{n}{2}$; $r^{\frac{\gamma}{2}} \|f\|_{L_p(K_{2r})} \leqslant 1$ where $\gamma = 1 - \frac{n}{2p}$; $r \leqslant c$.

When $\frac{r^\gamma}{M - m} \leqslant 1$

$$\operatorname{osc}(u; K_{\frac{r}{2}}) \leqslant \eta \cdot \operatorname{osc}(u; K_r) \quad \text{where} \quad \eta \in (0, 1) \text{ is constant.}$$

(If $\frac{r^\gamma}{M - m} > 1$ then obviously $\operatorname{osc}(u; K_{\frac{r}{2}}) \leqslant \operatorname{osc}(u; K_r) = M - m < r^\gamma$).

Proof of the lemma. Suppose that $K_{2r} \subset \Omega$. Let $\psi(x) \geqslant 0$ be a function satisfying the Lipschitz's condition, $\operatorname{supp}\psi(x) \subset K_{2r}$.

$$\eta(|x|) = \begin{cases} 1 & \text{at} \quad 0 \leqslant |x| \leqslant r \\ 0 & \text{at} \quad |x| \geqslant 2r \\ \dfrac{2r - |x|}{r} & \text{at} \quad r \leqslant |x| \leqslant 2r\,. \end{cases}$$

Take one more function $G(\sigma)$ defined at $\sigma > 0$ and satisfying the conditions

1) $G''(\sigma) \geqslant G'^2(\sigma)$, $G'(\sigma) \leqslant 0$, $G(\sigma) \geqslant 0$
2) $G(\sigma) \sim -\ln\sigma$ at $\sigma \to +0$
3) $G(\sigma) = 0$ at $\sigma \geqslant 1$.

There exists such a function, e.g.

$$G(\sigma) = \begin{cases} -\dfrac{\sigma^2}{2} + 2\sigma - \ln\sigma - \dfrac{3}{2} & \text{at} \quad \sigma \in (0, 1] \\ 0 & \text{at} \quad \sigma \geqslant 1\,. \end{cases}$$

There are two cases:

a) $u \geqslant \dfrac{M+m}{2}$ on a certain set $N \subset K_r$ with $\operatorname{mes} N \geqslant \frac{1}{2} \operatorname{mes} K_r$ or

b) $u \leqslant \dfrac{M+m}{2}$ on the set N.

If the case a) is true the function $v^{(1)} = 1 + \dfrac{u - \frac{M+m}{2}}{\frac{M-m}{2}}$ on N is not less then 1. It is clear that $v^{(1)} \geqslant 0$.

Let $\epsilon > 0$ be an arbitrary number, we shall choose it later. The substitution

$$V = u - G'(v^{(1)} + \epsilon)\psi$$

is allowed in (11). Having substituted we have

$$\int (a v_x^{(1)}, (G'(v^{(1)} + \epsilon)\psi)_x)\, dx \leqslant \frac{2}{M-m} \left| \int f \cdot G'(v^{(1)} + \epsilon)\psi\, dx \right|.$$

(Here and further the integrals are taken on $\operatorname{supp}\psi$.)

In the case b) we take the function $v^{(2)} = 1 - \dfrac{u - \frac{M+m}{2}}{\frac{M-m}{2}}$. It is obviously that $v^{(2)} \geqslant 1$ on N and $v^{(2)} \geqslant 0$.

Verify that the substitution

$$V = u + \lambda G'(v^{(2)} + \epsilon)\psi\,,$$

when $\lambda \geqslant 0$ is sufficiently small, is alloved in (11).

Indeed, when $v^{(2)} \geq 1$, $G'(v^{(2)} + \epsilon) = 0$ due to the definition G. And if $v^{(2)} < 1$, then $M - \frac{M-m}{2} v^{(2)} \geq \frac{M+m}{2} > 0$ and when $\lambda > 0$ is sufficiently small $\lambda |G'(v^{(2)} + \epsilon)| < \frac{M+m}{2}$ since $G'(v^{(2)} + \epsilon)$ is bounded.

Substitute V in (11) and divide both sides by $\lambda > 0$:

$$\int (av_x^{(2)}, (G'(v^{(2)} + \epsilon)\psi)_x)\, dx \leq \frac{2}{M-m} \left| \int f \cdot G'(v^{(2)} + \epsilon)\psi\, dx \right| .$$

Thus, both functions $v = v^{(1)}$ and $v = v^{(2)}$ satisfy the inequality

$$\int [(av_x, G''(v+\epsilon) v_x \psi) + (av_x, G'(v+\epsilon)\psi_x)]\, dx \leq \frac{2}{M-m} \int |f \cdot G'\psi|\, dx . \tag{21}$$

As $G'' \geq G'^2$ and $|G'(v+\epsilon)| \leq \frac{1}{\epsilon}$, from (21) follows

$$\int [aG'(v+\epsilon) v_x, G'(v+\epsilon) v_x)\psi + (aG'(v+\epsilon) v_x, \psi_x)]\, dx \leq \frac{2}{\epsilon(M-m)} \int |f|\psi\, dx . \tag{22}$$

If one omit the first component on the left side of (22) (which is non-negative), then one can see from the obtained inequality that the function $z = G(v + \epsilon)$ is the non-negative generalized subsolution of the equation

$$-\sum_{i,j=1}^{n} (a_{ij}(x) z_{x_i})_{x_j} = \frac{2}{\epsilon(M-m)} |f| .$$

It is known (cf. [2]) that if $\frac{r^\gamma}{M-m} < 1$, $\frac{r^\gamma}{\epsilon} < 1$, $r^{\gamma/2} \| f \|_{L_p(K_{2r})}$ then

$$\operatorname*{vrai\,max}_{K_{\frac{r}{2}}} z^2 \leq c(r^{-n} \int_{K_r} z^2\, dx + 1) . \tag{23}$$

Set in (22) $\psi(x) = \eta^2(|x|)$ and $z = G(v + \epsilon)$

$$\int (az_x\eta, z_x\eta)\, dx + 2\int (az_x\eta, \eta_x)\, dx \leq \frac{2}{\epsilon(M-m)} \int |f|\eta^2\, dx . \tag{24}$$

Apply the inequalities $(az_x\eta, z_x\eta) \geq \lambda^{-1} |z_x\eta|^2$ and $2|(az_x\eta, \eta_x)| \leq \Lambda(E|z_x\eta|^2 + \frac{1}{E}\eta_x^2)$ where $E > 0$ is an arbitrary number; we have from (21)

$$\frac{1}{\lambda} \int |z_x\eta|^2\, dx \leq \Lambda E \int |z_x\eta|^2\, dx + \frac{\Lambda}{E} \int \eta_x^2\, dx + \frac{2}{(M-m)\epsilon} \int |f|\eta^2\, dx .$$

Set $E = \frac{1}{2\lambda\Lambda}$:

$$\int |z_x\eta|^2\, dx \leq 4\lambda^2\Lambda^2 \int \eta_x^2\, dx + \frac{4\lambda}{\epsilon(M-m)} \int |f|\eta^2\, dx . \tag{25}$$

Estimate the right-hand side of (25): $4\lambda^2\Lambda^2 \int_{K_{2r}} \eta_x^2\, dx \leq cr^{n-2}$ since $|\eta_x| \leq \frac{1}{r}$.

$$\frac{4\lambda}{(M-m)\epsilon} \int_{K_{2r}} |f| \eta^2 \, dx \leqslant \frac{4\lambda}{(M-m)\epsilon} \|f\|_{L_p(K_{2r})} (\text{mes } K_{2r})^{1-\frac{1}{p}} =$$

$$= c \cdot \frac{r^\gamma}{M-m} \frac{r^{\frac{\gamma}{2}}}{\epsilon} \cdot r^{\frac{\gamma}{2}} \cdot \|f\|_{L_p(K_{2r})} \cdot r^{n-2}$$

since $2\gamma + n - 2 = n(1 - \frac{1}{p})$.

Due to the conditions of the lemma, we have

(26) $$\int_{K_r} z_x^2 \, dx \leqslant c r^{n-2}.$$

It is known (cf. [3] or [2]) that if $w(x) \in W_2^1(K_r)$, $N_1 \subset K_r$, $\text{mes } N_1 \geqslant \frac{1}{2} \text{mes } K_r$ then $r^{-n} \int_{K_r} |w|^2 dx \leqslant c(r^{-n+2} \int_{K_r} w_x^2 \, dx + r^{-n} \int_{N_1} w^2 \, dx)$.

Set here $N_1 = N$, $w = z$. As $v = 1$ on N, $z = G(v + \epsilon) = 0$ on N and consequently

$$\int_{K_r} z^2 \, dx = c r^2 \int_{K_r} z_x^2 \, dx ,$$

and this inequality combined with (26) gives the inequality

(27) $$\int_{K_r} z^2 \, dx \leqslant c r^n .$$

Having combined (27) and (23) we obtain

$$\underset{K_{\frac{r}{2}}}{\text{vrai max}} \, z^2 \leqslant c_1 \quad \text{i.e.} \quad -z \geqslant -\sqrt{c_1} = -c_2$$

or due to the definition of the function $z = G(v + \epsilon)$: $\ln(v + \epsilon) \geqslant -c_2$ that is

(28) $$v + \epsilon \geqslant e^{-c_2} = c_3 \quad \text{in} \quad K_{\frac{r}{2}}, \; c_3 \in (0, 1) .$$

Beside the condition $\epsilon^{-1} \cdot r^{\frac{\gamma}{2}} \leqslant 1$ we require that $\epsilon \leqslant \frac{c_3}{2}$ were fulfilled, which can be reached by decreasing r, since c_3 doesn't depend on r. If it is valid, then from (28) follows that in $K_{\frac{r}{2}}$ $v \geqslant \frac{c_3}{2} = c_4$, $c_4 \in (0, \frac{1}{2})$. This means in case a) that

$$1 + \frac{u - \frac{M+m}{2}}{\frac{M-m}{2}} \geqslant c_4 \quad \text{i.e.} \quad u \geqslant \frac{M-m}{2} c_4 + m \quad \text{in} \quad K_{\frac{r}{2}}$$

$$\operatorname{osc}(u; K_{\frac{r}{2}}) \leqslant M - \frac{M-m}{2} c_4 - m = (1 - \frac{c_4}{2})(M-m) = \eta \cdot \operatorname{osc}(u; K_r) .$$

In case b)

$$1 - \frac{u - \frac{M+m}{2}}{\frac{M-m}{2}} \geqslant c_4 \quad \text{i.e.} \quad u \leqslant M - \frac{M-m}{2} c_4 \quad \text{in} \quad K_{\frac{r}{2}} ,$$

so

$$\operatorname{osc}(u; K_{\frac{r}{2}}) \leqslant M - \frac{M-m}{2} c_4 - m \equiv \eta \operatorname{osc}(u; K_r) .$$

The lemma is proved.

The theorem 2 can be easily obtained from this lemma (cf. [2]) if we set $\alpha = \min(\log_{\frac{1}{2}} \eta, \gamma)$.

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Summary

On the regularity of the solutions of the variational inequalities

The paper deals with apriori estimates of the solutions of the so-called variation inequalities. It proves that on adequate conditions the solutions have bounds and satisfy the Hölder condition.

Р е з ю м е

О регулярности решения вариационных неравенств

В работе рассматриваются априорные оценки решений так называемых вариационных неравенств. Доказывается, что при подходящих условиях решения ограниченные и удовлетворяют условию Гельдера.