Discrete & Computational Geometry
https://doi.org/10.1007/s00454-023-00531-y

Orientation Preserving Maps of the Square Grid II

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Received: 10 June 2022 / Revised: 20 March 2023 / Accepted: 25 March 2023

Abstract
For a finite set $S \subset \mathbb{R}^2$, a map $\varphi \colon S \to \mathbb{R}^2$ is orientation preserving if for every non-collinear triple $u, v, w \in S$ the orientation of the triangle u, v, w is the same as that of the triangle $\varphi(u), \varphi(v), \varphi(w)$. Assuming that $\varphi \colon G_n \to \mathbb{R}^2$ is an orientation preserving map where G_n is the grid $\{0, \pm 1, \ldots, \pm n\}^2$ and n is large enough we prove that there is a projective transformation $\mu \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\|\mu \circ \varphi(z) - z\| = O(1/n)$ for every $z \in G_n$.

Keywords Order types · Orientation preserving maps · $n \times n$ grid

Mathematics Subject Classification 52B20 · 11H06

1 Introduction

This paper is about orientation preserving maps of the $n \times n$ grid and is a continuation of the results in [1]. We denote by G_n the grid $\{(i, j) \in \mathbb{Z}^2 : -n \le i, j \le n\}$ and by G_n^* the grid $\{(i, j) \in \mathbb{Z}^2 : 1 \le i, j \le n\}$. A map $\varphi \colon G_n \to \mathbb{R}^2$ is *orientation preserving* if for every non-collinear triple $u, v, w \in G_n$ the orientation of the triangle u, v, w is the same as that of the triangle $\varphi(u), \varphi(v), \varphi(w)$, or with a formula

Dedicated to the memory of Eli Goodman.

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Published online: 05 October 2023

$$\operatorname{sign}\det\begin{bmatrix} u & v & w \\ 1 & 1 & 1 \end{bmatrix} = \operatorname{sign}\det\begin{bmatrix} \varphi(u) & \varphi(v) & \varphi(w) \\ 1 & 1 & 1 \end{bmatrix}.$$

Our main result is that given an orientation preserving map $\varphi\colon G_n \to \mathbb{R}^2$ (and n is large enough) there is a projective transformation μ such that $\|\mu \circ \varphi(z) - z\|$ is small for every $z \in G$, namely, it is of order $1/n$. Precisely we have the following result.

Theorem 1.1 *Assume that $n \in \mathbb{N}$ is large and $\varphi\colon G_n \to \mathbb{R}^2$ an orientation preserving map. Then there is a projective transformation $\mu\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\|\mu\circ\varphi(z)-z\| = O(1/n)$ for every $z \in G_n$.*

The bound $O(1/n)$ is best possible which is shown by an example in Sect. 5.

A corollary to our main result is a sharp version of [1, Thm. 1.1] stating that if $\varphi\colon G_n \to \mathbb{R}^2$ is an orientation preserving map, then G_n contains a large subgrid on which an affine image of φ is very close to the identity. Our main result implies a stronger form of this theorem, namely the following.

Theorem 1.2 *For every $k \in \mathbb{N}$ and for every $\varepsilon \in (0, 0.1)$ there is $n = n(k, \varepsilon)$ such that if $\varphi\colon G_n \to \mathbb{R}^2$ is an orientation preserving map, then there is an affine transformation $\alpha\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $a \in \mathbb{Z}^2$ such that $a + G_k \subset G_n$ and for every $z \in a + G_k$*

$$\|\alpha \circ \varphi(z) - z\| < \varepsilon.$$

Here $n(k, \varepsilon) = O(k^2/\varepsilon)$ and this estimate is best possible.

We prove this theorem in Sect. 4. An example showing that the bound $n(k, \varepsilon) = O(k^2/\varepsilon)$ is best possible was given in [1].

The existence of $n(k, \varepsilon)$ in Theorem 1.2 was already proved by Nešetřil and Valtr [7, Lem. 10] as the key tool for proving several Ramsey-type results. However, the proof in the paper [7] relied on repeated compactness arguments, thus it gave no upper bound on n. The first explicit bound $n(k, \varepsilon) = O(k^4/\varepsilon^2)$ was given in [1]. From the (discrete and) computational geometry point of view, the most interesting consequences of any explicit bound, including our new bound $n(k, \varepsilon) = O(k^2/\varepsilon)$, in Theorem 1.2 might be those which are connected with the study of order types, as described in the next section.

2 Motivation and Rigidity

An *order type* of size n is the equivalence class of all ordered n-point sets that can be mapped into each other by a strongly orientation preserving map, where a map $\varphi\colon A \to \mathbb{R}^2$ from a finite planar point set A to $\mathbb{R}^2$ is *strongly orientation preserving* if it is orientation preserving and, additionally, it maps collinear triples of A to collinear triples.

Order types have been studied from various perspectives. For example, a famous result of Goodman et al. [4] and of Kratochvíl and Matoušek [5], states that there are order types of size n with double exponential span. Here the *span* of a finite point set

$A \subset \mathbb{R}^2$ is the ratio between the maximum distance in A and the minimum distance in A. Note that due to projective transformations the supremum of the spans of the sets of any fixed order type (of size at least three) is ∞. We define the *span* of an order type as the infimum of the spans of the point sets in the corresponding equivalence class. A neat and recent result of Goaoc and Welzl [2] states that the expected number of extreme points of a random order type (in $\mathbb{R}^2$) is slightly smaller than 4 when random order types are chosen from the uniform distribution on all order types of size n.

For more information on order types see the paper of Goodman and Pollack [3] for classical results and the recent paper of Pilz and Welzl [8] for further references. Our companion paper [1] explains further connections between order types and orientation preserving maps of G_n.

Our main result says that a certain order type (namely that of G_n^*) is "projectively rigid" meaning that every representation of that order type can be mapped by a projective (and orientation preserving) map to another representation which is "very close" to a fixed order type. Without giving the precise definition, what Theorem 1.1 states in this form is that the order type of G_n^* is projectively rigid with an error of $O(1/n)$. Or in other words, its "projective rigidity" is $O(1/n)$, at least for large n.

Some caution is in place here though. It is clear that the projective rigidity of (the order type of) G_2^* is zero. However one can show that the projective rigidity of G_3^* and G_4^* is infinite. It is not clear what the largest n is for which the projective rigidity of G_n^* is infinite, perhaps it is $n = 5$. We mention further that the projective rigidity of (the order type of) a convex n-gon is again infinite for large enough n. We plan to return to these questions in a follow-up paper to this one.

The paper is organized in the following way. The next section starts with some preparations, states several lemmas, and gives a sketch of the proof of the main theorem. The fairly simple proof of Theorem 1.2 is in Sect. 4, followed by an argument showing that the bound $O(n^{-1})$ is best possible in Theorem 1.1. The lemmas stated in Sect. 3 are proved in later sections, together with some necessary auxiliary results. Lemma 3.2 is proved in Appendix A. The last three sections present some arguments, often routine calculations, that are needed in the previous proofs.

3 Preparations and Sketch of Proof

Throughout the paper we will consider orientation preserving maps φ of some grid G and denote them as the pair (G, φ). Given a map $\varphi\colon G \to \mathbb{R}^2$ and a point $A = A(i, j) = (i, j) \in G$, we denote by $x(i, j), y(i, j)$ or by $\varphi(A)_x, \varphi(A)_y$ the x and y coordinates of $\varphi(i, j)$, that is,

$$\varphi(A) = \varphi(i, j) = (x(i, j), y(i, j)) = (\varphi(A)_x, \varphi(A)_y) \in \mathbb{R}^2 \ .$$

A *vertical block* of G is the set of lattice points in G on a vertical line, a *horizontal block* is the set of lattice points in G on a horizontal line, and a *diagonal block* is the set of lattice points in G on a line whose slope is 1. Given (G, φ) the φ image of a block is called a φ block. We say that (G, φ) is parallel separated, or *p-separated* for short, if the vertical, horizontal, and diagonal φ blocks are separated by parallel lines.

The points $A, B \in \mathbb{R}^2$ are ε-close if their x and y coordinates differ in at most ε, that is, $\|A - B\| < \varepsilon$. Here and throughout the paper $\|\cdot\|$ is the maximum norm. For distinct $A, B \in \mathbb{R}^2$, $L(A, B)$ denotes the line they span, $[A, B]$ the segment they define, and AB their Euclidean distance.

Assume that ε is a positive real. We say that (G, φ) is ε-close if for every point A of the grid $\varphi(A)$ and A are ε-close. Note that this definition makes sense even when ε is not small. Soon we will work with a 16-close (G, φ) pair. With this notation another form of Theorem 1.1 is the following.

Theorem 3.1 *If n is large enough and (G_n, φ) is an orientation preserving pair, then there exists a projective transformation μ such that $(G_n, \mu \circ \varphi)$ is ε-close, where $\varepsilon = O(n^{-1})$.*

We will frequently work with a general $k \times k$ grid G^* which is of the form $\{a, a+1, \ldots, a+k-1\} \times \{b, b+1, \ldots, b+k-1\}$ for some integers $a, b \in \mathbb{N}$. Given such a grid G^* we write $G^*[t]$ for the grid $\{a-t, \ldots, a+k-1+t\} \times \{b-t, \ldots, b+k-1+t\}$ where t is a positive integer. So $G^*[t]$ is a $(k+2t) \times (k+2t)$ grid. We define similarly $G[-t] = \{a+t, \ldots, a+k-1-t\} \times \{b+t, \ldots, b+k-1-t\}$ which is a $(k-2t) \times (k-2t)$ grid. Here we assume that $k > 2t > 0$.

Some further notation: If G^* is a subgrid of G_n and $\varphi \colon G_n \to \mathbb{R}^2$ is a map (orientation preserving or not), then the restriction of φ to G^* is denoted invariably by φ. So it makes sense to say that the pair (G^*, φ) is orientation preserving.

In the proofs we use constants $c, c_0, c_1, c_2, c_3 > 0$, they are universal and explicitly computable. Often we need a lower bound on k (or n) which is always of the form $k > k_0$ where k_0 is again universal and explicit. Frequently we will just say that something holds for large enough k. We will also use the convenient $O(\cdot)$ notation. In such cases the implied constants are again universal and explicitly computable.

The starting point of the proof of Theorem 3.1 is a lemma whose simple proof is given in Appendix A.

Lemma 3.2 *If (G_n, φ) is an orientation preserving pair, $n > 2m^2 - m$ and $m > 2$, then (G_m, φ) is p-separated.*

The proof of Theorem 1.1 has many ingredients and is based on the next five lemmas.

Lemma 3.3 *Let $k - 2$ be a multiple of 4, and G^* be a $k \times k$ grid. If (G^*, φ) is p-separated, then there exists an affine transformation ν such that $(G^*[-1], \nu \circ \varphi)$ is 16-close.*

Lemma 3.4 *If (G_k, φ) is orientation preserving, 16-close, and k is large enough, then there exists a projective transformation μ such that $\mu \circ \varphi$ is the identity on the four vertices of G_k and $(G_k, \mu \circ \varphi)$ is orientation preserving and c-close, where c is an explicit constant.*

Let (G^*, φ) be an orientation preserving pair with G^* a $k \times k$ grid. A *unit cell*, or simply a *cell*, $Q(i, j)$, of G^* is the set of four points $(i, j), (i+1, j), (i, j+1), (i+1, j+1) \in G^*$. Given $\varepsilon > 0$ the cell $Q(i, j)$ is said to be *ε-close* if for some real numbers x, y

the following holds:

$$\|\varphi(i,j)-(x,y)\|<\varepsilon, \qquad \|\varphi(i+1,j)-(x+1,y)\|<\varepsilon,$$
$$\|\varphi(i,j+1)-(x,y+1)\|<\varepsilon, \qquad \|\varphi(i+1,j+1)-(x+1,y+1)\|<\varepsilon.$$

Lemma 3.5 *Assume G^* is a $k\times k$ grid, and (G^*,φ) is an orientation preserving pair which is c-close and $k>k_0$ where k_0 depends only on c. Then every unit cell in $G^*[-1]$ is ε-close with $\varepsilon=c_1/k$.*

Lemma 3.6 *Assume k is large and is a multiple of 8, and (G_k,φ) is an orientation preserving pair which is c-close and every cell in G_k is (c_1/k)-close. If φ is the identity on the vertices of G_k, then $\varphi(i,j)$ is (c_2/k)-close to (i,j) for every $(i,j)\in G_k$.*

Lemma 3.7 *Assume that $\delta<0.01$ and $n>k$, k even and large, and $\varphi\colon G_n\to\mathbb{R}^2$ is a map such that its restriction to $G_k[20]$ is orientation preserving and the pair (G_k,φ) is δ-close. Then the restriction of φ to $G_k[20]$ is 22δ-close.*

The proof of Theorem 1.1 goes via a recursion during which we will often change φ to $\mu\circ\varphi$ where μ is a projective (or affine) map. It will be convenient to keep the same notation, that is, to rename $\mu\circ\varphi$ as φ. We hope that this will not cause any confusion.

Here comes a quick sketch of the proof. For the starting step we assume n is large and choose the largest odd $m\in\mathbb{N}$ with $n>2m^2-m$. Lemma 3.2 shows that (G_m,φ) is p-separated. Using Lemma 3.3 we find an affine map ν such that $(G_{m-2},\nu\circ\varphi)$ is 16-close and is of course orientation preserving. Set $k=m-2$ which is odd. The starting point of the recursion is the pair $(G_k,\nu\circ\varphi)$, or with our convenient convention (G_k,φ) which is 16-close and orientation preserving and (G_n,φ) well-defined (k is odd).

In a general step of the recursion we have a pair (G_k,φ) (with k odd) which is 16-close and orientation preserving and φ is defined on G_n (but may not be orientation preserving on the whole G_n). We apply Lemma 3.4. Then the new pair $(G_k,\mu\circ\varphi)$, or rather (G_k,φ) is orientation preserving and c-close (with the constant c from the lemma) and φ is the identity on the four vertices of G_k. The map φ is still defined on G_n and we show (in Lemma 8.1) that it is orientation preserving on the subgrid $G_k[20]=G_{k+20}$.

Several properties of the pair (G_k,φ) have to be established next. For instance, Lemma 10.1 says that the φ image of the four vertices of the cell $Q(i,j)\subset G_k[-1]$ are very close to the vertices of a unit square. The midpoint lemma (Lemma 11.1) shows that when $M\in G^*$ is the midpoint of the segment $[A,B]$ where $A,B\in G^*$, then $\varphi(M)$ is very close to the line $L(\varphi(A),\varphi(B))$; this holds when the direction of $L(A,B)$ is one of eight special directions; for the details see Sect. 11. We will encounter the case when (G_{2k},φ) is c-close and its cells are ε-close and we have to estimate how far $\varphi(0,0)$ deviates from $(0,0)$ as a function of the deviations at the four vertices of G_{2k}. This, together with similar deviation estimates, is carried out in Sect. 12. These estimates are used in the proof of Lemma 3.6 in Sect. 13.

Next we apply Lemma 3.6 to show that (G_{k^*},φ) is (c_2/k)-close for the largest $k^*\le k-2$ which is divisible by 8. Lemma 3.7 shows that the pair $(G_{k^*}[20],\varphi)=$

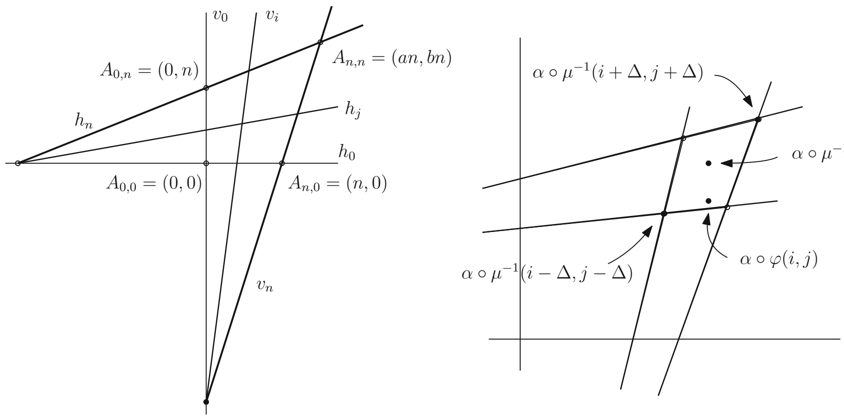

Fig. 1 Lines h_j, v_i, and the $\alpha \circ \mu^{-1}$ image of the square $[i \pm \Delta, j \pm \Delta]$

(G_{k^*+20}, φ) is orientation preserving and 16-close (because $22\delta < 16$), and (G_n, φ) is well defined. So we can move to the next step of the recursion with (G_{k^*+19}, φ). Here of course $k^* + 19$ is odd and larger than k, actually $k^* \geq k - 10$. In the final step (G_{k^*+20}, φ) is $O(1/n)$-close by Lemma 3.7.

4 Proof of Theorem 1.2

It will be convenient to work with the grid $\{0, 1, \ldots, n\}^2$, to be denoted by G^*. Suppose (G^*, φ) is an orientation preserving pair, and φ maps points of G^* to points in the so called φ-plane P. Originally only the points $\varphi(A)$, $A \in G^*$, are known in P. Together with the projective map μ from Theorem 1.1 we also have, in the image space of $\mu \circ \varphi$ (which we call the μ-plane) the vertical lines $V_i = \{(x, y) : x = i\}$ and the horizontal lines $H_j = \{(x, y) : y = j\}$ for $i, j \in \{0, 1, \ldots, n\}$. These lines form a chessboard like structure.

The lines $v_i = \mu^{-1}(V_i)$ lie in P and are concurrent, and so are the lines $h_j = \mu^{-1}(H_j) \subset P$. The lines h_0, h_n, v_0, v_n bound a convex quadrilateral with vertices $A_{i,j} = h_i \cap v_j$ for $i, j \in \{0, n\}$. Because of the symmetry of the chessboard we may assume that $A_{0,0}$ is contained in both segments $[A_{n,0}, h_0 \cap h_n]$ and $[A_{0,n}, v_0 \cap v_n]$; see Fig. 1, left.

Now we introduce a coordinate system in P by setting $A_{0,0} = (0, 0)$, $A_{n,0} = (n, 0)$, $A_{0,n} = (0, n)$, see Fig. 1 as well. Then $A_{n,n}$ is a well-defined point $v_n \cap h_n = (an, bn)$ where $1 < a, b$. With this convention we see that

$$\mu^{-1}(x, y) = \frac{n(ax, by)}{(1 - b)x + (1 - a)y + n(a + b - 1)}.$$

We want to show that (i, j) and $\varphi(i, j)$ (for $i, j \leq k$) are close to each other in the φ-plane P. For that we need another affine, or in this case linear, map $\alpha \colon P \to P$, namely the one given by

$$\alpha(x, y) = (a + b - 1)\left(\frac{x}{a}, \frac{y}{b}\right).$$

This is the affine map we are looking for: (i, j) and $\alpha \circ \varphi(i, j)$ are close to each other (for suitable values of i, j), as we shall see soon. Setting $D = a + b - 1$ we determine first

$$(X, Y) = \alpha \circ \mu^{-1}(i, j) = \frac{(nDi, nDj)}{nD - (b-1)i - (a-1)j}.$$

Introduce the notation $E = (b-1)i + (a-1)j$. As $0 \le i, j \le k$ we have $E \le (a + b - 2)k \le (D-1)k$ and

$$\begin{aligned} 0 \le X - i &= \frac{nDi}{nD - E} - i \le k\frac{E}{nD - E} \\ &< k\frac{(D-1)k}{nD - (D-1)k} < \frac{k^2}{n} \cdot \frac{D-1}{D - (D-1)k/n} < \frac{2k^2}{n} \end{aligned}$$

because $k/n < 1/2$ and $D - 1 < 2(D - (D-1)k/n)$ as one can check. The same way $0 < Y - j < 2k^2/n$, showing that

$$\|\alpha \circ \mu^{-1}(i, j) - (i, j)\| < \frac{2k^2}{n}.$$

The points (i, j) and $\mu \circ \varphi(i, j)$ in the μ-plane are within distance $O(1/n)$ of each other because of Theorem 1.1. For concreteness we assume $\|(\mu \circ \varphi(i, j) - (i, j)\| < c_0/n =: \Delta$. So these points lie in the square with vertices $(i \pm \Delta, j \pm \Delta)$ in the μ-plane. Consequently the $\alpha \circ \mu^{-1}$ images of (i, j) and $\mu \circ \varphi(i, j)$ lie in the $\alpha \circ \mu^{-1}$ image of this square, see Fig. 1, right. We compute the x and y component of the vector

$$v := \alpha \circ \mu^{-1}(i + \Delta, j + \Delta) - \alpha \circ \mu^{-1}(i - \Delta, j - \Delta).$$

This will give an upper bound on $\|\alpha \circ \varphi(i, j) - \alpha \circ \mu^{-1}(i, j)\|$ because $\alpha \circ \varphi(i, j) = \alpha \circ \mu^{-1} \circ \mu \circ \varphi(i, j)$.

Recall the definition of E and note that $Dn - E > Dn/2$ because $D > 1$ and $k < n/2$. The x component of v is

$$\begin{aligned} v_x &= \frac{nD(i + \Delta)}{nD - E - (D-1)\Delta} - \frac{nD(i - \Delta)}{nD - E + (D-1)\Delta} \\ &= 2nD\Delta\frac{(D-1)i + (nD - E)}{(nD - E)^2 - (D-1)^2\Delta^2} \\ &< 2c_0 D\frac{(D-1)k + (nD - E)}{(nD - E)^2} < \frac{2c_0}{n}\left(4\frac{D-1}{D} \cdot \frac{k}{n} + 2\right) < \frac{8c_0}{n}. \end{aligned}$$

The same estimate holds for the y component of v. It follows that $\|\alpha\circ\varphi(i,j)-\alpha\circ\mu^{-1}(i,j)\| < 8c_0/n$. Finally we have

$$\begin{aligned}&\|\alpha\circ\varphi(i,j)-(i,j)\|\\ &\quad\le \|\alpha\circ\varphi(i,j)-\alpha\circ\mu^{-1}(i,j)\|+\|\alpha\circ\mu^{-1}(i,j)-(i,j)\|\\ &\quad\le \frac{2k^2+8c_0}{n}<\varepsilon\end{aligned}$$

if $n=3k^2/\varepsilon$ for $k\ge k_0=\sqrt{8c_0}$. For $k<k_0$ define $n_0=(2k_0^2+8c_0)/\varepsilon=(24c_0)/\varepsilon$. Then $G^*_{n_0}$ contains a copy of $G^*_{k_0}$ such that with a suitable affine map α, we have $\|\alpha\circ\varphi(z)-z\|<\varepsilon$ for every z in that copy. Every G^*_k in that $G^*_{k_0}$ satisfies the requirements, and $k\ge 2$. Thus $n=(6c_0k^2)/\varepsilon$ works for all $k<k_0$.

5 The Bound in Theorem 1.1 is Best Possible

Here (and later) we need a simple claim about properties of projective maps. Assume $\eta>0$ is small and let $A_{0,0},A_{0,1},A_{1,0},A_{1,1}\in\mathbb{R}^2$ be points such that $\|A_{i,j}-(i,j)\|<\eta$ for all $i,j\in\{0,1\}$. There is a unique projective map $\mu\colon\mathbb{R}^2\to\mathbb{R}^2$ with $\mu(A_{i,j})=(i,j)$ for all $i,j\in\{0,1\}$.

Claim 5.1 *For every $z\in[-\eta,1+\eta]^2$ we have $\|\mu(z)-z\|<12\eta+O(\eta^2)$.*

The proof is in Appendix B. We mention that with more effort one can show $\|\mu(z)-z\|<2\eta+O(\eta^2)$ but that is not important for us.

Now we begin the proof that the bound in Theorem 1.1 is best possible. Given two distinct points of G_n, their line either contains the origin, or intersects the x-axis at a point $(x,0)$ with $|x|\ge 1/(2n)$. This is true because x is the solution of a linear equation with every coefficient an integer between $-2n$ and $2n$. The same applies to the intersection point with the y-axis. So every line determined by two points of G_n either passes through the origin or avoids the interior of the convex hull of the four points $(\pm1/(2n),\pm1/(2n))$.

For the example showing that $O(1/n)$ is best possible in Theorem 1.1 we take φ to be the identity at every point of G_n except at the origin where $\varphi(0,0)=z:=(1/(3n),0)$. The previous argument shows that (G_n,φ) is an orientation preserving pair. Assume now that ν is a projective map with $\|\nu\circ\varphi(z)-z\|<\varepsilon$ for every $z\in G_n$. We are going to show that $\varepsilon>1/(50n)$.

We set $\nu(-1,1)=A_1$, $\nu(1,1)=A_2$, $\nu(-1,-1)=A_3$, $\nu(1,-1)=A_4$. These values determine the projective map ν uniquely. We *claim* that $u=\nu^{-1}(z)$ lies in the square $[-1,1]^2$. Observe that A_1 is in the square of side length 2ε centred around $(-1,1)$, and the other A_is also are in the corresponding small squares. As ν is one-to-one, it maps $[-1,1]^2$ to the quadrilateral conv$\{A_1,A_2,A_3,A_4\}$ which contains the square $[-1+\varepsilon,1-\varepsilon]^2$. So u lies indeed in the square $[-1+\varepsilon,1-\varepsilon]^2\subset[-1,1]^2$.

Set $\mu=\nu^{-1}$ and apply Claim 5.1 to μ, this time in the square $[-1,1]^2$ (instead of $[0,1]^2$ but that does not matter). This shows that $\|\mu(u)-u\|<12\varepsilon+O(\varepsilon^2)$. This is

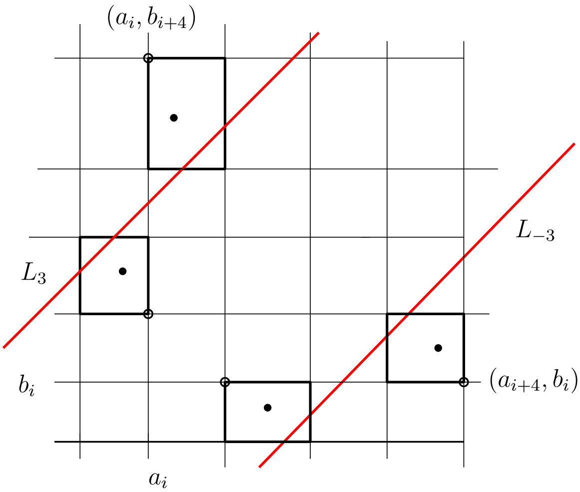

Fig. 2 The lines L_3 and L_{-3}

the same as $\|\nu(z) - z\| < 12\varepsilon + O(\varepsilon^2)$ because $\mu(u) - u = z - \nu(z)$. On the other hand, $\|\nu(z) - (0,0)\| = \|\nu \circ \varphi(0,0) - (0,0)\| < \varepsilon$. Thus

$$\frac{1}{3n} = \|z - (0,0)\| \le \|z - \nu(z)\| + \|\nu(z) - (0,0)\| < 13\varepsilon + O(\varepsilon^2)$$

implying that $\varepsilon > 1/(50n)$.

6 Proof of Lemma 3.3

We set $[k] = \{1, \ldots, k\}$. Assume that $G^* = [k] \times [k]$ and apply an affine transformation ν such that, for the $\nu \circ \varphi$ blocks, the horizontal separator lines become horizontal, the vertical separators become vertical and the diagonal separators have slope 1. With our convenient convention we assume that this is already the case for φ. After that we can still shift/scale the image without changing the slope of any line.

We have $k - 1$ vertical and horizontal separator lines and $2k - 2$ diagonal separator lines. Let $a_1, \ldots, a_{k-1}$ be the x-coordinates of the vertical separators and $b_1, \ldots, b_{k-1}$ be the y-coordinates of the horizontal separators. Let both of these sequences be increasing. Apply a shift/scale such that $a_1 = b_1 = 1$ and $a_{k-1} + b_{k-1} = 2k - 2$. Observe that for $1 \le i, j \le k$ the image $\varphi(i, j)$ is in the open rectangle $(a_{i-1}, a_i) \times (b_{j-1}, b_j)$, where $a_0 = b_0 = -\infty$ and $a_k = b_k = \infty$.

Let L_3 be the diagonal separator that separates the φ image of the diagonal blocks G^* on the grid line $y = x + 3$ and on the grid line $y = x + 2$; see Fig. 2. Similarly let L_{-3} be the diagonal separator separating the φ image of the diagonal blocks on the grid line $y = x - 3$ and on the grid line $y = x - 2$.

The rectangle $(a_{i-1}, a_i) \times (b_{i+1}, b_{i+2})$ contains $\varphi(i, i+2)$ which is below the line L_3 and therefore (a_i, b_{i+1}) is also below the line L_3 (see Fig. 2). Similarly we get that the point (a_i, b_{i+4}) is above the line L_3, the point (a_{i+1}, b_i) is above the line L_{-3}, the point (a_{i+4}, b_i) is below the line L_{-3}. Let $\Delta a_i = a_i - a_{i-1}$ and $\Delta b_i = b_i - b_{i-1}$ and $\Delta_i = \Delta a_i + \Delta b_i$.

Let $d/\sqrt{2}$ be the distance of the two lines L_3 and L_{-3}. Then the length of any axis-aligned monotone decreasing polygonal path from L_3 to L_{-3} is d. Based on the position of the grid points relative to the two diagonals L_3 and L_{-3} we see that

$$\Delta_i = \Delta a_i + \Delta b_i < d < \sum_{j=0}^{3} (\Delta a_{i+j} + \Delta b_{i+j}) = \Delta_i + \Delta_{i+1} + \Delta_{i+2} + \Delta_{i+3}.$$

Observe that there are $k - 2$ Δ_i, namely $\Delta_2, \dots, \Delta_{k-1}$, and they add up to

$$\sum \Delta_i = a_{k-1} + b_{k-1} - a_1 - b_1 = 2k - 2 - 1 - 1 = 2(k-2)$$

Let D be the maximum of Δ_i for $2 \le i \le k-1$. Their average is 2 therefore $2 \le D$. We have $\Delta_i < d$ for every i so $D < d$. Finally $k - 2$ is a multiple of 4 so we can group the sum $\sum \Delta_i$ into groups of size 4, each adding up to more than d, so $d < 8$. So we have $2 \le D < d < 8$. The strip between L_3 and L_{-3} contains all the points (a_i, b_i) including $(1, 1) = (a_1, b_1)$, therefore $|a_i - b_i| < d < 8$. Particularly $|a_{k-1} - b_{k-1}| < 8$, and since $a_{k-1} + b_{k-1} = 2k-2$ we have $k-5 < a_{k-1}, b_{k-1} < k+3$. Let $E_i = a_i + b_i - 2i$ for $1 \le i \le k-1$. By our choice of shift/scale we have $E_1 = E_{k-1} = 0$. Let E be the maximum of all $|E_i|$ for $1 \le i \le k-1$, say $E = |E_j|$. We claim that $E \le 22$.

We can assume here that E_j is positive because reflecting the rectangle $[a_1, a_{k-1}] \times [b_1, b_{k-1}]$ with respect to its centre would produce the same situation just the values E_j are changed to $-E_j$. Assume on the contrary that $E = E_j > 22$. Observe that $E_i - E_{i-1} = \Delta_i - 2$ and therefore $-2 < E_i - E_{i-1} < 6$. Particularly $|E_2|, |E_{k-2}| < 6$, and we may assume that $3 \le j \le k-3$. We distinguish two cases based on whether j is "closer" to 1 or $k-1$.

Consider the line $x + y = 2j + E$, which contains the point (a_j, b_j). Clearly $\varphi(j+1, j+1)$ is above that line. But then one of the points (if they exist) $\varphi(1, 2j+2)$ or $\varphi(2j+2, 1)$ has to be above that line as well, otherwise φ would reverse the orientation of the triangle $(2j+2, 1), (j+1, j+1), (1, 2j+2)$. Similarly, one of the two points (if they exist) $\varphi(k-1, 2j+4-k)$ or $\varphi(2j+4-k, k-1)$ has to be above that line as well. The first two points exist if $2j \le k-3$ and the latter two exist if $2j \ge k-3$ and $j \le k-3$.

Case 1: $2j \le k-3$. Without loss of generality we can assume that $\varphi(1, 2j+2)$ is above the line $x + y = 2j + E$. Then (a_1, b_{2j+2}) is above that line and therefore $b_{2j+2} > 2j + E - 1$. Then $a_{2j+2} > b_{2j+2} - 8 > 2j + E - 9$ and

$$\begin{aligned} E_{2j+2} &= a_{2j+2} + b_{2j+2} - 4j - 4 \\ &> 2j + E - 1 + (2j + E - 9) - 4j - 4 = 2E - 14 > E, \end{aligned}$$

which is a contradiction.

Case 2: $2j \geq k-3$. We can assume again that $\varphi(k-1, 2j+4-k)$ is above the line $x+y = 2j+E$. Then (a_{k-1}, b_{2j+4-k}) is above that line and therefore $b_{2j+4-k} > 2j+E-a_{k-1} > 2j+E-(k+3)$. Then $a_{2j+4-k} > b_{2j+4-k} - 8 > 2j+E-k-11$ and

$$\begin{aligned} E_{2j+4-k} &= a_{2j+4-k} + b_{2j+4-k} - 4j - 8 + 2k \\ &> (2j+E-k-3) + (2j+E-k-11) - 4j - 8 + 2k = 2E - 22 > E, \end{aligned}$$

which is a contradiction.

For $1 \leq i \leq k-1$ we have $-22 \leq a_i + b_i - 2i \leq 22$ and since a_i and b_i are less than 8 apart, we get that $-15 < a_i - i, b_i - i < 15$. The claim of the lemma follows from the fact that for $2 \leq i, j \leq k-1$ the point $\varphi(i, j)$ has x and y-coordinates in the intervals $[a_{i-1}, a_i]$ and $[b_{j-1}, b_j]$. So $i - 16 < a_{i-1} < x(i, j) < a_i < i + 15$.

Remark 6.1 This proof uses an argument that we like to call "doubling": if E_i is large, then so is E_{2i}, or rather E_{2i+2}. Assuming E_j is maximal, E_{2j+2} is large but at most E_j, giving upper bound on E_j. We will use this doubling argument later as well.

7 Proof of Lemma 3.4

This proof is fairly simple. We are going to use Claim 5.1 for the square $[-k, k]^2$ instead of the unit square $[0, 1]^2$. The parameter η is replaced by 16 which is fine because k is large. In this case the claim says that for the unique projective map μ satisfying

$$\begin{aligned} \mu(\varphi(-k, -k)) &= (-k, -k), \qquad & \mu(\varphi(k, -k)) &= (k, -k), \\ \mu(\varphi(-k, k)) &= (-k, k), \qquad & \mu(\varphi(k, k)) &= (k, k), \end{aligned}$$

and $\|\mu(z) - z\| < 12 \cdot 16 + O(k^{-1})$ for every $z \in [-k-16, k+16]^2$. Since $\varphi(A) \in [-k-16, k+16]^2$,

$$\|\mu \circ \varphi(A) - A\| \leq \|\mu \circ \varphi(A) - \varphi(A)\| + \|\varphi(A) - A\| < 192 + O(k^{-1}) + 16 < 210$$

for large enough k. The lemma holds with $c = 210$.

To complete the proof we have to see that $(G_k, \mu \circ \varphi)$ is orientation preserving. Let ℓ be the line that μ maps to the line at infinity. We are going to show that ℓ avoids the square $[-k-16, k+16]^2$ containing $\varphi(G_k)$. Assume that (G_k, φ) is c-close (with k much larger than $3c$).

The intersection point of the lines $L_1 = L(\varphi(-k, -k), \varphi(k, -k))$ and $L_2 = L(\varphi(-k, k), \varphi(k, k))$ is $P = (P_x, P_y)$, see Fig. 3 where the four small squares are centred around $(\pm k, \pm k)$ and their side lengths are 32. It is not hard to check that $|P_x| \geq k(k-c)/c$. The lines $\ell_1 = L((-k+c, -k+c), (k-c, -k-c))$ and $\ell_2 = L((-k+c, k-c), (k-c, k+c))$ split the plane into four cones. Let C be the one that contains the origin. Again it is easy to see that if $P_x > 0$, then

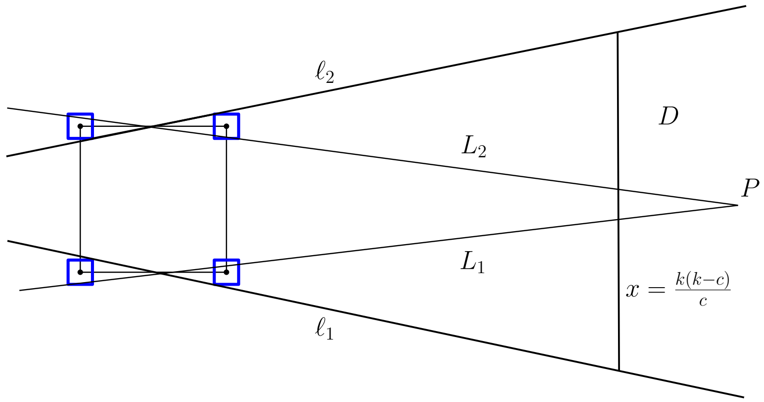

Fig. 3 The set D contains P

$P \in C$. This shows that when $P_x > 0$, then P is contained in the convex set $D = \{(x, y) \in C : x \geq k(k-c)/c\}$, see Fig. 3. Symmetrically, $P \in -D$ if $P_x < 0$.

Let $Q = (Q_x, Q_y)$ be the intersection point of $L(\varphi(-k,-k), \varphi(-k,k))$ and $L(\varphi(k,-k), \varphi(k,k))$. Let E be the copy of D, rotated around the origin by angle $+\pi/2$; E is not shown in Fig. 3. By symmetry $Q \in E$ if $Q_y > 0$, otherwise $Q \in -E$. A simple computation reveals that every line passing through a point of D and a point of E is at least at distance $k(k-3c)/(2c)$ from the origin. Then by symmetry, the line $L(P, Q)$ is at least this far away from the origin and avoids the square $[k-16, k+16]^2$.

The map μ carries the line $\ell = L(P, Q)$ to the line at infinity, and so μ does not change the orientation of a triangle in G_k if its vertices are on the same side of ℓ as the origin.

8 Lemma 8.1 and Its Proof

On a step of the recurrence, after the application of the projective map μ, there is an orientation preserving and c-close pair (G_k, φ). Let ℓ be the line in the plane of G_n that μ sends to the line at infinity. The next lemma is about this situation.

Lemma 8.1 *Under the above conditions $(G_k[20], \varphi)$ is also orientation preserving provided k is large enough.*

Proof Let ℓ^+ denote the halfplane (with bounding line ℓ) that contains G_k. It is evident that the orientation of a triangle with vertices $A, B, C \in G_n$ is the same as that of the triangle $\varphi(A), \varphi(B), \varphi(C)$ if and only if ℓ^+ contains one or three of the points A, B, C. So it suffices to prove that $(a, b) \in \ell^+$ for every $(a, b) \in G_k[20] \setminus G_k$. Define $A = (a, b)$. Because of symmetry we may assume that $k < a \leq 20$ and $0 \leq b \leq k + 20$.

Assume that, on the contrary, $A \notin \ell^+$. Then $\varphi(A)$ and $\varphi(k, k)$ are on opposite sides of the line $L(\varphi(B), \varphi(C))$ for every pair of points $B, C \in G_k$ such that A and (k, k) are on the same side of $L(B, C)$. Define $B_1 = (k/2, -k)$, $B_2 = (k/2, k)$, and

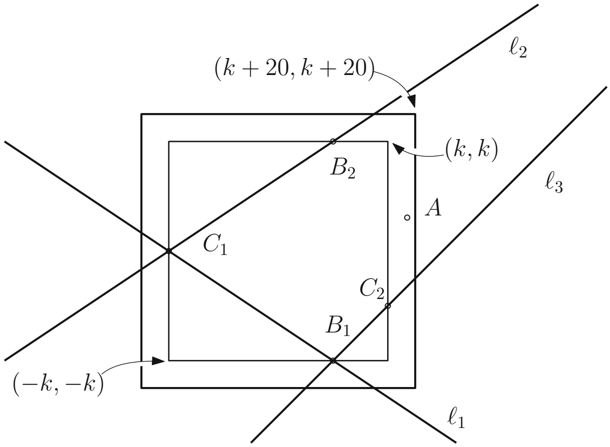

Fig. 4 Illustration for Lemma 8.1

$C_1 = (-k, 0)$, $C_2 = (k, -k/2)$, assuming that k is even; if it is not then replace $k/2$ by $(k+1)/2$. Set $L_1 = L(\varphi(B_1), \varphi(C_1))$, $L_2 = L(\varphi(B_2), \varphi(C_1))$, and $L_3 = L(\varphi(B_1), \varphi(C_2))$, and denote by L_i^+ the halfplane with bounding line L_i that contains $\varphi(k, k)$ for $i = 1, 2, 3$, and by L_i^- the complementary (open) halfplane. Since A and (k, k) are on the same side of each one of the lines $\ell_1 := L(B_1, C_1)$, $\ell_2 := L(B_2, C_1)$, $\ell_3 := L(B_1, C_2)$ (see Fig. 4), it follows that $\varphi(k, k) \in \bigcap_1^3 L_i^+$ and $\varphi(A) \in \bigcap_1^3 L_i^-$. For lack of space Fig. 4 does not show the lines L_i.

The points $\varphi(B_i)$ and $\varphi(C_i)$ lie in the small squares of side lengths $2c$ centred at B_i and C_i, respectively. As k is much larger than c, the direction of the line L_i is very close to that of ℓ_i for $i = 1, 2, 3$. Denote by ℓ_i^+ the halfplane with bounding line ℓ_i that contains (k, k) for $i = 1, 2, 3$. The intersection $\bigcap_1^3 \ell_i^+$ is a triangle Δ containing (k, k). It follows that $\bigcap_1^3 L_i^+$ is also a triangle which is close to Δ. But then $\bigcap_1^3 L_i^-$ is the empty set, contradicting $\varphi(A) \in \bigcap_1^3 L_i^-$. □

Remark 8.2 This proof shows that $(G_k[m], \varphi)$ is also orientation preserving for m larger than 20, up to probably $m \le k/10$ but we don't need this.

9 Preparations for Lemma 3.5

This section gives an auxiliary result for Lemma 3.5. Recall that the target there is to show that every cell in $G^*[-1]$ is ε-close with $\varepsilon = O(k^{-1})$ provided k is large enough (and (G^*, φ) is c-close). We prove first that the φ-image of a unit cell is close to an aligned square.

The x and y components of $\varphi(i, j)$ will be written as $x(i, j)$ and $y(i, j)$, respectively. One more piece of notation giving the x and y components of the vectors $\varphi(i+t, j) - \varphi(i, j)$ and $\varphi(i, j+t) - \varphi(i, j)$:

$$\begin{aligned}
Hx_t(i,j) &= x(i+t,j) - x(i,j),\\
Vx_t(i,j) &= x(i,j+t) - x(i,j),\\
Hy_t(i,j) &= y(i+t,j) - y(i,j),\\
Vy_t(i,j) &= y(i,j+t) - y(i,j),\\
Hx(i,j) &= Hx_1(i,j) = x(i+1,j) - x(i,j),\\
Vx(i,j) &= Vx_1(i,j) = x(i,j+1) - x(i,j),\\
Hy(i,j) &= Hy_1(i,j) = y(i+1,j) - y(i,j),\\
Vy(i,j) &= Vy_1(i,j) = y(i,j+1) - y(i,j).
\end{aligned}$$

So for instance $Hx_t(i,j)$ is the horizontal component of the vector $\varphi(i+t,j)-\varphi(i,j)$ while $Vx_t(i,j)$ is its vertical component.

Lemma 9.1 *If (G^*,φ) is c-close and k is large enough, then for all $1 < i, j < k-1$ and for all $t \in \mathbb{N}$ with $i+t, j+t < k$*

$$\left|\frac{Hy_t(i,j)}{Hx_t(i,j)}\right|, \left|\frac{Vx_t(i,j)}{Vy_t(i,j)}\right| < \frac{3+2c}{k} \quad \textit{and} \tag{9.1}$$

$$\left|\frac{Vy(i,j)}{Hx(i,j)} - 1\right| < \frac{13+12c}{k} = O(k^{-1}). \tag{9.2}$$

Note the condition $1 < i, j < k-1$ says, for instance, that (9.2) only applies to unit cells $Q(i,j)$ that are in $G^*[-1]$.

Proof The ratio $Hy_t(i,j)/Hx_t(i,j)$ is equal to the slope of the line through $\varphi(i,j)$ and $\varphi(i+t,j)$. Consider the segment $[(i,j),(i+t,j)]$, where $2 \le j \le k-1$. Compare its position to the four points $(1,j-1)$, $(1,j+1)$, $(k,j-1)$, $(k,j+1)$. The slope s of the line $L(\varphi(i,j),\varphi(i+t,j))$ is more than that of $L(\varphi(1,j+1),\varphi(k,j-1))$ but less than that of $L(\varphi(1,j-1),\varphi(k,j+1))$, that is,

$$-\frac{2-2c}{k-1-2c} < s < \frac{2+2c}{k-1-2c} \quad \text{implying} \quad |s| \le \frac{3+2c}{k}.$$

By switching the roles of the x and y coordinates we get the same bounds for the vertical segments. Therefore the image of each unit cell is already close to some rectangle. We remark here that $Hx(i,j) > 0$ follows. Indeed, assuming $j \le k/2$, $Hx(i,j) \le 0$ would imply that $\varphi(i,j+1)$ and $\varphi(k,k)$ are on opposite sides of $L(\varphi(1,k),\varphi(i,j))$ contradicting the orientation preserving property. A similar argument works when $j > k/2$. It follows further that $Hx_t(i,j) = \sum_{s=i}^{i+t-1} Hx(s,j) > 0$. Completely analogously $Vy(i,j), Vy_t(i,j) > 0$ follows as well.

Observe that for every unit cell in G^* one of its two diagonals intersects G^* in at least $k/2$ points. Consider the diagonal segment $[(i+1,j),(i,j+1)]$, where $k/2 \le i+j+1 \le k$. Compare its position to the four points $(1,i+j-1)$, $(1,i+j+1)$, $(i+j-1,1)$, $(i+j+1,1)$. So the slope s of the line $L(\varphi(i+1,j),\varphi(i,j+1))$ is more than that of $L(\varphi(1,i+j-1),\varphi(i+j+1,1))$ but less than that of

$L(\varphi(1, i+j+1), \varphi(i+j-1, 1))$. Therefore $-(i+j+2c)/(i+j-2-2c) < s < -(i+j-2c)/(i+j+2+2c)$. Since k is big enough, $|s+1| < (8c+6)/k$.

By symmetry the argument can be extended to $k/2 \leq i+j+1 \leq 3k/2$. Again by symmetry we get that the slope s of the line $L(\varphi(i, j), \varphi(i+1, j+1))$ is closer to 1 than $(8c+6)/k$. The above estimates on the slopes imply that in the φ-image of the cell $Q(i, j)$ (which is a convex quadrilateral) the sides $[\varphi(i, j), \varphi[i+1, j]$ and $[\varphi(i, j+1), \varphi[i+1, j+1]$ are almost horizontal, the sides $[\varphi(i, j), \varphi[i, j+1]$ and $[\varphi(i, j), \varphi[i+1, j+1]$ are almost vertical, the slope of the diagonal $[\varphi(i, j), \varphi[i+1, j+1]$ is almost 1, and the slope of the diagonal $[\varphi(i, j+1), \varphi[i+1, j]$ is almost -1. We show next that these conditions imply (9.2).

Consider the triangle $\varphi(i, j), \varphi(i+1, j), \varphi(i, j+1)$. Assume that $\Delta = Hx(i, j) \geq Vy(i, j)$, then $\Delta = \max\{Hx(i, j), Vy(i, j)\}$ and write $\delta := \Delta - Vy(i, j) \geq 0$. Then the slope of the diagonal times -1 is

$$1 - \frac{8c+6}{k} < \frac{Vy(i, j) - Hy(i, j)}{Hx(i, j) - Vx(i, j)} < \frac{\Delta - \delta + (3+2c)\Delta/k}{\Delta - (3+2c)\Delta/k}$$

which yields

$$\frac{\delta}{\Delta} < \frac{12+12c}{k}.$$

Using the values of Δ and δ we get

$$0 \leq \frac{\delta}{\Delta} = 1 - \frac{Vy(i, j)}{Hx(i, j)} < \frac{12+12c}{k}$$

proving (9.2) when $\Delta = Hx(i, j)$. The same argument works when $\Delta = Vy(i, j)$ and gives

$$0 \leq \frac{\delta}{\Delta} = 1 - \frac{Hx(i, j)}{Vy(i, j)} < \frac{12+12c}{k}.$$

Then $1 - (12+12c)/k < Hx(i, j)/Vy(i, j)$ which implies that

$$\frac{Vy(i, j)}{Hx(i, j)} < \left(1 - \frac{12+12c}{k}\right)^{-1} < 1 + \frac{13+12c}{k},$$

at least for large k. This proves (9.2) when $\Delta = Vy(i, j)$. □

Symmetry implies that, analogously to (9.2), we have

$$\left|\frac{Vy(i, j)}{Hx(i, j+1)} - 1\right|, \left|\frac{Vy(i+1, j)}{Hx(i, j)} - 1\right| = O(k^{-1}). \tag{9.3}$$

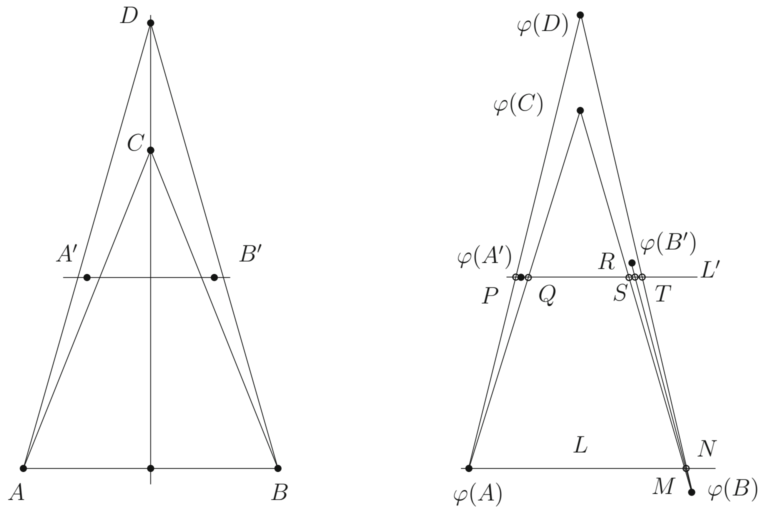

Fig. 5 Illustration for Lemma 10.1

This shows that the sides of the quadrilateral $\varphi(Q(i, j))$ are almost equal. Moreover, since $Q(i-1, j)$ and $Q(i, j)$ share a side,

$$\left|\frac{Hx(i-1, j)}{Hx(i, j)} - 1\right| = O(k^{-1}). \tag{9.4}$$

holds as well provided $i - 1 > 1$. We can see now that the image of a unit cell is very close to an aligned square. We want to show that it is close to a unit square. This is the content of the next section.

10 Proof of Lemma 3.5

Lemma 10.1 *Under the conditions of the previous lemma and for large enough k,*

$$1 - O(k^{-1}) < Hx(i, j), Vy(i, j) < 1 + O(k^{-1}),$$

where the constant in $O(\cdot)$ depends only on c. Therefore every unit cell is ε-close with $\varepsilon = c_1/k$ where c_1 depends only on c.

The lemma shows via (9.1) and (9.2) that the vectors $\varphi(i+1, j) - \varphi(i, j)$ are almost horizontal, and the vectors $\varphi(i, j+1) - \varphi(i, j)$ are almost vertical and both have length almost one, implying that the unit cell $Q(i, j)$ is ε-close with $\varepsilon = O(k^{-1})$. This completes the proof of Lemma 3.5.

Proof Let i, j, t be positive integers such that $2 \le i-2, i+2, j-t, j+t+1 \le k-1$ and that $5t > k$. Consider the following two line segments: one of length 4, $[A(i-2, j-t), B(i+2, j-t)]$ and another one of length 2, $[A'(i-1, j), B'(i+1, j)]$; see Fig. 5, left. Define $C(i, j+t-1)$ and $D(i, j+t+1)$. The triangle DAB contains the line segment $[A', B']$ but the triangle CAB does not contain either of the points A', B'.

Let L be the horizontal line through $\varphi(A)$ and L' the horizontal line through $\varphi(A')$. Let M and N be the intersection of L the with lines $L(\varphi(C), \varphi(B))$ and $L(\varphi(D), \varphi(B))$ respectively. Let L' intersect the four sides of the (non-convex) quadrilateral $\varphi(A), \varphi(D), \varphi(C), \varphi(B)$ in the points P, Q, R, T in that order. Let S be the intersection of L' and $L(\varphi(B), \varphi(B'))$. Since φ is orientation preserving we have the following six points on the line L' from left to right in order: $P, \varphi(A'), Q, R, S, T$ and the points $\varphi(A), M, N$ are on the line L. The order of M and N depends whether $\varphi(B)$ is above or below the line L.

There are five almost vertical lines, namely the four sides of the (non-convex) quadrilateral whose vertices are $\varphi(A), \varphi(D), \varphi(B), \varphi(C)$ and the line $L(\varphi(B), \varphi(B'))$. The absolute value of the slope of these five lines is at least $(t-2c)/(2+2c) > k/(12+12c)$. This implies using Lemma 9.1 that

$$\begin{aligned} Hx_4(i-2, j-t) &= \varphi(A)M(1+O(k^{-2})) \quad \text{and} \\ Hx_4(i-2, j-t) &= \varphi(A)N(1+O(k^{-2})) \quad \text{and} \\ Hx_2(i-1, j) &= \varphi(A')S(1+O(k^{-2})). \end{aligned} \tag{10.1}$$

Because of the order of the six points on L' we have

$$QR < \varphi(A')S < PT. \tag{10.2}$$

The distance of C to L' is at least $t-1-2c$ and to L is at most $2t-1+2c$. So

$$QR \geq \frac{t-1-2c}{2t-1+2c}\varphi(A)M > \frac{1}{2}\left(1-\frac{3+18c}{k}\right)\varphi(A)M,$$

where the last inequality holds for large enough k, we omit the straightforward calculations. Similarly using the point D instead of C we get that

$$PT < \frac{1}{2}\left(1+\frac{3+18c}{k}\right)\varphi(A)N.$$

Combining all these inequalities gives that

$$\frac{1}{2}\left(1-\frac{3+18c}{k}\right)Hx_4(i-2, j-t) < Hx_2(i-1, j)(1+O(k^{-2}))$$

and

$$Hx_2(i-1, j)(1+O(k^{-2})) < \frac{1}{2}\left(1+\frac{3+18c}{k}\right)Hx_4(i-2, j-t).$$

We use next (9.4) twice to see that

$$\frac{Hx_2(i-1,j)}{Hx(i,j)} = 2 + O(k^{-1}).$$

Similarly

$$\frac{Hx_4(i-2,j-t)}{Hx(i,j-t)} = 4 + O(k^{-1}),$$

which implies that

$$\frac{Hx(i,j-t)}{Hx(i,j)} = 1 + O(k^{-1}).$$

From this it follows that for every $i = 2, \dots, k-2$, $Hx(i,j)$ differs from $Hx(i,k/2)$ by $O(k^{-1})$ if j is farther from $k/2$ than $k/5$. Otherwise we can compare $Hx(i,j)$ with $Hx(i,k/2)$ in two steps. If, for example $3k/10 < j < k/2$, then $Hx(i,j)$ differs from $Hx(i,2)$ by at most $O(k^{-1})$, which differs from $Hx(i,k/2)$ by the same amount. It follows that for a fixed i the numbers $Hx(i,j)$ differ from $Hx(i,k/2)$ by at most $O(k^{-1})$.

Switching the roles of x and y, an identical argument shows that for a fixed j the numbers $Vy(i,j)$ differ from $Vy(k/2,j)$ by at most $O(k^{-1})$. Since for every i,j the numbers $Vy(i,j)$ and $Hx(i,j)$ differ by at most $O(k^{-1})$, we get that for every i,j the numbers $Hx(i,j)$, $Hx(i,k/2)$, $Vy(i,k/2)$, $Vy(k/2,k/2)$ differ by at most $O(k^{-1})$. Thus the numbers $Hx(i,j)$ are almost equal: any two of them differ by at most $O(k^{-1})$.

We are nearly finished with the proof. Observe that $Hx_{k-4}(2,j) = \sum_{i=2}^{k-2} Hx(i,j)$ for every $j \in \{2,\dots,k-2\}$ and $k-4-2c < Hx_{k-4}(2,j)k-4+2c$ because (G^*,φ) is c-close. Thus the average of $Hx(i,j)$ for $i \in \{2,\dots,k-2\}$ is $1+O(k^{-1})$, showing that for every i,j

$$Hx(i,j) = 1 + O(k^{-1}) \quad \text{and} \quad Vy(i,j) = 1 + O(k^{-1})$$

by symmetry. □

We just proved that $\varphi(G^*[-1])$ is locally very close to a unit grid. This implies that the distance of the images of (i,j) and $(i+t,j)$ is $t \cdot (1 \pm O(k^{-1}))$. This is great for small values of t but is useless when t is comparable to k, since we get a $O(1)$ error. So it can happen that the distance between the images of $(1,1)$ and $(k,1)$ is $k+100$, say, and the distance between the images of $(1,k)$ and (k,k) is $k-100$, which is 200 apart, too large for our purposes.

The next step is crucial, where the local property (of $\varphi(G^*[-1])$ being locally very close to a unit grid) is extended to the whole grid. The next section is devoted proving the lemmas needed for that.

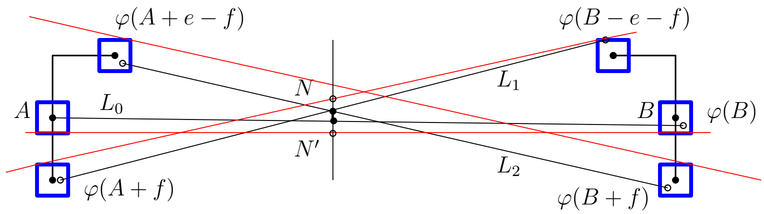

Fig. 6 Case $e = (1, 0)$ of the lemma

11 The Midpoint Lemma

We need some further preparation. We will call the primitive vectors shorter than 3 *short vectors*. They are the following ones:

$$(0, \pm 1), (\pm 1, 0), (\pm 1, \pm 1), (\pm 2, \pm 1), (\pm 1, \pm 2).$$

Lemma 11.1 *Let (G^*, φ) be an orientation preserving and c-close pair (with G^* a $k \times k$ grid, and c from Lemma 3.4). Suppose that every unit cell in G^* is ε-close, where $\varepsilon = c_1/k$. Let A, B, M be grid points of G^* on a line parallel to a short vector e such that M is the midpoint of the segment AB and $t \geq 3c$ is the integer such that $AB = 2te$. Then the distance of $\varphi(M)$ to the line $L(\varphi(A), \varphi(B))$ is less than $7\varepsilon + 1/(2t - 2c - 1)$.*

Proof Apart from symmetries there are only three distinct cases for the short vector e, namely $e = (1, 0)$ or $(1, 1)$ or $(1, 2)$. There are two further cases depending on the orientation of the triangle with vertices $\varphi(A), \varphi(M)$, and $\varphi(B)$: $\varphi(M)$ is either "above" or "below" the line $L_0 := L(\varphi(A), \varphi(B))$. These two cases are again symmetric. So we can assume that $\varphi(M)$ is above L_0; see Fig. 6.

Define $f = (0, -1)$. In order to make the computations simpler we assume (using translations), that $A = \varphi(A) = (0, 0)$, so in the figure A and $\varphi(A)$ coincide. Then $\varphi(B) = 2te + (a, b)$ where $|a|, |b| < 2c$ because (G^*, φ) is c-close. The line $L_1 := L(A + f, B - f - e)$ is above M and so is the line $L_2 := L(A - f + e, B + f)$ because (G^*, φ) is an orientation preserving pair.

Therefore the three lines L_0, L_1, L_2 determine a triangle that contains $\varphi(M)$. See Fig. 6; this triangle is so small that the figure does not show the point $\varphi(M)$. Clearly the distance of $\varphi(M)$ to the line L_0 is less than the distance of L_0 to the opposite vertex of the triangle, which is the intersection point, say N, of the lines L_1 and L_2.

The line through N and parallel with f intersects L_0 in N'. It is evident that the distance of N from L_0 is less than the length of $[N, N']$. We are going to establish an upper bound on the length of $[N, N']$ in the three cases. Note first that $\varphi(A + f)$ is ε-close to $\varphi(A) + f$ because the cells in G^* are ε-close. Similarly $\varphi(B - f - e)$ is ε- or 2ε- or 3ε-close to $(\varphi(B) - f - e)$, depending on the value of e.

Suppose for a moment that $\varepsilon = 0$. Then $\varphi(A + f), \varphi(B - f - e)$, etc all coincide with $\varphi(A) + f, \varphi(B) - f - e$, etc, respectively. The length NN' can be computed easily: one determines the coordinates of the point $N = L_1 \cap L_2$, $N = (x_0, y_0)$ say.

Then one computes y-component of $N' \in L_0$ so $N' = (x_0, y_1)$. The length of $[N, N']$ is equal to $y_0 - y_1$. The details of this computation are explained in Appendix C. The outcome is that

$$NN' = y_1 - y_0 = \frac{1}{2t+a-1} - \frac{\Delta^2}{(2t+a)(2t+a-1)} < \frac{1}{2t+a-1},$$

where $\Delta = b/2$ when $e = (1,0)$, $\Delta = a - b/2$ when $e = (1,1)$, $\Delta = a + b/2$ when $e = (1,2)$.

When $\varepsilon > 0$ the points $\varphi(A) + f, \varphi(B), \varphi(A - f + e)$, etc are in the aligned and small squares of side length 2ε (or 4ε or 6ε, depending on e) centred at $\varphi(A + f), \varphi(B), \varphi(A) - f + e$, etc. It is not hard to check (see Fig. 6) that NN' is maximal when the line L_1 is tangent from above to the small squares centred at $\varphi(A) + f$, $\varphi(B) - f - e$, and L_2 is tangent from above to the corresponding small squares around $\varphi(B) + f, \varphi(A) - f + e$, while L_0 is tangent from below to the small squares centred at $\varphi(A)$ and $\varphi(B)$. The red lines in the figure show these lines tangent to some squares. The three red lines determine a triangle Δ that contains the triangle determined by L_0, L_1, L_3, so NN' is shorter than the length of the segment which is the intersection of $L(N, N')$ and Δ.

The lines L_1 and L_2 have to be translated upwards (from their position when $\varepsilon = 0$), and L_0 downwards in direction f. The amount of these translations is half the width of an aligned square of side length 6ε orthogonal to the direction of the lines L_1 and L_2 and L_0. As $t > 3c$ is large, the directions of these lines are close to that of $L(\varphi(A), \varphi(B))$, which is also close to that of $L(A, B)$. The maximal amount can be determined directly in all three cases. It turns out to be always smaller than 3.5ε; details are left to the interested reader. Translation goes once up and once down implying that

$$NN' < 7\varepsilon + \frac{1}{2t-a-1},$$

and the result follows as $|a| < 2c$. □

If a 1-Lipschitz function f on $[n]$ has the property that the value at every midpoint is close to the average of the values at the two endpoints, and the value at the ends are 0, then the function is bounded. More precisely:

Lemma 11.2 *Let $n > 100$ and c^* be some positive constant. Assume the function $f\colon \{0, 1, \ldots, n\} \to \mathbb{R}$ satisfies the following properties:*

- *$f(0) = f(n) = 0$ and $|f(i) - f(i-1)| \le 1$ for every $i \in [n]$.*
- *For any positive i, t such that $0 \le i - t < i + t \le n$ and $10t \ge n$ we have $|(f(i-t) + f(i+t))/2 - f(i)| \le c^* n/t$.*

Then $|f(i)| < 62c^* + 4$ for all $0 \le i \le n$.

The proof uses a version of the doubling argument and is explained in Appendix D. The lemma is needed for the following result, where the constants c and c_1 come from Lemmas 3.4 and 10.1.

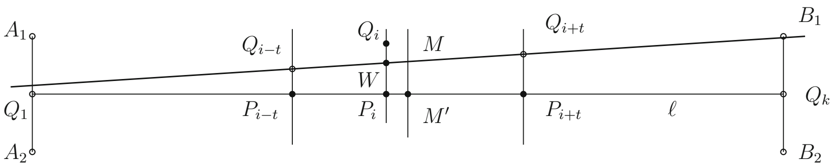

Fig. 7 Points and lines in Lemma 11.3

Lemma 11.3 *Assume* (G^*, φ) *is a c-close and orientation preserving pair where* G^* *is the grid* $[k] \times [k]$ *(with k large enough) and every unit cell is* ε*-close with* $\varepsilon = c_1/k$. *Let* $j \in [k]$ *and set* $Q_1 = \varphi(1, j)$ *and* $Q_k = \varphi(k, j)$ *and define* $\ell = L(Q_1, Q_k)$. *Then* $\varphi(i, j)$ *is at distance* $O(k^{-1})$ *from* ℓ *for every* $(i, j) \in G^*$.

Proof Let $d(i)$ be the vertical and signed distance of $Q_i = \varphi(i, j)$ from the line ℓ and let $f(i) = kd(i)/10$. We claim that the function $f(i)$ fulfills the conditions of Lemma 11.2; f is defined on $[k]$ and not on $\{0, 1, \ldots, n\}$ but that does not matter. Evidently $f(1) = f(k) = 0$.

To check the other properties we need some preparations. Let $A_1 = \varphi(1, j+1)$, $A_2 = \varphi(1, j-1)$, $B_1 = \varphi(k, j+1)$, $B_2 = \varphi(k, j-1)$. Let s^+ and s^- be the slope of the line $L(A_2, B_1)$ and $L(A_1, B_2)$, respectively; see Fig. 7.

Claim 11.4 *For large enough k we have* $0 \le s^+ - s^- \le 9/k$.

Proof The vector $Q_k - Q_1$ equals $(k+\Delta_x, \Delta_y)$ with $|\Delta_x|, |\Delta_y| < 2c$ because (G^*, φ) is c-close, and because the cells are ε-close, both vectors $A_2 - A_1$ and $B_2 - B_1$ are within distance 2ε from the vector $(0, 2)$ in the maximum norm. Thus

$$s^+ \le \frac{\Delta_y + 2 + 2\varepsilon}{k + \Delta_x - 2\varepsilon} \quad \text{and} \quad s^- \ge \frac{\Delta_y - 2 - 2\varepsilon}{k + \Delta_x + 2\varepsilon}.$$

From this $s^+ - s^-$ can be estimated the usual way (we omit the straightforward details) and the bound $9/k$ follows for large enough k. □

Let P_i be the intersection of ℓ and the vertical line through $Q_i = \varphi(i, j)$. Assume $i, h \in [k]$ and $i < h$. Because of the orientation preserving property the line $L(Q_i, Q_h)$ intersects both segments $[A_1, A_2]$ and $[B_1, B_2]$, therefore its slope, s^*, is between s^+ and s^-, and so is the slope s of $\ell = L(Q_1, Q_k)$. It follows that, with $(P_h - P_i)_x$ denoting the x-component of the vector $P_h - P_i$,

$$d(h) - d(i) = d(Q_h, P_h) - d(Q_i, P_i) = (s^* - s)(P_h - P_i)_x. \tag{11.1}$$

We use this with $h = i + 1$ first, combined with Claim 11.4:

$$|d(i+1) - d(i)| \le \frac{9}{k}(P_{i+1} - P_i)_x < \frac{9}{k}(1 + 2\varepsilon).$$

So $|f(i+1) - f(i)| \le 1$ indeed; f is indeed Lipschitz-1.

The last condition to be checked is the midpoint property. Let $t \geq k/10$ and let $1 \leq i - t < i < i + t \leq k$, so $2t \leq k$. Let M be the midpoint of the segment $[Q_{i-t}, Q_{i+t}]$ and let M' the intersection of ℓ and the vertical line through M, see Fig. 7. We know that

$$d(M, M') = \frac{d(i-t) + d(i+t)}{2}.$$

The horizontal distance between M and Q_i, $(M - Q_i)_x = (M' - P_i)_x$, is at most $2c$ in absolute value, since G^* is c-close. Let W be the point of intersection of the vertical line through Q_i and $L(Q_{i-t}, Q_{i+t})$. So $d(i) = d(Q_i, W) + d(W, P_i)$; see Fig. 7. In view of Lemma 11.1, $|d(Q_i, W)| < 7\varepsilon + 1/(2t - 2c - 1)$. Here $1/(2t - 2c - 1) < 1/t$ if $t < 2t - 2c - 1$ or, what is the same, if $2c + 1 < t$. This holds if k is large enough, namely larger than $10(2c + 1)$ because $t > k/10$. Thus

$$|d(Q_i, W)| < 7\varepsilon + \frac{1}{t}.$$

The slope of the line $L(Q_{i-t}, Q_{i+t})$ is between s^+ and s^-. Equation (11.1) shows that

$$|d(W, P_i) - d(M, M')| < \frac{9}{k} \cdot 2c = \frac{18c}{k}.$$

Define

$$D := \frac{d(i-t) + d(i+t)}{2} - d(i) = d(M, M') - (d(Q_i, W) + d(W, P_i)).$$

Then

$$\begin{aligned} |D| &\leq |d(M, M') - d(W, P_i)| + |d(Q_i, W)| \\ &< \frac{18c}{k} + 7\varepsilon + \frac{1}{t} \leq \frac{18c}{k} + \frac{7c_1}{k} + \frac{1}{t}. \end{aligned}$$

Since $2 \leq k/t$, we have

$$\begin{aligned} \left| \frac{f(i-t) + f(i+t)}{2} - f(i) \right| &= \frac{k}{10} \left| \frac{d(i-t) + d(i+t)}{2} - d(i) \right| \\ &< \frac{18c}{10} + \frac{7c_1}{10} + \frac{k}{10t} = \frac{9c + 3.5c_1 + 1}{10} \cdot \frac{k}{t}. \end{aligned}$$

So the midpoint property is true with $c^* = (9c + 3.5c_1 + 1)/10$.

By Lemma 11.2 we know that for all i, $|f(i)| < 62c^* + 4$ and therefore

$$|d(i)| = \frac{10}{k} |f(i)| < \frac{10(62c^* + 4)}{k} = O(k^{-1}).$$

□

12 Auxiliary Lemmas

In this section we will work with the grid G_{2k}, and we assume throughout that (G_{2k}, φ) is c-close and its cells are ε-close. In this section $\varepsilon > 0$ is considered a parameter, but the reader may think it is just c_1/k. We set $A_1 = (-2k, 2k)$, $A_2 = (2k, 2k)$, $A_3 = (2k, -2k)$, and $A_4 = (2k, -2k)$. They are the vertices of the underlying $[-2k, 2k]^2$ square. Given $A \in G_{2k}$ and $B \in \mathbb{R}^2$ we say that $\varphi(A) = B$ with error η if $\|\varphi(A) - B\| < \eta$. The quadrilateral Q with vertices $\varphi(A_1), \varphi(A_2), \varphi(A_3), \varphi(A_4)$ is close to the square $[-2k, 2k]^2$. We show next that $\varphi(0, 0)$ is close to the point where the diagonals of this quadrilateral meet.

Claim 12.1 *Assume $\varphi(A_i) = A_i + (x_i, y_i)$ for $i = 1, 2, 3, 4$ and set $x = \sum_1^4 x_i$, $y = \sum_1^4 y_i$. Then*

$$\varphi(0,0) = \frac{1}{4}(x + y - 2y_2 - 2y_3, x + y - 2x_2 - 2x_3)$$

with an error $24\varepsilon + O(k^{-1})$.

We call x_i the *x-deviation* at $\varphi(A_i)$, or simply the x-deviation, and y_i is the y-deviation. The claim states that the deviation at $(0, 0)$ is a linear function of the deviations at the vertices plus a small error term.

Proof The intersection point of the lines $L_1 := L(\varphi(A_1), \varphi(A_4))$ and $L_2 = L(\varphi(A_2), \varphi(A_3))$ is the solution of two linear equations in variables X and Y:

$$\begin{aligned}
\left(1 - \frac{y_4 - y_1}{4k}\right)\left(X - \frac{x_1 + x_4}{2}\right) + \left(1 + \frac{x_4 - x_1}{4k}\right)\left(Y - \frac{y_1 + y_4}{2}\right) &= 0,\\
\left(1 + \frac{y_2 - y_3}{4k}\right)\left(X - \frac{2_1 + x_3}{2}\right) - \left(1 + \frac{x_2 - x_3}{4k}\right)\left(Y - \frac{y_2 + y_3}{2}\right) &= 0.
\end{aligned}$$

Direct checking shows that the solution is

$$X = \frac{x + y - 2y_2 - 2y_3}{4} \quad \text{and} \quad Y = \frac{x + y - 2x_2 - 2x_3}{4}$$

with an error $O(k^{-1})$ in both cases. The midpoint of the diagonal $[A_1, A_4]$ is $(0, 0)$. The midpoint lemma (Lemma 11.1) states that the point $\varphi(0, 0)$ is at distance at most $7\varepsilon + O(k^{-1})$ from the line L_1. The same upper bound applies to its distance from L_2. Then $\varphi(0, 0)$ lies in the parallelogram whose sides are parallel with L_1 and L_2, and opposite sides are at distance at most $14\varepsilon + O(k^{-1})$. The diameter of this parallelogram is at most $2(14\varepsilon + O(k^{-1}))$ because the slope of L_1 and L_2 is close to -1 and 1. Finally, we note that (X, Y), the intersection point of L_1 and L_2, also lies in this parallelogram, and then $\|\varphi(0, 0) - (0, 0)\| < 28\varepsilon + O(k^{-1})$, indeed. □

We define next $B_1 = (-2k, k)$, $B_2 = (2k, k)$, $B_3 = (-2k, -k)$, $B_4 = (2k, -k)$. The quadrilateral Q_2 with vertices $\varphi(B_1), \varphi(B_2), \varphi(B_3), \varphi(B_4)$ is close to the (horizontal) domino $[-2k, 2k] \times [-k, k]$. The intersection of the diagonals is again close to $\varphi(0, 0)$.

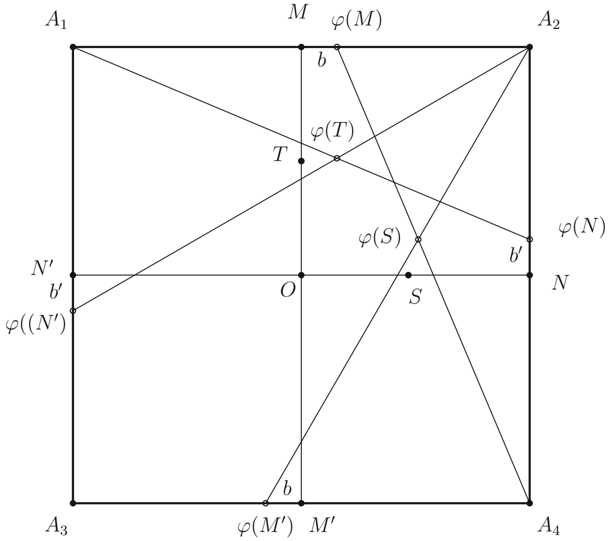

Fig. 8 Some diagonals in G_{2k}

Claim 12.2 *Assume* $\varphi(B_i) = B_i + (x_i, y_i)$ *for* $i = 1, 2, 3, 4$ *and set* $x = \sum_1^4 x_i$, $y = \sum_1^4 y_i$. *Then*

$$\varphi(0,0) = \frac{1}{8}(2x + 2y - 4y_2 - 4y_3, x + y - 2x_2 - 2x_3)$$

with an error $28\varepsilon + O(k^{-1})$.

The *proof* is completely analogous to that of Claim 12.1 and is therefore omitted. By symmetry the analogous statement, with the roles of x and y exchanged, holds for the vertical domino $[-k, k] \times [-2k, 2k]$. The next lemma is important. It says that the deviation at $(0, 2k)$ is a well-defined linear function of the deviations at the vertices of G_{2k}.

Lemma 12.3 *Under the conditions of Claim* 12.1,

$$\varphi(0, 2k) = (0, 2k) + \left(\frac{x_1 + x_2}{2} + \frac{y_1 - y_2 - y_3 + y_4}{4}, \frac{y_1 + y_2}{2}\right)$$

with an error $O(\varepsilon) + O(k^{-1})$.

Proof We set $M = (0, 2k)$, $M' = (0, -2k)$, $N = (2k, 0)$, $N' = (-2k, 0)$, and $O = (0, 0)$, $S = (0, k)$, $T = (k, 0)$; see Fig. 8. Claim 12.1 gives the coordinates of $\varphi(O)$ with $O(\varepsilon) + O(k^{-1})$ precision.

Let b be the x-coordinate of $\varphi(M)$ and b' be the y-coordinate of $\varphi(N)$. The midpoint lemma shows that

$$\varphi(M)=\varphi(0,2k)=\left(b,2k+\frac{y_1+y_2}{2}\right),$$
$$\varphi(N)=\varphi(2k,0)=\left(2k+\frac{x_2+x_4}{2},b'\right),$$

with an error of $7\varepsilon+O(k^{-1})$ in both cases. Moreover, O is the midpoint of $[M,M']$, so by the midpoint lemma again

$$\varphi(M')=\left(-b+\frac{x+y-2y_2-2y_3}{4},-2k+\frac{y_3+y_4}{2}\right)$$

with an error of $O(\varepsilon)+O(k^{-1})$. The same method shows that, with the same error term,

$$\varphi(N')=\left(-2k+\frac{x_1+x_3}{2},-b'+\frac{x+y-2x_2-2x_3}{4}\right).$$

The vertical version of Claim 12.2 applies to the domino $[0,2k]\times[-2k,2k]$ and we get the y-coordinate of $\varphi(T)$ (to be denoted by $\varphi(T)_y$):

$$\varphi(T)_y=b+\frac{-2x_1-6x_2-2x_3+2x_4-y_1+5y_2+3y_3+y_4}{8}$$

with an error of $O(\varepsilon)+O(k^{-1})$. The midpoint of the segment $[O,N]$ is T and by the midpoint lemma $\varphi(T)_y=(\varphi(O)_y+\varphi(N)_y)/2$ holds with the same error term. We now have two formulae for $\varphi(T)_y$ giving the following equation:

$$b'=2b+\frac{-3x_1-5x_2-x_3+x_4-2y_1+4y_2+2y_3+0y_4}{4}$$

with an error of $O(\varepsilon)+O(k^{-1})$.

The same way we are going to express $\varphi(S)_x$ (the x-coordinate of $\varphi(S)$) in two different ways. Claim 12.2 shows that in the domino $[-2k,2k]\times[0,2k]$,

$$\varphi(S)_x=b'+\frac{x_1+5x_2+3x_3-x_4+2y_1-6y_2-2y_3-2y_4}{8},$$

with an error of $O(\varepsilon)+O(k^{-1})$. Since S is the midpoint of $[O,M]$, $\varphi(S)_x=(\varphi(O)_x+\varphi(M)_x)/2$ with the usual error term. This gives another expression for $\varphi(S)_x$ which implies that

$$b=2b'+\frac{0x_1+4x_2+2x_3-2x_4+y_1-5y_2-y_3-3y_4}{4}$$

with an error of $O(k^{-1})$. We omit the tedious but straightforward calculations. Solving the system of two linear equations in the two variables b, b' we get

$$b = \frac{x_1 + x_2}{2} + \frac{y_1 - y_2 - y_3 + y_4}{4}$$

with an error of $O(\varepsilon) + O(k^{-1})$. □

13 Proof of Lemma 3.6

In this section (G_{8k}, φ) is an orientation preserving pair, which is c-close, every cell is ε-close with $\varepsilon = c_1/k$, and φ coincides with the identity on the vertices of G_{8k}. We are going to show that every $(i, j) \in G_{8k}$ is $O(k^{-1})$-close. Claims 12.1 and 12.2 imply that this is true for the points $(\alpha k, \beta k)$ where α and β are integers in $[-8, 8]$. For concreteness we assume that in every statement with a $O(k^{-1})$ estimate, the statement holds with c_3/k where c_3 is a constant that depends only on c. In particular, $\|\varphi(\alpha k, \beta k) - (\alpha k, \beta k)\| < c_3/k$ for integers $\alpha, \beta \in [-8, 8]$.

The proof is based again on a doubling argument (like in Lemmas 3.3 and 11.2) but this time in dimension 2.

Let M be the maximum error in the x and y coordinates, that is,

$$M = \max \|\varphi(i, j) - (i, j)\|.$$

Assume that this maximum error is in the x coordinate at the point (i, j). The target is to prove that $M = O(k^{-1})$. Lemma 11.3 shows that the x-deviations at $(i, 8k)$ and $(i, -8k)$ cannot be both less than $M - c_3/k$ in absolute value. We assume that the x-deviation at $(i, 8k)$ is $M^* := \varphi(i, 8k)_x - i$. Then either

$$M^* > M - \frac{c_3}{k} \ \text{ if } \ M^* > 0, \qquad \text{or} \qquad M^* < -M + \frac{c_3}{k} \ \text{ if } \ M^* < 0.$$

Without loss of generality we suppose that $i < 0$.

Case 1: $\alpha k + k < i < \alpha k + 2k$ and $-8 \le \alpha \le -2$. Set $k' = i - \alpha k > 0$ and consider the pair (G^*, φ) where G^* is the usual grid on the square $[\alpha k, \alpha k + 2k'] \times [8k - 2k', 8k]$. The side of this square, $2k' := 2(i - \alpha k)$, is at least $2k$ but at most $4k$ long. We are going to use Lemma 12.3 on G^*. Let A_1, A_2, A_3, A_4 be the four corners of G^* and assume that $\varphi(A_j) - A_j = (x_j, y_j)$ for $j \in [4]$. Here both $|x_1|, |y_1| < c_3/k$ because the error at $(\alpha k, 8k)$ is small. In view of Lemma 11.3 $|y_2| < c_3/k$. The y-deviation at the endpoint of the segment $[(-8k, 8k - 2k'), (8k, 8k - 2k')]$ is at most M. Then according to Lemma 11.3 the difference of the y-deviations at the bottom vertices of the square is at most $2M \cdot 4k'/(16k) < M/2$ plus an error term $2c_3/k$, that is, $|y_4 - y_3| < M/2 + 2c_3/k$. Moreover, $-M \le x_2 \le M$ because M is the maximum deviation, so $|x_2/2| < M/2$. By Lemma 12.3 the x-deviation at the midpoint $(i, 8k)$ of the top side of square is

$$M^* = \varphi(i, 8k)_x - i = \frac{x_1 + x_2}{2} + \frac{y_1 - y_2 - y_3 + y_4}{4}$$

with an error term which is at most c_3/k. Assuming $M^* > 0$ term by term estimation in the previous equation gives that

$$\begin{aligned} M - \frac{c_3}{k} < M^* &\leq \frac{c_3}{2k} + \frac{x_2}{2} + \frac{c_3}{4k} + \frac{c_3}{4k} + \frac{|y_3 - y_4|}{4} + \frac{c_3}{k} \\ &< \frac{5c_3}{2k} + \frac{5M}{8}, \end{aligned}$$

implying that $M < 28c_3/(3k)$. The case when $M^* < 0$ is similar:

$$-M + \frac{c_3}{k} > M^* \geq -\frac{5c_3}{2k} + \frac{x_2}{2} - \frac{|y_3 - y_4|}{4} > -\frac{5M}{8} - \frac{5c_3}{2k},$$

and then $M < 28c_3/(3k)$.

Case 2: $-8k < i < -7k$. Here we use first the square whose top left vertex is $(i, 8k)$ and the midpoint of the top side is $(-6k, 8k)$. Setting $k' = -6k - i > k$ this square is $[i, i + 2k'] \times [8k - 2k', 8k]$, with vertices A_1, A_2, A_3, A_4. Its side length is between $2k$ and $4k$ and G^* is the usual $2k' \times 2k'$ grid on this square. Suppose that $\varphi(A_j) - A_j = (x_j, y_j)$ for all j. Then $|y_1|, |y_2| < c_3/k$ and $|y_3 - y_4| \leq M/2 + 2c_3/k$ follow the same way as before. Here $x_1 = M^*$ by definition, and again

$$M^* > M - \frac{c_3}{k} \text{ if } M^* > 0, \quad \text{and} \quad M^* < -M + \frac{c_3}{k} \text{ if } M^* < 0.$$

Lemma 12.3 gives the following expression for the x-deviation at $(-6k, 8k)$:

$$\varphi(-6k, 8k)_x - (-6k) = \frac{x_1 + x_2}{2} + \frac{y_1 - y_2 - y_3 + y_4}{4}$$

with an error term which is at most c_3/k. Our first target is to show that $|x_2|$ is large. Assume first that $x_1 = M^* > 0$. As the x-deviation at $(-6k, 8k)$ is small, at most c_3/k, we see that

$$x_2 < -x_1 + \frac{|y_3 - y_4|}{2} + \frac{3c_3}{k} \leq -M + \frac{M}{4} + \frac{5c_3}{k} = -\frac{3M}{4} + \frac{5c_3}{k},$$

so x_2 is negative and smaller than $-3M/4$ plus an error term of order k^{-1}. Analogously we get, when $M^* < 0$, that

$$x_2 > \frac{3M}{4} - \frac{5c_3}{k}.$$

The rest of the proof is a repetition of the argument in Case 1 in the square whose top side is the segment $[B_1, B_2]$ where $B_1 = (-6k, 8k)$, $B_2 = (-6k + 2k', 8k)$. Let B_3, B_4 denote the other two vertices of this square. The deviations are $\varphi(B_i) - B_i = (x_i^*, y_i^*)$ for $i \in 1, 2, 3, 4$.

The midpoint of the $[B_1, B_2]$ side is $(-6k + k', 8k)$, exactly A_2. The x-deviation at A_2 is x_2. We just proved that $|x_2| > 3M/4 - 5c_3/k$. Lemma 12.3 states that

$$x_2 = \frac{x_1^* + x_2^*}{2} + \frac{y_1^* - y_2^* - y_3^* + y_4^*}{4}$$

with an error term which is at most c_3/k. Here $|x_1^*|, |y_1^*|, |y_2^*| < c_3/k$, and $|y_3^* - y_4^*| < M/2 + 2c_3/k$ follows the same way as before. We can estimate x_2^* from here. When $M^* < 0$ we have with the previous method

$$\frac{3M}{4} - \frac{5c_3}{k} < x_2 < \frac{x_2^*}{2} + \frac{M}{8} + \frac{5c_3}{2k}.$$

But then $M \geq x_2^* > 5M/4 - 15c_3/k$ showing that $M < 60c_3/k$. The case $M^* > 0$ works the same way and is left to the reader.

14 Proof of Lemma 3.7

We have to see how far $\varphi(a, b)$ is from $(a, b) \in G_k[20] \backslash G_k$. By symmetry we can assume that $k < a \leq k + 20$ and $0 \leq b \leq k + 20$.

Observe that the points $(a-40, b-20)$ and $(a-2k, b-k)$ are in G_k. The orientation preserving property implies that $\varphi(a, b)$ is below the line $L_1 := L((a - 2k + \delta, b - k - \delta), (a - 40 - \delta, b - 20 + \delta))$ because $\varphi(a - 2k, b - k)$ lies in the small square of side length 2δ centred around $(a - 2k, b - k)$ and, similarly, $\varphi(a - 40, b - 20)$ lies in the corresponding small square, see Fig. 9. The same way, $\varphi(a, b)$ is above the line $L_2 := L((a - 2k - \delta, b - k + \delta), (a - 40 + \delta, b - 20 - \delta))$. Analogously, the points $(a - k/2, b - k)$ and $(a - 20, b - 40)$ are also in G_k, and $\varphi(a, b)$ is above the line $L_3 := L((a-k/2-\delta, b-k+\delta), (a-20+\delta, b-40-\delta))$ and below the corresponding line $L_4 := L((a - k/2 + \delta, b - k - \delta), (b - 20 + \delta))$. Consequently $\varphi(a, b)$ lies in the quadrilateral, Q say, determined by these four lines. Clearly $(a, b) \in Q$. Then $\|\varphi(a, b) - (a, b)\|$ is at most the diameter of Q measured in max norm.

For simpler calculation we introduce a new coordinate system where $(a-2k, b-k)$ is the origin and (a, b) is the point $(2k, k)$, then $(a-k/2, b-k)$ becomes to $(3k/2, 0)$. This way we get rid of the parameters a, b which is fine. We keep the names of L_i and Q.

Translate L_1 and separately L_2 so that they contain the (new) origin and let L_1^* and L_2^* denote the translated copies. These two line split the plane into four cones, define C_1 as the cone containing $(2k, k)$. It is clear that $Q \subset C_1$. Translate the lines L_3 and L_4 so that they go through the point $(3k/2, 0)$ and let L_3^* and L_4^* denote the translated copies. These lines again define four cones. Let C_2 be the one that contains $(2k, k)$. Again, $Q \subset C_2$ follows. Thus $Q \subset C_1 \cap C_2$ and diam $Q <$ diam $(C_1 \cap C_2)$.

A simple inspection shows that the diameter of $C_1 \cap C_2$ is the segment connecting the points $(x_1, y_1) := L_1^* \cap L_4^*$ and $(x_2, y_2) := L_2^* \cap L_3^*$. So we have to solve two systems of linear equations, each with two equations in two variables. Straightforward and generous (yet tedious) calculations show that $0 < x_1 - x_2 < 6\delta$ and $0 < y_1 - y_2 < 22\delta$

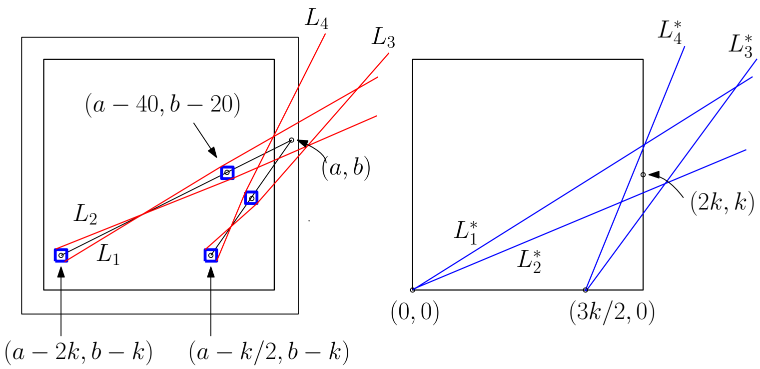

Fig. 9 The lines L_i are red the lines L_i^* are blue

if k is large enough and $\delta < 0.01$. So diam $Q < 22\delta$ showing that $(G_k[20], \varphi)$ is indeed 22δ-close.

Remark 14.1 The same method works when $k < n < k + 20$ and shows that in that case G_n is 22δ-close, too.

15 Proof of Theorems 1.1 and 3.1

As promised in Sect. 2 we begin with n large and choose the largest odd m satisfying $n > 2m^2 - m$. Lemma 3.2 shows that (G_m, φ) is p-separated. Let G^* be the grid on $[-(m-1), m]^2$; it is a $2m \times 2m$ grid, a subgrid of G_m. Since m is odd, $2m$ is 2 mod 4. Lemma 3.3 applies and shows that $(G^*[-1], \nu \circ \varphi)$ is 16-close with a suitable affine map ν. As $G_{m-2} \subset G^*$, $(G_{m-2}, \nu \circ \varphi)$ is 16-close. After renaming the recursion starts with (G_{m-2}, φ), which is an orientation preserving and 16-close pair with (G_n, φ) well defined (and orientation preserving), $m - 2$ odd.

In the general step of the recursion we have a pair (G_k, φ) which is 16-close and orientation preserving and φ is defined on the whole G_n, k odd. Lemma 3.4 gives a projective map μ such that $(G_k[-1], \mu \circ \varphi)$ is orientation preserving and c-close (with the constant c from the lemma) and $\mu \circ \varphi$ is the identity on the four vertices of G_k.

Let ℓ be the line that μ maps to the line at infinity. Then $\mu \circ \varphi$ is still defined on G_n unless $\varphi(A)$ lies on ℓ for some point $A \in G_n$. Even if this happens, $\mu \circ \varphi(A)$ is a well-defined point at infinity for which later projective maps can be applied with no trouble. Alternatively we can start by assuming that the points $\varphi(i, j)$ for all $(i, j) \in G_n$ are in algebraically independent position. Then $\mu \circ \varphi(A) \in \ell$ never happens, and we can use a limit or approximation argument in the end.

As $G_k[-1]$ is exactly G_{k-1}, after renaming we have the orientation preserving and c-close pair (G_{k-1}, φ) and φ is the identity on the four vertices of G_{k-1} and remains so for the rest of this step of the recursion. Here φ is defined on G_n and may not be orientation preserving on the whole G_n.

Next Lemma 10.1 states that the φ-image of every cell of $G_{k-1}[-1] = G_{k-2}$ is ε-close with $\varepsilon = c_1/k$. Using the results of Lemmas 11.1, 11.3, and 12.3 we apply

Lemma 3.6 to show that (G_{k^*}, φ) is (c_2/k)-close for the largest $k^* \leq k-2$ which is divisible by 8.

According to Lemma 8.1 the map φ is orientation preserving on $G_{k-2}[20]$. So we can use Lemma 3.7 to show that the pair $(G_{k^*}[20], \varphi) = (G_{k^*+20}, \varphi)$ is 22δ-close and $22\delta < 16$. So we can move to the next step of the recursion with (G_{k*+19}, φ); of course k^*+19 is odd and larger than k, actually $k^*+19 \geq k+10$ because $k^* \geq k-9$.

The last step of the recursion comes when $k^* + 19 \geq n$ and one has to be a bit careful. At this point (G_k^*, φ) is (c_2/k)-close, and every unit cell is $\delta = c_1/k$-close. An obvious modification of Lemma 8.1 shows that (G_n, φ) is orientation preserving. We can apply Lemma 3.7 as explained in Remark 14.1 to show directly that (G_n, φ) is 22δ-close. In this case, obviously, $22\delta = O(n^{-1})$ and there is no need to carry out the recursion step again. The proof is finished.

Acknowledgements The first author (IB) was partially supported by Hungarian National Research Grants No. 131529, 132696, and 133819.

Appendix A: Proof of Lemma 3.2

Set $A = (0, n)$, $A' = (0, -n)$. Let S_i be the vertical block $\{(i, j) : j \in [-m, m]\}$ of G_m. If $0 < i \leq m$ we define $B = (i, m)$ and $B' = (i, -m)$. The line $L(A, B')$ separates the neighbouring blocks S_{i-1} and S_i since $n > 2m^2 - m$ and therefore

$$\frac{n+m}{i} \geq \frac{n+m}{m} > 2m.$$

Similarly the line $L(A', B)$ also separates the same two blocks. Therefore both lines $L(\varphi(A), \varphi(B'))$ and $L(\varphi(A'), \varphi(B))$ separate the φ blocks $\varphi(S_i)$ and $\varphi(S_{i+1})$. Consequently there is a separator line through their intersection that is parallel to the line $L(\varphi(A), \varphi(A'))$. Similarly,for $-m \leq i < 0$, the φ blocks $\varphi(S_i)$ and $\varphi(S_{i+1})$ can be separated by a line parallel to $L(\varphi(A), \varphi(A'))$. So the vertical φ blocks of G_k can be separated by parallel lines. An analogous argument shows that the horizontal φ blocks can be separated by parallel lines as well.

Define next $C = (n, n)$, $C' = (-n, -n)$, and for any $0 \leq j \leq 2m$ let $D_j = \{(t, t+j) : t \in [-m, m-j]\}$ be a diagonal block of the central grid G_m. For $-2m \leq j \leq 0$ let $D_j = \{(t, t+j) : t \in [-m-j, m]\}$ be a diagonal block of G_m. Let $0 < i \leq 2m$ and let $F = (m-i, m)$, $F' = (-m, i-m) \in D_i$, $Q = (m-i+1, m) \in D_{i-1}$. Observe that $n > 2m^2 - m$ implies that $4n > 4m^2 + 1$ and therefore $(2m+1)^2 - 4(n+m) < 0$ which is the discriminant of the quadratic polynomial $x^2 - (2m+1)x + n + m$. Therefore for every positive integer i we have $i^2 - (2m+1)i + n + m > 0$. After rearranging the inequality we get $n+m-i > 2mi - i^2 = i(2m-i)$, so

$$\frac{i}{n+m-i} < \frac{1}{2m-i} \quad \text{implying that} \quad \frac{n+m}{n+m-i} < \frac{2m-i+1}{2m-i},$$

which shows that

$$\frac{n+m-i}{n+m} < \frac{2m-i}{2m-i+1}.$$

Here $(n+m-i)/(n+m)$ is the slope of $L(F', C)$ and $(2m-i)/(2m-i+1)$ is the slope of $L(F', Q)$. This implies that Q is below the line $L(F', C)$ and therefore $L(C, F')$ separates the diagonal blocks D_i and D_{i-1}. Similar arguments as before finish the proof namely, that the images of all the diagonal φ blocks $\varphi(D_i)$ can be separated by lines parallel to $L(\varphi(C), \varphi(C'))$.

We mention that a similar argument with a similar purpose was used in [1].

Appendix B: Proof of Claim 5.1

The projective map μ can be written as $\mu = \tau_3 \circ \tau_2 \circ \tau_1$ where τ_1 is the translation by $-A_{0,0}$, τ_2 is the linear map with $\tau_2 \circ \tau_1(A_{1,0}) = (1,0)$ and $\tau_2 \circ \tau_1(A_{0,1}) = (0,1)$, and τ_3 is the simple projective map that keeps $(0,0)$, $(1,0)$, $(0,1)$ fixed and carries $\tau_2 \circ \tau_1(A_{1,1})$ to $(1,1)$. Define $z = (x, y)$, $\tau_1(z) = z_1$, $\tau_2(z_1) = z_2$, $\tau_3(z_2) = z_3$, and $z_i = (x_i, y_i)$ for $i = 1, 2, 3$.

Evidently $\|z_1 - z\| \leq 2\eta$ for all $z \in \mathbb{R}^2$. From now on we assume $z \in [-\eta, 1+\eta]$. Set $\tau_1(A_{1,0}) = (1+a_1, b_1)$ and $\tau_1(A_{0,1}) = (a_2, 1+b_2)$. Of course $|a_i|, |b_i| < 2\eta$ for every $i = 1, 2$. The inverse of τ_2 is given by the matrix

$$P = \begin{pmatrix} 1+a_1 & b_1 \\ a_2 & 1+b_2 \end{pmatrix}, \quad \text{so} \quad P^{-1} = \frac{1}{\det P}\begin{pmatrix} 1+b_2 & -a_2 \\ -b_1 & 1+a_1 \end{pmatrix},$$

where $\det P = (1+a_1)(1+b_2) - a_2 b_1 > 0$. Now $(x_2, y_2) = \tau_2(x_1, y_1) = P^{-1}(x_1, y_1)$ and

$$\begin{aligned} |x_2 - x_1| &= \frac{1}{\det P}|(\det P x_1 - [(1+b_2)x_1 - a_2 y_1])| \\ &= \frac{1}{\det P}|((1+b_2)(a_1 + b_1 a_2)x_1 - a_2 y_1)| \\ &< \frac{2\eta}{\det P}[(1+2\eta)^2 + 1] \max\{|x_1|, |y_1|\} < 4\eta + O(\eta^2), \end{aligned}$$

where the last inequality holds for $z_1 = (x_1, y_1) \in [-2\eta, 1+2\eta]^2$. Analogously $|y_2 - y_1| < 4\eta + O(\eta^2)$, and then $\|z_2 - z_1\| < 4\eta + O(\eta^2)$ and $z_2 \in [-6\eta, 1+6\eta]^2$ holds with error $O(\eta^2)$.

Let $\tau_2 \circ \tau_1(1,1) = (1+a_3, 1+b_3) =: B$ and $|a_3|, |b_3| < 6\eta + O(\eta^2)$ follows. The line $L((1,0), B)$ intersects the x-axis at the point N_x and $L((1,0), B)$ intersects the

y-axis at N_y. The projective map τ_3 carries the line $L((N_x, 0), (0, N_y))$ to the line at infinity. Direct computation shows that $N_x = -(1 + a_3)/b_3$ and $y_3 = N_x y_2/(N_x - x_2) = y_2 + x_2 y_2/(N_x - x_2)$. Thus when $(x_2, y_2) \in [-6\eta, 1 + 6\eta]^2$ holds with error $O(\eta^2)$, we have

$$|y_3 - y_2| = \left|\frac{x_2 y_2}{N_x - x_2}\right| < 6\eta + O(\eta^2),$$

and the same bound holds for $|x_3 - x_2|$. Finally we have for all $z \in [-\eta, 1 + \eta]^2$ that

$$\begin{aligned}
\|\mu(z) - z)\| &= \|\tau_3 \circ \tau_2 \circ \tau_1(z) - z\| \leq \|\tau_3 \circ \tau_2(z_1) - z_1\| + \|z_1 - z\| \\
&\leq \|\tau_3(z_2) - z_2\| + \|z_2 - z_1\| + \|z_1 - z\| \\
&< 6\eta + 4\eta + 2\eta + O(\eta^2) = 12\eta + O(\eta^2).
\end{aligned}$$

Appendix C: Determining $y_1 - y_0$ in Lemma 11.1

We explain the case when $e = (1, 0)$, see Fig. 6. Recall that $B = (2t + a, b)$ and $|a|, |b| < 2c$. The equations of L_0, L_1, L_2 are

$$y = \frac{b}{2t + a}x, \quad y = \frac{b + 2}{2t + a - 1}x - 1, \quad y = \frac{b - 2}{2t + a - 1}x + 1,$$

respectively. The common point of L_1 and L_2 is $N' = (x_0, y_0)$ where $x_0 = t + a/2 - b/4$. Then

$$\begin{aligned}
y_1 - y_0 &= \frac{b + 2}{2t + a - 1}x_0 - 1 - \frac{b}{2t + a}x_0 \\
&= \left(\frac{b + 2}{2t + a - 1} - \frac{b}{2t + a}\right)x_0 - 1 \\
&= \frac{4t + 2a + b}{(2t + a - 1)(2t + a)}\left(t + \frac{a}{2} + \frac{b}{4}\right) - 1 \\
&= \frac{1}{2t + a - 1} - \frac{b^2}{4(2t + a - 1)(2t + a)} < \frac{1}{2t + a - 1}.
\end{aligned}$$

The computation in the other two cases is similar and is omitted.

Appendix D: Proof of Lemma 11.2

Let $M = |f(j)|$ be the maximum of the $|f(i)|$. Assume without loss of generality that $f(j) = M$ is positive and by symmetry we may assume that $j \leq n/2$. We distinguish two cases based on the how large j is.

Case 1: $j \geq n/10$. The midpoint property with $i = t = j$ shows that $|(f(2j) + f(0))/2 - f(j)| \leq cn/j$, and so

$$\frac{f(2j) + f(0)}{2} - f(j) \geq -\frac{c^* n}{j}.$$

Since $f(0) = 0$ and $n/j \leq 10$, after rearranging this implies that

$$M \geq f(2j) \geq 2f(j) - \frac{2c^* n}{j} \geq 2M - 20c^*$$

and therefore $M \leq 20c^*$.
Case 2: $j < n/10$. Let $n = 4k + r$ where $r = 0, 1, 2, 3$. By the 1-Lipschitz property, $|f(4k)| \leq 3$. Using the midpoint property first with $i = t = 2k$ we get $|f(2k)| \leq 1.5 + c^* n/(2k)$. The midpoint property again with $i = t = k$ gives

$$|f(k)| \leq \frac{3}{4} + \frac{c^* n}{4k} + \frac{c^* n}{k} \leq 1 + \frac{5}{4} \cdot \frac{n}{k} c^* \leq 1 + 6c^*,$$

since $4 \leq n/k \leq 103/25 < 24/5$. Here $k - j > n/8$ because $j < n/10 < k$ and $k \geq 25n/103$. The midpoint property with $i = k$, $t = k - j$ shows that

$$\begin{aligned}
\frac{f(2k-j) + f(j)}{2} - f(k) \leq \frac{c^* n}{k - j} \qquad &\text{implying} \\
f(2k - j) \leq \frac{2c^* n}{k - j} + 2f(k) - f(j) &\leq 16c^* + 2 + 12c^* - M \\
&\leq 28c^* + 2 - M.
\end{aligned}$$

Finally we use the midpoint property with $i = t = 2k - j > n/3$,

$$\begin{aligned}
\frac{f(4k-2j) + f(0)}{2} - f(2k - j) \leq \frac{c^* n}{2k - j} \qquad &\text{implying} \\
f(4k - 2j) \leq \frac{2c^* n}{2k - j} + 2f(2k - j) - f(0)& \\
\leq 6c^* + 2(28c^* + 2 - M) &= 62c^* + 4 - 2M,
\end{aligned}$$

and $-M \leq f(4k - 2j) \leq 62c^* + 4 - 2M$ implies $M \leq 62c^* + 4$.

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