

Crossing lemma for the odd-crossing number

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ABSTRACT

A graph is 1-planar, if it can be drawn in the plane such that there is at most one crossing on every edge. It is known, that 1-planar graphs have at most $4n - 8$ edges.

We prove the following odd-even generalization. If a graph can be drawn in the plane such that every edge is crossed by at most one other edge *an odd number of times*, then it is called 1-odd-planar and it has at most $5n - 9$ edges. As a consequence, we improve the constant in the Crossing Lemma for the odd-crossing number, if adjacent edges cross an even number of times. We also give upper bound for the number of edges of k -odd-planar graphs.

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1. Introduction

By a *graph* we always mean a *simple graph*, that is, a graph with no loops and no parallel edges. We use the term *multigraph* if loops and parallel edges are allowed. A *drawing* of a (multi)graph in the plane is a representation such that vertices are represented by distinct points and its edges by curves connecting the corresponding points. We assume that no edge passes through any vertex other than its endpoints, no two edges touch each other (i.e., if two edges have a common interior point, then at this point they properly cross each other), no three edges cross at the same point, and two edges cross only finitely many times.

The *crossing number* of a graph G , $\text{CR}(G)$, is the minimum number of crossings (crossing points) over all drawings of G . The *pair-crossing number*, $\text{PCR}(G)$, is the minimum number of pairs of crossing edges over all drawings of G . In an optimal drawing for $\text{CR}(G)$, any two edges cross at most once [17]. Therefore, it is not easy to see the difference between these two definitions. Indeed, there was some confusion in the literature between these two notions, until the systematic study of their relationship [12]. Clearly, $\text{PCR}(G) \leq \text{CR}(G)$, and in fact, we cannot rule out the possibility, that $\text{CR}(G) = \text{PCR}(G)$ for every graph G . Probably it is the most interesting open problem in this area. From the other direction, the best known bound is $\text{CR}(G) = O(\text{PCR}(G)^{3/2} \log \text{PCR}(G))$ [16], [8].

The *odd-crossing number*, $\text{OCR}(G)$, is the minimum number of pairs of edges that cross an odd number of times, over all drawings of G . Clearly, (as non-crossing edges cross an even number of times) for every graph G , $\text{OCR}(G) \leq \text{PCR}(G) \leq \text{CR}(G)$. According to the (weak) Hanani-Tutte theorem [6], [14], if $\text{OCR}(G) = 0$, then G is planar, that is, $\text{OCR}(G) = \text{PCR}(G) = \text{CR}(G) = 0$. It was shown in [14] that for $k = 1, 2, 3$, if $\text{OCR}(G) = k$, then $\text{OCR}(G) = \text{PCR}(G) = \text{CR}(G) = k$. There are examples where OCR is different from PCR and CR , namely, there is an infinite family of graphs with $\text{OCR}(G) < 0.855 \cdot \text{PCR}(G)$ [18], [15]. From the other direction we only have $\text{PCR}(G) < 2\text{OCR}(G)^2$ [12].

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Table 1
Nine versions of the crossing number.

Rule +	$\text{OCR}_*(G) \leq \text{OCR}_+(G)$	$\text{PCR}_+(G)$	$\text{CR}(G)$
Rule 0	$\text{OCR}(G)$	$\text{PCR}(G)$	
Rule -	$\text{OCR}_-(G)$	$\text{PCR}_-(G)$	$\text{CR}_-(G)$

In [13] some further variants were introduced, in order to study the role of crossings between adjacent edges. For each of CR, PCR, and OCR, they introduced three counting rules.

Rule +: Only those drawings are considered, where adjacent edges cannot cross.

Rule 0: Adjacent edges can cross and their crossings are counted as well.

Rule -: Adjacent edges can cross and their crossings are not counted.

Combining these rules with the three crossing numbers, we get nine possibilities. But it is easy to see that $\text{CR}_+ = \text{CR}$ [13]. On the other hand, regarding Rule + for the odd-crossing number, it seems more natural to assume that adjacent edges cross an *even number of times* than to assume that they do not cross at all. So, let $\text{OCR}_*(G)$ be the minimum number of odd-crossing pairs of edges over all drawings of G where adjacent edges cross an even number of times (these drawings are called *weakly semisimple* in [5]). Therefore, we have nine versions, see Table 1. In this table, values do not decrease if we move to the right or up, and it was shown in [15] that $\text{CR}(G) < 2\text{OCR}_-(G)^2$. On the other hand, there are graphs G such that $\text{OCR}_-(G) < \text{OCR}(G)$ [7].

The Crossing Lemma, discovered by Ajtai, Chvátal, Newborn, Szemerédi [4] and independently by Leighton [9] is definitely the most important inequality for crossing numbers.

Crossing Lemma. *If a simple graph G of n vertices has $m \geq 4.5n$ edges, then $\text{CR}(G) \geq \frac{1}{60.75} \frac{m^3}{n^2}$ edges.*

The bound is tight, apart from the value of the constant [11]. The constant above follows from the beautiful probabilistic argument of Chazelle, Sharir and Welzl [3]. This argument works for all nine versions of the crossing number [13]. For the classical crossing number, $\text{CR}(G)$, the constant was improved in three steps [11], [10], the best bound is due to Ackerman [1], $\text{CR}(G) \geq \frac{1}{29} \frac{m^3}{n^2}$, when $m \geq 7n$.

The only improvement for any other version is a result of Ackerman and Schaefer [2], $\text{PCR}_+(G) \geq \frac{1}{34.2} \frac{m^3}{n^2}$, when $m \geq 6.75n$. For all other versions of the crossing number, the constant 60.75 is the best we have.

In this note we get an improvement for two other versions, OCR_+ and OCR_* .

Theorem 1. *Suppose that G has n vertices and $m \geq 6n$ edges. Then $\text{OCR}_+(G) \geq \text{OCR}_*(G) \geq \frac{1}{54} \frac{m^3}{n^2}$.*

Our approach is very similar to all previous improvements mentioned above. The first step is to find many odd-crossing pairs in sparse graphs. Then this bound is applied for a random subgraph of G to get the general bound.

A graph G is called *k-planar* if it can be drawn in the plane such that there are at most k crossings on each edge. Such a drawing is called a *k-plane* drawing. Let $m_k(n)$ denote the maximum number of edges of a *k-planar* graph of n vertices.

Clearly, $m_0(n) = 3n - 6$. It is known that $m_1(n) = 4n - 8$ for $n \geq 12$, $m_2(n) \leq 5n - 10$ and it is tight for infinitely many values of n , [11], $m_3(n) \leq 5.5n - 11$, $m_4(n) \leq 6n - 12$, which are tight up to an additive constant [10], [1].

We prove an odd-even version of these results. A graph G is called *k-odd-planar* if it can be drawn in the plane such that any edge is crossed an *odd number of times* by at most k other edges (edges crossing an even number of times do not count). Such a drawing is called a *k-odd-plane* drawing.

Let $m_k^{\text{odd}}(n)$ denote the maximum number of edges of a *k-odd-planar* graph with n vertices. Clearly, we have $m_k^{\text{odd}}(n) \geq m_k(n)$ and by the weak Hanani-Tutte theorem [6], [14], we have $m_0^{\text{odd}}(n) = 3n - 6$.

Theorem 2. *For any $n, k \geq 1$ we have*

$$m_k^{\text{odd}}(n) \leq m_k(n) + k(n - 1).$$

This result is interesting only for small k . For k (and n) large enough, we have an easy better bound.

Theorem 3. *For any $n, k \geq 1$ we have*

$$m_k^{\text{odd}}(n) \leq \sqrt{32} \sqrt{kn}.$$

We do not think that our bounds are tight. We cannot even rule out the possibility that $m_k^{\text{odd}}(n) = m_k(n)$ for every n, k .

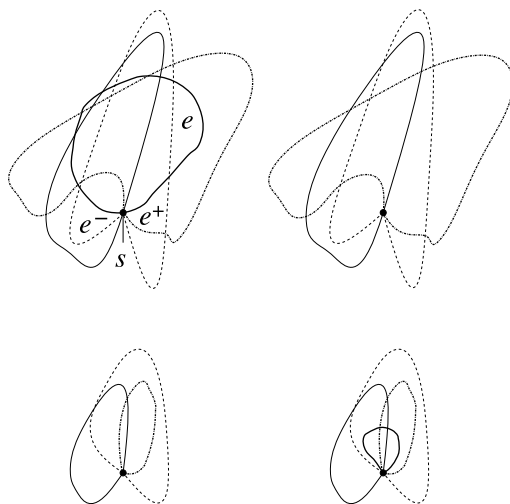


Fig. 1. Steps in the proof of Lemma 1.

2. Proofs

A (multi)graph G , together with its drawing D in the plane, is called *topological (multi)graph*. The points (resp. curves), representing the vertices (resp. edges) of G are called *vertices* (resp. *edges*) of the topological (multi)graph. If the drawing is obvious from the context, we do not make any notational distinction between the topological (multi)graph and the underlying abstract (multi)graph. Let G be a topological multigraph and e an edge. The pieces of e in small neighborhoods of its endpoints are called *endings* of e and denoted by e^+ and e^- . As e is an undirected edge, the $+$ and $-$ signs have no special meaning, either ending can be e^+ and the other one is e^- . The *rotation system* is the cyclic order of adjacent edges, or endings, at each vertex. A cyclic order is always clockwise. Two edges form an *odd pair* (resp. *even pair*) if they cross an *odd* (resp. *even*) number of times. An edge is called *even* if it is crossed an even number of times by every other edge and it is *odd* otherwise.

According to the weak Hanani-Tutte theorem, if a graph can be drawn so that any two edges cross an even number of times, then it is planar. This result has many proofs, one of the nicest and simplest is due to Pelsmajer, Schaefer and Štefankovič [14]. The proof is based on the following lemma.

Lemma 0 ([14]). *Let G be a topological multigraph that has one vertex and n edges (loops). Suppose that every edge is even. Then, G can be redrawn such that the rotation system is the same and there is no edge crossing.*

First we prove the following generalization.

Lemma 1. *Let G be a topological multigraph that has one vertex and m edges (loops). Then, G can be redrawn such that (i) the rotation system is the same (ii) even pairs do not cross, (iii) odd pairs cross once, and (iv) there are no self-crossings.*

Proof. The proof is by induction on the number of edges. If there is only one loop, the statement is trivial.

Suppose that G has one vertex v and $m > 1$ loops, and the statement has been proved for a smaller number of loops. Let D denote the present drawing of G . We can assume immediately that there are no self-crossings, since we can eliminate self-crossings of each loop without changing the number of crossings between different loops.

Let s be a very short segment ending in v and not crossing any of the loops.

Assume without loss of generality that for any loop e , s , e^- and e^+ are in this clockwise order at v (otherwise we can switch the notation for e^- and e^+).

For two loops e and f , we say that $f < e$ if the clockwise order of endings at v is s , e^- , f^- , f^+ , e^+ . The relation $<$ defines a partial order on the loops. Let e be a minimal loop with respect to $<$. That is, any other loop f has at most one ending between e^- and e^+ .

Delete e and apply the induction hypothesis. We get a drawing D' satisfying the conditions. In particular, the cyclic order of endings is the same as in D . Insert e^- and e^+ back to their original places and connect them by an arc in a small neighborhood of v , going clockwise from e^- to e^+ . We can do it such that it crosses only those endings, each exactly once, which are between e^- and e^+ . In the obtained drawing D_1 , the rotation system is the same as in D and there are no self-crossings. For any two loops f , $g \neq e$, (ii) and (iii) are satisfied by the induction hypothesis. If e, f is an odd pair in D , then it has exactly one ending between e^- and e^+ so in D_1 they cross once. If it is an even pair in D , then f has no ending between e^- and e^+ so in D_1 they do not cross. This concludes the proof of Lemma 1. See Fig. 1. \square

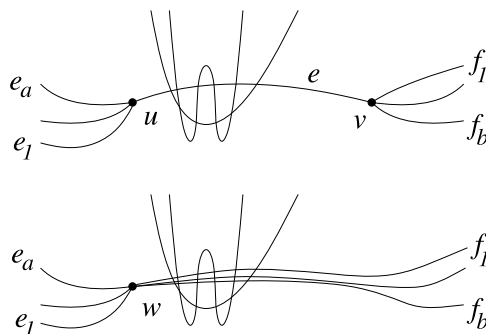


Fig. 2. Contracting edge $e = uv$ to u . The resulting vertex is called w .

Remarks. 1. The statement of Lemma 1 can be found implicitly in [14], p. 492, as a remark.

2. Another possible proof is the following. Take a drawing of G which has the same rotation system and under this condition the minimum number of crossings. It can be shown that this drawing satisfies the conditions, otherwise we could have a better drawing of G . But we did not find this method easier.

Lemma 2. Let $k \geq 0, l \geq 1$. Suppose that $n_1, \dots, n_l > 0$ and $n_1 + \dots + n_l = n$. Then

$$m_k(n) \geq m_k(n_1) + \dots + m_k(n_l).$$

Proof. For every i take a k -plane graph of n_i vertices and $m_k(n_i)$ edges. Their disjoint union is a k -plane graph with n vertices and $m_k(n_1) + \dots + m_k(n_l)$ edges. \square

Definition. Suppose that G is a topological multigraph. Let $e = uv$ be an edge. The contraction of e to u is the following procedure. Move v to u , along the edge e . At the same time, extend all other edges incident to v by curves along e , from a neighborhood of v to u , without creating any crossing among them. See Fig. 2.

Suppose that $e = uv$ is an even edge of G . Contract e to u and let w be the new vertex and G' the resulting topological graph. There is a natural bijection between the edges of $G \setminus e$ and G' , and the odd pairs are exactly the same. Any vertex $x \neq u, v$ of G remains a vertex of G' with the same cyclic order of incident edges. Suppose that in G , e, e_1, \dots, e_a and e, f_1, \dots, f_b are the cyclic orders of incident edges at u and v , respectively. Then in G' , the cyclic order of edges at w is $e_1, \dots, e_a, f_1, \dots, f_b$. We can “reverse” the contraction as follows. If we replace w in G' by two very close vertices u' and v' , connected by an edge, u' incident to e_1, \dots, e_a and v' incident to f_1, \dots, f_b , we get a topological graph with exactly the same rotation system as G .

Proof of Theorem 2. Suppose that G has n vertices and m edges, and it is drawn in the plane such that any edge is crossed by at most k other edges an odd number of times. We can assume that G is connected, as an abstract graph, otherwise, by Lemma 2, we can argue separately for each component. Suppose first that G contains a spanning tree F whose edges cross each other an even number of times. Remove all edges of G that cross an edge of F an odd number of times. We get the topological graph G_1 in the inherited drawing. Since F has $n - 1$ edges and each of them is crossed by at most k other edges an odd number of times, for the number of m_1 edges of G_1 , $m_1 \geq m - k(n - 1)$. In G_1 , all edges of F are even. Contract all edges of F . We get a topological multigraph G_2 , which has only one vertex v . The odd pairs are exactly the same as in G_1 . Apply Lemma 1 for G_2 and get the topological multigraph G_3 . Since the rotation system is the same in G_3 as in G_2 , we can reverse the contractions without creating any additional crossing. This way we get G_4 . Observe that G_4 is a redrawing of G_1 , with the property that if two edges form an even (odd) pair in G , then they do not cross (cross once) in G_4 . By the assumption, every edge in G_1 is part of at most k odd pairs. Therefore, G_4 is a k -plane drawing of G_1 so it has at most $m_k(n)$ edges. Consequently, for the number of edges of G we have $m \leq m_k(n) + k(n - 1)$.

Suppose now that G does not contain a spanning tree whose edges cross each other an even number of times. Let F be a maximal forest whose edges cross each other an even number of times. Let V_1, \dots, V_l be the vertices of the connected components of F , $|V_i| = n_i$. Since F is not a tree, $l \geq 2$ and $|E(F)| = n - l$. By the maximality of F , those edges of G that connect two components cross some edge of F an odd number of times. Remove all edges of G that cross some edge of F an odd number of times. We removed at most $k(n - l)$ edges. The resulting graph has l components G_1, G_2, \dots, G_l , on vertices V_1, \dots, V_l . Let $|E(G_i)| = m_i$. By the construction, each G_i contains a spanning tree F_i whose edges are even. Contract the edges of F_i and argue as above. We get that

$$m \leq \sum_{i=1}^l m_i + k(n-l) \leq \sum_{i=1}^l m_k(n_i) + k(n-l) \leq m_k(n) + k(n-1). \quad \square$$

Proof of Theorem 1. We have $m_0^{\text{odd}}(n) = 3n - 6$, and by Theorem 2, $m_1^{\text{odd}}(n) \leq m_1(n) + n - 1 = 5n - 9$.

First we show, by induction on the number of edges that for any graph with n vertices and m edges, $\text{ocr}(G) \geq m - 3n$. If $m \leq 3n$, the statement is trivial. Suppose that $m > 3n$ and we have proved the statement for $m - 1$. Take any drawing D of G . Since $m_0^{\text{odd}}(n) = 3n - 6$, there is an odd pair e, f . Remove e from G , and then the obtained drawing has one less edges and at least one less odd pairs. Therefore, the number of odd pairs in D is at least $1 + (m - 1) - 3n = m - 3n$.

Now we show, again by induction on the number of edges that for any graph with n vertices and m edges, $\text{ocr}(G) \geq 2m - 8n$. If $m \leq 5n$, then $\text{ocr}(G) \geq m - 3n \geq 2m - 8n$. Suppose that $m > 5n$ and we have proved the statement for $m - 1$. Take any drawing D of G , since $m_1^{\text{odd}}(n) \leq 5n - 9$, there is an edge e in two odd pairs, e, f and e, g . Remove e from G , and then the obtained drawing has one less edges and at least two less odd pairs. Therefore, the number of odd pairs in D is at least $2 + 2(m - 1) - 8n = 2m - 8n$.

Let G be a graph of n vertices and $m \geq 6n$ edges, drawn in the plane realizing $\text{ocr}_*(G)$, that is, any two adjacent edges cross an even number of times and there are $\text{ocr}_*(G)$ pairs of edges that cross an odd number of times. Take a random subgraph G' such that we take each vertex independently with probability $p = 6n/m$. Let n', m' , and $x(G')$ denote the number of vertices (resp. edges) of G' , and the number of odd-crossing pairs of edges in G' , in the inherited drawing. We have

$$E(n') = pn, \quad E(m') = p^2m, \quad E(\text{ocr}_*(G')) \leq E(x(G')) = p^4\text{ocr}_*(G).$$

For G' we have $\text{ocr}_*(G') \geq \text{ocr}(G') \geq 2m' - 8n'$, taking expected values,

$$p^4\text{ocr}_*(G) \geq 2p^2m - 8pn.$$

For $p = 6n/m$, we have

$$\text{ocr}_*(G) \geq \frac{1}{54} \frac{m^3}{n^2}. \quad \square$$

Remark. Combining Theorem 2 and the bounds for $m_k(n)$ we obtain that $m_1^{\text{odd}}(n) \leq 5n - 9$ and $m_2^{\text{odd}}(n) \leq 7n - 12$. In the proof of Theorem 1 we used only the first inequality, the second would not help. However, if we could prove that $m_2^{\text{odd}}(n) \leq 6.8n + c$ for some constant c , then we would get an improvement in Theorem 1 as well.

Proof of Theorem 3. We apply a version of the Crossing Lemma for the odd-crossing number, from [12]: If G has n vertices and m edges, and $m \geq 4n$, then $\text{ocr}(G) \geq \frac{1}{64} \frac{m^3}{n^2}$.

Let $k \geq 1$. Suppose that G is k -odd-planar with n vertices and $m = m_k^{\text{odd}}(n)$ edges. We can assume that $m \geq 4n$, otherwise we are done.

Take a k -odd-plane drawing of G . Every edge participates in at most k odd pairs. Therefore, there are at most $km/2$ odd pairs, so $\text{ocr}(G) \leq km/2$. On the other hand, we can apply the Crossing Lemma for the odd-crossing number, so

$$km/2 \geq \text{ocr}(G) \geq \frac{1}{64} \frac{m^3}{n^2}.$$

It follows that $m \leq \sqrt{32} \sqrt{kn}$. \square

Remarks. 1. It is easy to see that Theorem 3 is better than Theorem 2 if $n, k \geq 40$. This threshold 40 can be improved, but it is not very interesting.

2. The analogue of Theorem 3 for $m_k(n)$ instead of $m_k^{\text{odd}}(n)$ was first proved in [11]. Using the best known constant for the Crossing Lemma from [1] we can get that for any $n, k \geq 2$, $m_k(n) \leq 3.81 \sqrt{kn}$.

3. In the proof of Theorem 1 we had to assume that adjacent edges cross an even number of times. Therefore, it holds only for $\text{ocr}_*(G)$ and $\text{ocr}_+(G)$. We believe that it is not necessary and Theorem 1 can be extended to $\text{ocr}(G)$. Similarly, we believe that the bound of Ackerman and Schaefer [2], $\text{PCR}_+(G) \geq \frac{1}{34.2} \frac{m^3}{n^2}$ can be extended to $\text{PCR}(G)$. We also believe that both of them are very interesting problems.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

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