Smaller Gershgorin disks for multiple eigenvalues for complex matrices

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ABSTRACT. Extending an earlier result for real matrices we show that multiple eigenvalues of a complex matrix lie in a reduced Gershgorin disk. One consequence is a slightly better estimate in the real case. Another one is a geometric application. Further results of a similar type are given for normal and almost symmetric matrices.

1. INTRODUCTION

Gershgorin's classic result [10] has been a major tool to estimate the eigenvalues of an $n \times n$ (complex) matrix $A = (a_{i,j})_{i,j=1}^{n}$ for the last 90 years. It says that if λ is an eigenvalue of A, then there is $i \in [n] := \{1, \ldots, n\}$ such that

$$|a_{i,i} - \lambda| \leqslant \sum_{j \neq i} |a_{i,j}|.$$

In other words λ lies in the *Gershgorin disk* $D(a_{i,i}, R_i)$ in the complex plane for some $i \in [n]$, where $a_{i,i}$ is the centre, and $R_i = \sum_{j \neq i} |a_{i,j}|$ is the radius of the disk. Equivalently, every eigenvalue λ of A satisfies

$$(1.1) \qquad \lambda \in \bigcup_{i=1}^{n} D(a_{i,i}, R_i).$$

In other words $\lambda \notin \bigcup_{i=1}^{n} D(a_{i,i}, R_i)$ implies that $\mathrm{rank}(A - \lambda I) = n$, where I stands for the $n \times n$ unit matrix.

In particular, if

$$(1.2) \qquad |a_{i,i}| > \sum_{j \neq i} |a_{i,j}| \text{ for every } i \in [n],$$

then $0 \notin \bigcup_{i=1}^{n} D(a_{i,i}, R_i)$ and so the matrix is not singular. Matrices satisfying (1.2) are called *diagonally dominant* matrices. This is a classical application of Gershgorin's theorem: diagonally dominant matrices have full rank, that is, zero is not an eigenvalue because no

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Gershgorin disk contains the origin. There are various generalizations of Gershgorin's theorem see for instance Alon [2] and Alon and Solymosi [3] that imply this case of diagonally dominant matrices.

Under some special conditions, the Gershgorin bound has recently been improved by Bárány and Solymosi [5] followed by Hall and Marsli [11]. Assume A is an $n \times n$ real matrix and let t_i denote the median of the numbers $a_{i,1}, \ldots, a_{i,i-1}, 0, a_{i,i+1}, \ldots, a_{i,n}$ (here $a_{i,i}$ is replaced by 0) and set $r_i = |t_i| + \sum_{j \neq i} |a_{i,j} - t_i|$. It is easy to see (check Section 3 for an argument) that both t_i and r_i can be determined explicitly.

The following result is implicit in [5]; see the inequality in the last lines of page 3 there. A detailed proof appears in Hall and Marsli [11].

Theorem 1.1. *Assume λ is an eigenvalue of the real matrix A whose geometric multiplicity is at least 2. Then $\lambda \in \bigcup_1^n D(a_{i,i}, r_i)$.*

A similar result was proved much earlier, in 1954, by Ky Fan and Hoffman [7]. They again require geometric multiplicity, but allow (geometric) multiplicity higher than 2 and the corresponding radius is given in different terms. They say that it was Hadamard who discovered the basic case of Gershgorin's theorem, namely that condition (1.1) implies rank $A = n$, .

The main aim of this paper is to extend Theorem 1.1 to complex matrices and, simultaneously, to get rid of the condition on "geometric multiplicity". The proof uses Gershgorin's original theorem and is fairly simple. It also gives a stronger result (Corollary 1.3 below) in the real case. Another target is the extension of Theorem 1.1 in the real case by replacing the smaller Gershgorin disks with slightly larger disks when the underlying matrix is normal, and when it is almost symmetric. This is carried out in Sections 5 and 6.

Here comes the statement in the complex case.

Theorem 1.2. *Let A be an $n \times n$ complex matrix and λ be a complex number. If there are $c_i \in \mathbb{C}$ satisfying the inequality*

$$(1.3) \qquad |a_{i,i} - c_i - \lambda| > \sum_{i \neq j} |a_{i,j} - c_i| \text{ for all } i \in [n],$$

then rank $(A - \lambda I) \geqslant n - 1$.

The relation to Gershgorin disks can be seen as follows. Fix the numbers c_i satisfying (1.3) (if they exist) and define $\rho_i = \sum_{i \neq j} |a_{i,j} - c_i|$ for $i \in [n]$. Theorem 1.2 states then, in a form analogous to (1.1), that if λ is an eigenvalue of A with multiplicity at least two, then

$\lambda \in \bigcup_1^n D(a_{i,i} - c_i, \rho_i)$. Note that in this formulation there is no need for any condition on geometric multiplicity.

Checking whether condition (1.3) holds is fairly easy in the real case as we shall see in Section 3. The complex case is more complex and we give some comments on it at the end of Section 3.

Choosing the optimal c_i often reduces the radius of the Gershgorin disk for eigenvalues with a multiplicity of at least two. The $n \times n$ all one matrix, J_n, shows that higher multiplicity might not guarantee a smaller radius in Theorem 1.1 because the zero eigenvalue of J_n has multiplicity $n - 1$ in J_n and still, it lies on the boundary of the disk with radius $r_i = 1$ with center $a_{i,i} = 1$.

Assume now that A is a real $n \times n$ matrix. For all $i \in [n]$ let t_i^* denote the median of the numbers $a_{i,1}, \ldots, a_{i,i-1}, a_{i,i+1}, \ldots, a_{i,n}$ and set $r_i^* = \sum_{j \neq i} |a_{i,j} - t_i^*|$. Theorem 1.2 when applied to the real matrix A gives the following result.

Corollary 1.3. *Assume A is a real $n \times n$ matrix. If $\lambda \notin \bigcup_1^n D(a_{i,i}, r_i^*)$, then* $\mathrm{rank}\,(A - \lambda I) \geqslant n - 1$. *In other words, if λ is an eigenvalue of A with multiplicity two or more, then $\lambda \in \bigcup_1^n D(a_{i,i}, r_i^*)$.*

This is a stronger version of Theorem 1.1 in the sense that λ is a usual (algebraic) eigenvalue of A. At the same time the t_i^* (and then r_i^*) given here are slightly different from the t_i in Theorem 1.1 because t_i is the median of n numbers $0, a_{i,1}, \ldots, a_{i,n}$, where $a_{i,i}$ is missing while t_i^* is the median of $n - 1$ numbers $a_{i,1}, \ldots, a_{i,n}$ ($a_{i,i}$ is missing again). However, Corollary 1.3 is stronger than Theorem 1.1 in this sense as well because $r_i^* \leqslant r_i$.

In general, bounding the rank is important in linear algebra, such bounds have various applications in different parts of mathematics. There are nice examples of applications to combinatorics in Alon's classical papers [1, 2] that provide rank bounds for real matrices where the diagonal elements are larger than other entries in their row, but not large enough for a direct application of Gershgorin's theorem.

2. Proof of Theorem 1.2

We begin with a simple reduction. When $\lambda = 0$ Theorem 1.2 takes the following form.

Lemma 2.1. *Let A be an $n \times n$ complex matrix. Assume there are numbers $c_i \in \mathbb{C}$ satisfying the inequality*

$$(2.1) \qquad\qquad |a_{i,i} - c_i| > \sum_{i \neq j} |a_{i,j} - c_i|,$$

for all $i \in [n]$. Then $\operatorname{rank} A \geqslant n - 1$. $\qquad\qquad\qquad\qquad$ □

The lemma implies the general case by applying it to the matrix $A - \lambda I$. Indeed, the diagonal entries of this matrix are $a_{i,i} - \lambda$ and the off-diagonal ones are simply $a_{i,j}$ so condition (1.3) for A is the same as condition (2.1) for $A - \lambda I$.

The **proof** of the lemma is simple and uses diagonally dominant matrices. Let C be the matrix whose every entry in row i is c_i for $i \in [n]$ and set $D = A - C$. Condition (2.1) says exactly that D is a diagonally dominant matrix. The original Gershgorin theorem shows then that $\operatorname{rank} D = n$. As $\operatorname{rank} A \geqslant \operatorname{rank} D - 1 = n - 1$, the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

We note that when condition (1.3) holds for $c_i = 0$ we get back Gershgorin's circle theorem because then C is the zero matrix that has rank zero.

Remark. One can go one step further. Define C as above (with parameters c_i) and let E be the $n \times n$ complex matrix with every entry in its jth column equal to e_j. We define $D = A - C - E$, so the entry i, j of D is $a_{i,j} - c_i - e_j$. Again, if the parameters c_i, e_j can be chosen so that D is diagonally dominated, then $\operatorname{rank} A \geqslant d - 2$.

3. The real case

For the application of Theorem 1.2 one has to decide if suitable c_i values exist or not. This is fairly easy for real matrices as explained below.

Assume A is a real $n \times n$ matrix and let $b_{i,1} \leqslant \ldots \leqslant b_{i,n-1}$ be the increasing rearrangement of the $n - 1$ numbers $a_{i,1}, \ldots, a_{i,n}$ where $a_{i,i}$ is missing. The function $f(t) = \sum_{i \neq j} |a_{i,j} - t|$ is piecewise linear and convex. It attains its minimum at the median of the numbers $b_{i,j}$. This median is the single point $t_i^* := b_{i,n/2}$ when n is even and can be taken for any point of the interval $J := [b_{i,(n-1)/2}, b_{i,(n+1)/2}]$ when n is odd. We remark that in this case, r_i^* is the same no matter what $t \in J$ is chosen for the median.

Claim 3.1. *In the real case condition (1.3) is satisfied by some c_i if and only if it is satisfied by $c_i = t_i^*$ when n is even and by one of the endpoints of J when n is odd.*

Proof. One direction is easy: there is nothing to check when (1.3) is satisfied at $c_i = t_i^*$ or when it is satisfied by one of the endpoints of J.
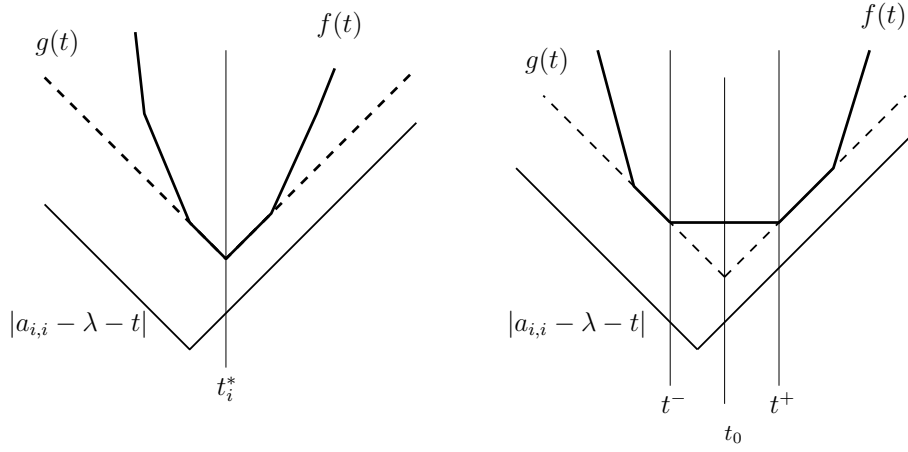
FIGURE 1. The functions $f(t), g(t)$ and $|a_{i,i} - \lambda - t|$.

For the other direction assume first that n is even and that condition (1.3) is not satisfied at t_i^*, that is $|a_{i,i} - t_i^* - \lambda| \leqslant f(t_i^*)$. The function $g(t) = |t - t_i^*| + f(t_i^*)$ is a translated copy of the function $t \to |t|$ and $|a_{i,i} - t - \lambda| \leqslant g(t)$ for all t follows from $|a_{i,i} - t_i^* - \lambda| \leqslant f(t_i^*)$. Moreover $g(t)$ has slope $+1$ for $t > t_i^*$ and slope -1 for $t < t_i^*$, while the slope of $f(t)$ is at least 1 for $t > t_i^*$ and and at most -1 for $t < t_i^*$; see Figure 1 left. Then $g(t) \leqslant f(t)$ for all $t \in \mathbb{R}$ because $g(t_i^*) = f(t_i^*)$ and $|a_{i,i} - \lambda - t| \leqslant f(t)$ for all t follows.

The case of odd n is similar. For simpler notation set $J = [t^-, t^+]$ and $t_0 = \frac{1}{2}(t^- + t^+)$. Suppose that (1.3) is not satisfied at $t = t^-$ and at $t = t^+$. Observe that $f(t^-) = f(t^+)$. Setting $g(t) = |t - t_0| + f(t^+) - \frac{1}{2}(t^+ - t^-)$ we see that $g(t^-) = f(t^-)$ and $g(t^+) = f(t^+)$, see Figure 1 right. Then $|a_{i,i} - \lambda - t| \leqslant g(t)$ for all t because this inequality holds for $t = t^+$ and $t = t^-$. Next one shows that $g(t) \leqslant f(t)$ for all $t \in \mathbb{R}$. This holds because for $t > t^+$ the slope of $g(t)$ is 1 and the slope of $f(t)$ is at least one and for $t < t^-$ the slope of $g(t)$ is -1 and that of $f(t)$ is at most -1. Moreover $f(t)$ is the constant $f(t^+) = f(t^-)$ for $t \in J$, and there $g(t)$ is below this constant. $\qquad\square$

So in the real case in order to check whether condition (1.3) holds and for finding the suitable c_is one has to compare the minimum of $f(t)$ with the value of $|a_{i,i} - \lambda - t|$ at one or two well-defined points.

Finding c_i in the complex case is a different task. We have to decide if the function $t \to |a_{i,i} - t|$ is below the function $t \to \sum_{i \neq j} |a_{i,j} - t|$ for all complex numbers t or not. One can check if this holds when $t = a_{i,j}$ for all $j \in [n]$. At every other $t \in \mathbb{C}$ both functions are differentiable

so one could, in principle, decide if the maximum of the function

$$|a_{i,i} - t| - \sum_{i \neq j} |a_{i,j} - t|$$

is positive for any fixed $i \in [n]$ or not. As the real case indicates, a good candidate for c_i is where the function $f(t) = \sum_{j \neq i} |a_{i,j} - t|$ attains its minimum on $t \in \mathbb{C}$. This point can be determined by convex programming. We remark without proof that $\arg\min f(t)$ is a single point unless all the points $a_{i,j}$ ($j \neq i$) are collinear. The latter case is covered by Claim 3.1.

4. AN EXAMPLE

We show a geometric application of Lemma 2.1 which is similar to an example of Alon [2] and of Bukh and Cox [4]. The unit ℓ_1-ball in $\mathbb{R}^d$ is the convex hull of the $2d$ points $\pm e_i$, i.e., the collection of $x \in \mathbb{R}^d$ such that $\|x\|_1 \leq 1$. Here e_i denotes the vector with a 1 in the ith coordinate and 0's elsewhere. The Euclidean distance between e_i and $-e_i$ is 2 (for all i), and the distance between two other vertices is exactly $\sqrt{2}$. What happens if we relax the distance constraints a bit? What is the maximum number of point pairs, (p_i, q_i) in $\mathbb{R}^d$, to be denoted by k, such that $(p_i - q_i)^2 = 4$ for every $i \in [k]$ and for every pair $i, j \in [k]$, $i \neq j$ we have

$$|(p_i - q_j)^2 - 2| \leq \varepsilon, \ |(p_i - p_j)^2 - 2| \leq \varepsilon, \ |(q_i - q_j)^2 - 2| \leq \varepsilon.$$

The answer is that there are at most two extra point pairs if ε is small enough. More precisely

Claim 4.1. *If $\varepsilon < 2/(3d + 5)$, then $k \leq d + 2$.*

Proof. Assume that $k > d + 2$ and consider the first $d + 3$ pairs (p_i, q_i). The $(d + 2) \times (d + 2)$ matrix M is defined by the dot products $m_{i,j} = (p_i - p_1) \cdot (q_j - p_1)$, for all $2 \leq i, j \leq d + 3$. The rank of M is at most d. The identity $(p_i - p_1)^2 + (q_j - p_1)^2 - (p_i - q_j)^2 = 2(p_i - p_1) \cdot (q_j - p_1)$ (the cosine theorem in trigonometry) and the conditions imply that

$$(2 - \varepsilon) + (2 - \varepsilon) - (2 + \varepsilon) \leqslant 2(p_i - p_1) \cdot (q_j - p_1) \leqslant (2 + \varepsilon) + (2 + \varepsilon) - (2 - \varepsilon).$$

Consequently

$$|m_{i,j} - 1| \leqslant \frac{3}{2}\varepsilon \text{ for } i \neq j \text{ and } |m_{i,i} - 1| \geqslant 1 - \varepsilon \text{ for } 2 \leqslant i \leqslant d + 3.$$

We check that Lemma 2.1 applies now with $c_i = 1$ for all $i = 2, 3, \ldots, d + 3$. Indeed $|m_{i,i} - c_i| \geqslant 1 - \varepsilon$ and $\sum_{j \neq i} |m_{i,j} - c_i| \leqslant (d + 1)\frac{3}{2}\varepsilon$ when $i \neq j$. Thus the inequality $|m_{i,i} - c_i| > \sum_{j \neq i} |m_{i,j} - c_i|$ follows from the

condition $\varepsilon < 2/(3d + 5)$ via a simple computation. By Lemma 2.1, rank $M \geqslant d+2-1 = d+1$, contradicting the fact that rank $M \leqslant d$. $\square$

The following construction shows that the result in Claim 4.1 is close to being sharp. Let us choose a d such that there is a Hadamard matrix, H_n of order $n = d + 2$. The row vectors of H_n and $-H_n$ form a scaled and rotated ℓ_1 ball in $\mathbb{R}^n$. The pointset we are going to consider in $\mathbb{R}^d$ consists of the projection of the $2n$ row vectors $\pm\frac{1}{\sqrt{d}}H_n$ by deleting the last two coordinates of every row. The squared distance between non-antipodal points is between $2 - 4/d$ and $2 + 4/d$ while the distance between the $d + 2$ antipodal points is exactly 2.

One can relax the condition $(p_i - q_i)^2 = 4$ to $|(p_i - q_i)^2 - 4| \leq \varepsilon$. In this case, the proof goes along the same steps as above and gives that for $\varepsilon < 2/(3d + 6)$ the maximal number of such p_i, q_i pairs is at most $d + 2$. We omit the details.

5. Normal Matrices

Here we show that if two eigenvalues of a real and normal matrix are close, then both of them lie in a smaller Gershgorin disk. A matrix is normal if its eigenvectors belonging to distinct eigenvalues are orthogonal. For instance, a symmetric and real matrix is always normal.

Theorem 5.1. *Assume λ and μ are distinct eigenvalues of a real and normal $n \times n$ matrix A, $n \geqslant 3$. Then both λ and μ lie in the disk $D(a_{i,i}, \rho_i)$ for some $i \in [n]$ where $\rho_i = r_i + \sqrt{n}|\lambda - \mu|$.*

Proof. Let A be a normal, real, $n \times n$, matrix. Let $v = (v_1, \ldots, v_n)$ resp. $w = (w_1, \ldots, w_n)$ be the eigenvectors corresponding to λ and μ, the coordinates $v_i, w_j \in \mathbb{C}$. As $n \geqslant 3$ there are $\alpha, \beta \in \mathbb{C}$, such that

$$\sum_{i=1}^{n} \alpha v_i + \beta w_i = 0,$$

and the largest coordinate is one, i.e. $\alpha v_i + \beta w_i = 1$, and $|\alpha v_j + \beta w_j| \leq 1$ for any $j \in [n]$.

Let $t = \alpha v + \beta w$. Then $At = \mu\alpha v + \lambda\beta w$ and

$$a_{i,1}t_1 + a_{i,2}t_2 \ldots + a_{i,i} + \ldots + a_{i,n}t_n = \mu\alpha v_i + \lambda\beta w_i = \mu + (\lambda - \mu)\beta w_i$$

because $t_i = 1$. Set $S = a_{i,1}t_1 + a_{i,2}t_2 \ldots + a_{i,i} + \ldots + a_{i,n}t_n - a_{i,i}$. Then

$$|\mu - a_{i,i}| \leqslant |S| + |\lambda - \mu||\beta w_i|.$$

Bounding S comes from Theorem 2 in [5] and, in a slightly more general form, from Lemma 2.4 of [11] in the form of the following fact.

Lemma 5.2. $|S| \leqslant r_i$.

The proof is the same as in [5] and we present it at the end of this section.

We bound the error term $|\beta w_i|$ using that A is normal. The two eigenvectors are orthogonal, so we have

$$n \geq |t|^2 = |\alpha v|^2 + |\beta w|^2,$$

which implies that $|\beta w_i| \leq \sqrt{n}$. $\qquad\qquad\square$

The **proof** of Lemma 5.2 is simple.

$$
\begin{aligned}
S & = a_{i,1}t_1 + \ldots + a_{i,i-1}t_{i-1} + 0t_i + a_{i,i+1}t_{i+1} + \ldots + a_{i,n}t_n \\
& = (a_{i,1} - x)t_1 + \ldots + (a_{i,i-1} - x)t_{i-1} + \\
& \quad + (0 - x)t_i + (a_{i,i+1} - x)t_{i+1} + \ldots + (a_{i,n} - x)t_n
\end{aligned}
$$

for every $x \in \mathbb{R}$ because $\sum t_i = 0$. As $|t_i| \leqslant 1$ for all $i \in [n]$ this implies that

$$|S| \leqslant |a_{i,1} - x| + \ldots + |a_{i,i-1} - x| + |0 - x| + |a_{i,i+1} - x| + \ldots + |a_{i,n} - x|.$$

The function $x \rightarrow |a_{i,1} - x| + \ldots + |a_{i,n} - x|$ is the sum of convex functions and, as we have seen, its minimum is reached on the median, t_i, of the numbers $a_{i,1}, \ldots, a_{i,i-1}, 0, a_{i,i+1}, \ldots, a_{i,n}$ and equals r_i. $\qquad\square$

6. ALMOST SYMMETRIC MATRICES

This section extends the previous results to almost symmetric real matrices.

There are different ways to define almost symmetric matrices. They appear in various contexts in applied linear algebra. For example in [6] and [8] a nearly symmetric matrix is defined as a matrix in which the overwhelming majority of entries are symmetric about its diagonal. In other examples, an almost symmetric matrix M is given as $M = S + E$ where S is a symmetric matrix and E is a small error or "noise". We can't repeat the arguments on Theorem 5.1 because even small changes can make the eigenvectors far from being orthogonal, like in the example below.

$$M = \begin{bmatrix} 1 & \varepsilon \\ 0 & 1 + \varepsilon \end{bmatrix}$$

We will not give a formal definition of what an almost symmetric matrix is. Instead we introduce a parameter, $\Delta(A)$, measuring the symmetry of a matrix A:

$$\Delta(A) = \max_{i \in [n]} \left| \sum_{j=1}^{n} (a_{i,j} - a_{j,i}) \right|.$$

In this definition the pairs $a_{i,j}$ and $a_{j,i}$ can be far from each other, the parameter $\Delta(A)$ is the maximum difference between the sum of the entries in row i and that of the entries in column i. For instance $\Delta(A) = 0$ for a doubly stochastic matrix A.

With this new matrix parameter, we generalize Theorem 5.1 to arbitrary real matrices. We are going to use the well-known fact that an eigenvector of a real matrix A which belongs to the eigenvalue λ is orthogonal to any eigenvector of A^T which belongs to a different eigenvalue μ.

Theorem 6.1. *Assume that $A = \{a_{i,j}\}_{i,j=1}^n$ is a real matrix and $n \geqslant 3$. If λ and μ are two distinct eigenvalues of A, then there is $i \in [n]$ such that both λ and μ lie in the disk $D(a_{i,i}, \rho_i)$ where $\rho_i = r_i + \sqrt{n}(\Delta + |\lambda - \mu|)$ where r_i is the same as before.*

Proof. Let v be the eigenvector of λ in A and w be the eigenvector of μ in A^T. The two vectors are orthogonal, so they are linearly independent. Then there are $\alpha, \beta \in \mathbb{C}$ such that the coordinates of $\alpha v + \beta w$ add up to zero:

$$\sum_{i=1}^n \alpha u_i + \beta w_i = 0,$$

and the largest norm coordinate (indexed by $i \in [n]$) is one. We assume again that $\alpha v_i + \beta w_i = 1$ and so $|\alpha v_j + \beta w_j| \leqslant 1$ for all $j \in [n]$. With these notations we have the system of equations

$$A\alpha u + A^T \beta w = \lambda \alpha v + \mu \beta w = \lambda(\alpha v + \beta w) + (\mu - \lambda)\beta w,$$

Let us consider the ith row

$$\alpha v_1 a_{1,i} + \beta w_1 a_{i,1} + \ldots + a_{i,i} + \ldots + \alpha v_j a_{j,i} + \beta w_j a_{i,j} + \ldots = \lambda + (\mu - \lambda)\beta w_i.$$

From this we have

$$|\lambda - a_{i,i}| \leq \left| \sum_{j \neq i}^n (\alpha v_j + \beta w_j) a_{j,i} - \beta \sum_{j=1}^n w_j (a_{j,i} - a_{i,j}) - (\mu - \lambda)\beta w_i \right|$$

Using the bound $|\beta w_i| \leq \sqrt{n}$ from the previous proof we have

$$|\lambda - a_{i,i}| \leq \left| \sum_{j \neq i} (\alpha u_j + \beta w_j) a_{j,i} - \beta \sum_{j=1}^n w_j (a_{j,i} - a_{i,j}) - (\mu - \lambda)\beta w_i \right| \leq$$

$$\leq \left| \sum_{j \neq i} (\alpha u_j + \beta w_j) a_{j,i} \right| + \sqrt{n} \left| \sum_{j=1}^n (a_{j,i} - a_{i,j}) \right| + \sqrt{n} |(\mu - \lambda)| \leq$$

$$\leq r_i + \sqrt{n} \left(\Delta(A) + |\mu - \lambda| \right).$$

Here the estimate $\left|\sum_{j\neq i}(\alpha u_j + \beta w_j)a_{j,i}\right| \leqslant r_i$ is the same as in the proof of Theorem 5.1. $\qquad\square$

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