A positive Bondi–type mass in asymptotically de Sitter spacetimes

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Abstract

The general structure of the conformal boundary $\mathscr{I}^+$ of asymptotically de Sitter spacetimes is investigated. First we show that Penrose's quasi-local mass, associated with a cut $\mathcal{S}$ of the conformal boundary, can be zero even in the presence of outgoing gravitational radiation. On the other hand, following a Witten–type spinorial proof, we show that an analogous expression based on the Nester–Witten form is finite only if the Witten spinor field solves the 2-surface twistor equation on $\mathcal{S}$, and it yields a positive functional on the 2-surface twistor space on $\mathcal{S}$, provided the matter fields satisfy the dominant energy condition. Moreover, this functional is vanishing if and only if the domain of dependence of the spacelike hypersurface which intersects $\mathscr{I}^+$ in the cut $\mathcal{S}$ is locally isometric to the de Sitter spacetime. For non-contorted cuts this functional yields an invariant analogous to the Bondi mass.

1 Introduction

The simplest explanation of the root of the deviation of the observed red shift vs. luminosity diagram of distant type Ia supernovae from the expected one is probably the strict positivity of the cosmological constant [1, 2, 3]. Thus the history of our observed universe should be modeled by asymptotically de Sitter spacetimes, and hence a systematic study of these spacetimes, e.g. their asymptotic properties, is physically justified. The conformally cyclic cosmological model (or shortly CCC model) of Penrose [3] is based on the positivity of the cosmological constant. In this model the crossover hypersurface is just the (spacelike) future timelike infinity of the previous aeon and the big bang singularity of the present aeon.

In the study of asymptotic properties of spacetime one of the most important questions is that of the 'conserved' quantities, e.g. the energy-momentum. In asymptotically flat or asymptotically anti-de Sitter spacetimes the 'total' energy-momentum is thought of as associated with a *localized* gravitating source at an arbitrary but *finite* instant, represented by a spacelike hypersurface in the spacetime (or, rather, by a 'cut' of the null or timelike conformal boundary, respectively, at *finite* retarded time). This may be strictly conserved and total (like the ADM energy-momentum [4]), or may change in time but still be total (like the total mass associated with *closed* Cauchy surfaces in closed universes [9, 10]), or may change in time and be associated only with an (infinite) portion of spacetime (like the Bondi–Sachs [5, 6, 7] energy-momentum at the future null infinity, or the Abbott–Deser [8] energy-momentum at the conformal boundary of asymptotically anti de Sitter spacetime, which are changing from cut to cut in the presence of outgoing radiation or of in- or outgoing radiation, respectively). However, the analogous 'total' energy-momentum in asymptotically de Sitter spacetimes would be associated with the *spacelike* future conformal boundary (or a part of this boundary), and hence it would be interpreted as being associated with the *asymptotic final state* of the universe (or a part of it). Though it would not have any dynamical content (in contrast to the total mass of closed universes, the Bondi–Sachs or the Abbott–Deser energy-momenta), it could provide a good control on the spacetime geomery near the conformal boundary.

The idea of the energy-momentum associated with a cut of the future conformal boundary has already been raised by Penrose [11], where two potentially viable strategies were also discussed. One of these approaches is based on the charge integral of the curvature and the use of twistorial methods, and the second is analogous to the ideas behind the Bondi–Sachs energy-momentum. (The asymptotic properties of the spacetime geometry, including the 'conserved' quantities, are already well known when the spacetime is asymptotically flat [12], and also when the cosmological constant is *negative* [12, 14, 15]. For an analogous recent investigation in the presence of a *positive* cosmological constant, see [16]. For a different concept of 'total' mass in the presence of a positive Λ, see [8], which mass can, however, be negative [17, 18, 19, 20]. The second strategy of Penrose was also discussed in a nutshell by Frauendiener [21]. The conserved quantities of the linearized theory on de Sitter background are discussed in [22].)

In the present paper we investigate the general structure of the conformal boundary, and the notion of 'total' energy-momentum associated with a cut of the future conformal boundary and based both on the charge integral of the curvature and of the Nester–Witten 2-form, too. We refine (and at certain points improve) the previous analyses of the general structure of the conformal boundary itself, construct a coordinate system and complex null tetrad (that are analogous to the Bondi and Newman–Penrose ones, respectively, near the null infinity of asymptotically flat spacetimes), and determine the asymptotic geometry of the smooth spacelike hypersurfaces that extend to the (spacelike) conformal boundary. These provide the technical background for the investigation of the energy-momentum. We show that the construction for the energy-momentum based on the charge integral of the curvature and the use of the 2-surface twistors does *not* have the rigidity property: It may be vanishing even for non-trivial spacetime configurations, e.g. in vacuum spacetimes with non-vanishing rescaled conformal electric curvature (representing for example pure outgoing gravitational radiation). Hence it does not seem to provide an appropriate measure of the 'strength' of the gravitational 'field'.

On the other hand, we show that the expression based on the Nester–Witten 2-form has the positivity and rigidity properties: Following a Witten-type argument on a spacelike hypersurface Σ that intersects the future conformal boundary in a spacelike cut $\mathcal{S}$, the integral of the Nester–Witten 2-form on $\mathcal{S}$ defines a non-negative functional on the space of the boundary values for the Dirac spinor solution of the Witten equation on Σ, provided the matter fields satisfy the dominant energy condition; and this functional is vanishing if and only if the domain of dependence of Σ is locally isometric to the de

Sitter spacetime. Interestingly enough, we may *not* impose any boundary condition for the Witten spinors at the cut by hand. All the boundary conditions are determined by the Witten equation itself and the requirement of the finiteness of the resulting functional; and the boundary condition is that the spinor field on the cut *must solve the 2-surface twistor equation* of Penrose. On the other hand, the interpretation of the resulting (positive definite) functional is not trivial: What we could consider to be the energy-momentum 4-vector is only a part of a bigger multiplet of quantities which are 'mixed' by the symmetry group of the 2-surface twistor space. Energy-momentum, or at least mass, could be defined in an invariant way only in the presence of extra structures on the 2-surface twistor space. In particular, if the cut is non-contorted (e.g. when the whole conformal boundary is intrinsically locally conformally flat), then a positive measure of the strength of the gravitational 'field', which could be interpreted as mass, is found whose vanishing is equivalent to the local de Sitter nature of the domain of dependence of the hypersurface Σ above.

In the proof of the existence of solutions of the Witten equation we use functional analytic techniques. It turns out that, on asymptotically hyperboloidal hypersurfaces, the solutions of the Witten equations and their derivatives fall off with the *same* rate. Thus, the weighted Sobolev spaces do *not* seem to be the appropriate function spaces, and the classical Sobolev spaces with an overall weight function in front of the volume element should be used. The necessary technical details are also developed here.

The structure of the paper follows the logic above: In section 2 we discuss the structure of the conformal boundary, section 3 is devoted to the discussion of the properties of an expression based on the integral of the curvature on the cut. Then, in section 4, we consider the expression based on the Nester–Witten 2-form, prove the positivity and rigidity properties, and introduce the mass on non-contorted cuts. The Appendix is devoted to the introduction of the functional analytic tools and statements needed in the proof of the existence and uniqueness of solutions of the Witten equation on asymptotically hyperboloidal hypersurfaces.

We use the abstract index formalism and the sign conventions of [12]. In particular, the signature of the spacetime metric is $(+,-,-,-)$, and the Riemann tensor is defined according to $-R^a{}_{bcd}X^bV^cW^d := V^c\nabla_c(W^d\nabla_dX^a) - W^c\nabla_c(V^d\nabla_dX^a) - [V,W]^c\nabla_cX^a$ for any vector fields X^a, V^a and W^a. Thus, Einstein's equations take the form $G_{ab} := R_{ab} - \frac{1}{2}Rg_{ab} = -\varkappa T_{ab} - \Lambda g_{ab}$, where $\varkappa := 8\pi G$ and G is Newton's gravitational constant, and $\Lambda > 0$.

2 The general framework

2.1 The conformal boundary of spacetimes with positive Λ

Let the physical spacetime be denoted by $(\hat M, \hat g_{ab})$, which is assumed to admit a nontrivial smooth conformal completion (M, g_{ab}, Ω); i.e. there exist a manifold M with nonempty boundary ∂M, a Lorentzian metric g_{ab} on M and a function $\Omega : M \to [0,\infty)$, all of them smooth, such that

(i) $M - \partial M$ is diffeomorphic to (and hence identified with) $\hat M$;

(ii) $g_{ab} = \Omega^2\hat g_{ab}$ on $\hat M$;

(iii) the boundary is just $\partial M = \{\Omega = 0\}$, and $\nabla_a\Omega$ is nowhere vanishing on ∂M;

(iv) $\hat{g}_{ab}$ solves Einstein's equations $\hat{R}_{ab} - \frac{1}{2}\hat{R}\hat{g}_{ab} = -\varkappa\hat{T}_{ab} - \Lambda\hat{g}_{ab}$ with *positive* cosmological constant Λ and energy-momentum tensor $\hat{T}_{ab}$ satisfying the dominant energy condition;

(v) the physical energy-momentum tensor satisfies the fall-off condition $\hat{T}^a{}_b = \Omega^3 T^a{}_b$, where $T^a{}_b$ is some smooth tensor field on M.[1]

It is known [12] that $(\nabla_e\Omega)(\nabla^e\Omega) \approx \Lambda/3$, where $\approx$ means 'equal at the points of ∂M'. Thus the positivity of Λ is equivalent to the *spacelike* nature of ∂M, and we denote by $\mathscr{I}^+$ the part of ∂M whose points are *future* endpoints of inextensible non-spacelike curves in $\hat{M}$. (In the CCC model such a boundary hypersurface represents the crossover hypersurface between two successive aeons.) The hat on quantities and objects is referring to the physical spacetime, while the unhatted ones are in the unphysical (M, g_{ab}). (See e.g. [16], or for the analogous definition in the $\Lambda < 0$ case, see [14].)

A number of facts follow from this definition. In particular,

(1) $\hat{C}^a{}_{bcd} = C^a{}_{bcd} \approx 0$, i.e. the Weyl tensors are vanishing on $\mathscr{I}^+$ [12];

(2) there is a conformal factor Ω such that the extrinsic curvature χ_{ab} and the acceleration a_e of $\mathscr{I}^+$ in (M, g_{ab}) are vanishing ('conformal Bondi gauge'). The remaining conformal gauge freedom is $g_{ab} \mapsto \omega^2 g_{ab}$, where $\omega = \omega_0 + \Omega^2\Theta$ is a strictly positive function on M in which Θ is arbitrary on M, ω_0 is arbitrary but positive on $\mathscr{I}^+$ and is constant along its normals (at least in a neighbourhood of $\mathscr{I}^+$). Thus we can still rescale the *intrinsic* induced metric of $\mathscr{I}^+$ freely [12];

(3) if N^a denotes the future pointing, g_{ab}–unit normal to the $\Omega =$ const hypersurfaces (denoted henceforth by $\mathcal{H}_\Omega$), and $P^a_b := \delta^a_b - N^a N_b$, the g_{ab}–orthogonal projection to $\mathcal{H}_\Omega$, then the part $P^a_b\hat{T}^b{}_c N^c$ of the physical energy-momentum tensor tends to zero in the $\Omega \to 0$ limit *faster* than Ω^3, i.e. with Ω^4 [14].

However, the analysis behind these results can be refined further, yielding a slightly more detailed characterization of the structure of the conformal boundary and minor corrections of previous results. In particular, the same general formulae from which the faster fall-off $P^a_b\hat{T}^b{}_c N^c = O(\Omega^4)$ was derived in [14] imply that $P^a_b\hat{T}^b{}_c P^c_d = O(\Omega^4)$ also holds; or that the divergence equation (10) of [14] for the electric part of the rescaled Weyl tensor should be corrected. Thus, in the next subsection, a more detailed discussion of the structure of the conformal boundary will be given.

2.2 The structure of the conformal boundary

2.2.1 Consequences of Einstein's equations

By Einstein's equations, the assumption on the asymptotic form of the physical energy-momentum tensor and the conformal rescaling formulae, the Einstein tensor of the unphysical spacetime is

[1] It might be worth noting that this fall-off condition is stronger than is needed to retain the smoothness of the conformal boundary (cf. [13]).

$$
\begin{aligned}
R_{ab}-\frac{1}{2}Rg_{ab}&=\hat{R}_{ab}-\frac{1}{2}\hat{R}\hat{g}_{ab}+2\Omega^{-1}\big(\nabla_a\nabla_b\Omega-g_{ab}\nabla_c\nabla^c\Omega\big)+3\Omega^{-2}g_{ab}\big(\nabla_c\Omega\big)\big(\nabla^c\Omega\big)\\
&=-\varkappa\Omega T_{ab}+\Omega^{-2}g_{ab}\Big(3\big(\nabla_c\Omega\big)\big(\nabla^c\Omega\big)-\Lambda\Big)+2\Omega^{-1}\big(\nabla_a\nabla_b\Omega-g_{ab}\nabla_c\nabla^c\Omega\big). \qquad (2.1)
\end{aligned}
$$

Multiplying by Ω^2 and evaluating at $\Omega=0$, and multiplying by Ω, taking the trace and evaluating at $\Omega=0$, respectively, we obtain that $3(\nabla_c\Omega)(\nabla^c\Omega)\approx\Lambda$ and $4\nabla_a\nabla_b\Omega\approx g_{ab}(\nabla_c\nabla^c\Omega)$ (see also [12]). Thus, smoothness of the extension implies that there exist a smooth function ϕ and a tensor field Ψ_{ab} on M such that

$$
\big(\nabla_c\Omega\big)\big(\nabla^c\Omega\big)=\frac{1}{3}\Lambda+\Omega\phi, \qquad \nabla_a\nabla_b\Omega=\frac{1}{4}g_{ab}\big(\nabla_c\nabla^c\Omega\big)+\Omega\Psi_{ab}. \qquad (2.2)
$$

Taking the trace of the second we find that $\Psi_{ab}g^{ab}=0$. It is known (see e.g. [12]) that the conformal factor can always be chosen such that $\nabla_c\nabla^c\Omega\approx 0$ (the 'Bondi conformal gauge'), i.e. $\nabla_c\nabla^c\Omega=\Omega\Psi$ for some smooth Ψ on M. In the rest of the present paper we assume that Ω is such a conformal factor. Hence

$$
\nabla_a\nabla_b\Omega=\Omega\big(\frac{1}{4}g_{ab}\Psi+\Psi_{ab}\big). \qquad (2.3)
$$

Substituting the first of (2.2) and (2.3) into (2.1) we obtain

$$
R_{ab}-\frac{1}{2}Rg_{ab}=-\varkappa\Omega T_{ab}+2\Psi_{ab}-\frac{3}{2}\Psi g_{ab}+3\Omega^{-1}\phi g_{ab}.
$$

Multiplying by Ω and evaluating at $\Omega=0$, we obtain that $\phi\approx 0$, i.e.

$$
\big(\nabla_a\Omega\big)\big(\nabla^a\Omega\big)=\frac{1}{3}\Lambda+\Omega^2\Phi \qquad (2.4)
$$

for some smooth function Φ on M.

However, Ψ_{ab}, Ψ and Φ are not independent. To see this, let us define the lapse N of the foliation by the level sets $\mathcal{H}_\Omega$ of Ω by $1=:-NN^a\nabla_a\Omega$, by means of which the future pointing unit normal of the leaves is $N^a=-Ng^{ab}\nabla_b\Omega$ and $N=1/|\nabla_c\Omega|\approx\sqrt{3/\Lambda}$ (by (2.4)). (Since Ω is *decreasing* in the future direction and we want the lapse to be *positive*, it should be defined with the minus sign.) For later use, we need the acceleration of the leaves $\mathcal{H}_\Omega$. By the definition of the lapse and (2.4) this is

$$
a_e:=N^a\nabla_aN_e=-D_e(\ln N)=\frac{1}{2}N^2D_e\frac{1}{N^2}=\frac{1}{2}\Omega^2N^2D_e\Phi, \qquad (2.5)
$$

where D_e denotes the intrinsic Levi-Civita derivative operator on $\mathcal{H}_\Omega$ determined by the induced (negative definite) metric $h_{ab}:=P^c_aP^d_bg_{cd}$. Also, we can calculate the extrinsic curvature and its trace:

$$
\chi_{ab}:=P^c_aP^d_b\nabla_cN_d=-\Omega N\big(P^c_aP^d_b\Psi_{cd}+\frac{1}{4}\Psi h_{ab}\big), \qquad (2.6)
$$

$$
\chi:=\chi_{ab}h^{ab}=-\Omega N\big(h^{ab}\Psi_{ab}+\frac{3}{4}\Psi\big). \qquad (2.7)
$$

Note that $0=\Psi_{ab}g^{ab}=\Psi_{ab}h^{ab}+\Psi_{ab}N^aN^b$. In particular, (2.6) shows that $\mathscr{I}^+$ is extrinsically flat. On the other hand, by (2.3) and (2.4), we can calculate the mean curvature χ in an alternative way:

$$\Omega\Psi=\nabla_a\nabla^a\Omega=\nabla_a\bigl(-\frac{1}{N}N^a\bigr)=-\frac{1}{N}\chi-\frac{1}{2}NN^a\nabla_a\frac{1}{N^2}=-\frac{1}{N}\chi+\Omega\Phi-\frac{1}{2}\Omega^2NN^a\nabla_a\Phi,$$

from which

$$\chi=-\Omega N\bigl(\Psi-\Phi\bigr)-\frac{1}{2}\Omega^2N^2N^a\nabla_a\Phi$$

follows. Comparing this with (2.7) we see that

$$\frac{1}{4}\Psi=h^{ab}\Psi_{ab}+\Phi-\frac{1}{2}\Omega NN^a\nabla_a\Phi\approx h^{ab}\Psi_{ab}+\Phi. \tag{2.8}$$

Thus, Ψ is determined by Φ and the three-dimensional trace of Ψ_{ab}, and, in particular, on $\mathscr{I}^+$, it is an *algebraic* expression of them.

Substituting (2.3) and (2.4) into (2.1) and taking into account (2.8) we obtain

$$R_{ab}-\frac{1}{2}Rg_{ab}=-\Omega\bigl(\varkappa T_{ab}+3g_{ab}NN^c\nabla_c\Phi\bigr)+2\Psi_{ab}-6g_{ab}h^{cd}\Psi_{cd}-3\Phi g_{ab}. \tag{2.9}$$

On the other hand, it is known (e.g. from the initial value formulation of general relativity), that the various 3+1 pieces of the (unphysical) Einstein tensor are

$$N^cN^d\bigl(R_{cd}-\frac{1}{2}Rg_{cd}\bigr)=-\frac{1}{2}\bigl(\mathcal{R}+\chi^2-\chi_{cd}\chi^{cd}\bigr), \tag{2.10}$$

$$P^c_aN^d\bigl(R_{cd}-\frac{1}{2}Rg_{cd}\bigr)=-D_b\bigl(\chi^b{}_a-\chi\delta^b_a\bigr), \tag{2.11}$$

$$\begin{aligned}P^c_aP^d_b\bigl(R_{cd}-\frac{1}{2}Rg_{cd}\bigr)=&\,\mathcal{R}_{ab}-\frac{1}{2}h_{ab}\bigl(\mathcal{R}+\chi^2-\chi_{cd}\chi^{cd}\bigr)+\bigl(\pounds_N\chi_{cd}\bigr)P^c_aP^d_b+\chi\chi_{ab}\\&-2\chi_{ac}\chi^c{}_b+\frac{1}{N}D_aD_bN-h_{ab}\bigl(h^{cd}\pounds_N\chi_{cd}-\chi_{cd}\chi^{cd}+\frac{1}{N}D_cD^cN\bigr).\end{aligned} \tag{2.12}$$

Here $\mathcal{R}_{ab}$ is the Ricci tensor and $\mathcal{R}$ the intrinsic scalar curvature of D_a, and $\pounds_N$ denotes Lie derivative along the unit normal vector field N^a. Comparing the various 3+1 parts of (2.9) with (2.10), (2.11) and (2.12), respectively, and using the definitions, equations (2.6)-(2.8) and the fact $\Psi_{ab}N^aN^b=-h^{ab}\Psi_{ab}$, we find

$$\mathcal{R}\approx16h^{ab}\Psi_{ab}+6\Phi, \tag{2.13}$$

$$P^c_aN^d\Psi_{cd}\approx0, \tag{2.14}$$

$$\mathcal{R}_{ab}\approx P^c_aP^d_b\Psi_{cd}+5h_{ab}h^{cd}\Psi_{cd}+2\Phi h_{ab}. \tag{2.15}$$

Comparing (2.15) with (2.6), by (2.8) we obtain that

$$\sqrt{\frac{3}{\Lambda}}\lim_{\Omega\to0}\bigl(\Omega^{-1}\chi_{ab}\bigr)\approx-\bigl(\mathcal{R}_{ab}-\frac{1}{4}\mathcal{R}h_{ab}\bigr)-\frac{1}{2}\Phi h_{ab}. \tag{2.16}$$

Note that the first term on the right is just the Schouten tensor of the (three dimensional) Riemannian geometry $(\mathscr{I}^+,h_{ab})$.

2.2.2 Consequences of the Bianchi identity

By Einstein's equations and the assumption on the asymptotic form of the physical energy-momentum tensor, the Schouten tensor of the (four dimensional) physical spacetime is

$$\hat{S}_{ab} := -\big(\hat{R}_{ab} - \frac{1}{6}\hat{R}\hat{g}_{ab}\big) = \varkappa\Omega\big(T_{ab} - \frac{1}{3}Tg_{ab}\big) - \frac{1}{3}\Omega^{-2}\Lambda g_{ab}. \tag{2.17}$$

Hence, by the definition of the Weyl tensor and the conformal rescaling formulae, the contracted Bianchi identity for the physical curvature tensor yields

$$\begin{aligned}0 =& \hat{\nabla}_a\hat{C}^a{}_{bcd} + \frac{1}{2}\big(\hat{\nabla}_c\hat{S}_{db} - \hat{\nabla}_d\hat{S}_{cb}\big) \\ =& \nabla_a C^a{}_{bcd} - \Omega^{-1}\big(\nabla_a\Omega\big)C^a{}_{bcd} + \frac{1}{2}\varkappa\Omega\big(\nabla_c T_{db} - \nabla_d T_{cb}\big) - \frac{1}{6}\varkappa\Omega\big(g_{bd}\nabla_c T - g_{bc}\nabla_d T\big) \\ &- \varkappa\big(T_{bc}\nabla_d\Omega - T_{bd}\nabla_c\Omega\big) - \frac{1}{2}\varkappa\big(g_{bc}T_d{}^e - g_{bd}T_c{}^e\big)\nabla_e\Omega + \frac{1}{2}\varkappa T\big(g_{bc}\nabla_d\Omega - g_{bd}\nabla_c\Omega\big).\end{aligned} \tag{2.18}$$

Multiplying this equation by Ω and evaluating it at $\Omega = 0$, we obtain $(\nabla_a\Omega)C^a{}_{bcd} \approx 0$, which is known to imply $C_{abcd} \approx 0$ [12]. Thus we can write $C_{abcd} = \Omega K_{abcd}$ for some smooth tensor field K_{abcd} on M. Substituting this form of the non-physical Weyl tensor back into (2.18), we obtain

$$\begin{aligned}2\Omega\nabla^a K_{abcd} =& \varkappa\Omega\big(\nabla_d T_{cb} - \nabla_c T_{db}\big) + \frac{1}{3}\varkappa\Omega\big(g_{bd}\nabla_c T - g_{bc}\nabla_d T\big) \\ &+ 2\varkappa\big(T_{bc}\nabla_d\Omega - T_{bd}\nabla_c\Omega\big) + \varkappa\big(g_{bc}T_d{}^e - g_{bd}T_c{}^e\big)\nabla_e\Omega + \varkappa T\big(g_{bd}\nabla_c\Omega - g_{bc}\nabla_d\Omega\big).\end{aligned} \tag{2.19}$$

Evaluating this at $\Omega = 0$ and then contracting with N^b and $P^b_a N^c$, respectively, we obtain

$$T_{bc}P^b_a N^c \approx 0, \qquad T_{cd}P^c_a P^d_b \approx 0. \tag{2.20}$$

The first of these has already been derived in [14] (in the presence of a negative cosmological constant). Therefore, the asymptotic form of the various 3+1 pieces of T_{ab} is

$$T_{ab}N^a N^b = \nu + \Omega\mu, \quad T_{bc}P^b_a N^c = \Omega J_a, \quad T_{cd}P^c_a P^d_b = \Omega\Sigma_{ab}, \tag{2.21}$$

for some smooth ν, μ, J_a and Σ_{ab} on M. However, these quantities are not quite independent, because they should satisfy the local conservation law $\hat{\nabla}_a\hat{T}^a{}_b = 0$. Next we evaluate the consequences of this restriction.

By the conformal rescaling formulae and the asymptotic form of the energy-momentum tensor we have that

$$0 = \Omega^{-2}\hat{\nabla}_a\hat{T}^a{}_b = \Omega\nabla_a T^a{}_b - \nabla_a\Omega\big(T^a{}_b - T\delta^a_b\big). \tag{2.22}$$

(It might be worth noting that the contraction of this equation with NP^b_c and NN^b, respectively, yields $T_{ab}N^a P^b_c \approx 0$ and $T_{ab}h^{ab} \approx 0$. The former is just the first of (2.20), but the latter is only the trace of the second.) Then substituting (2.21) into (2.22) and contracting with N^b and P^b_c, respectively, and evaluating the resulting equations on $\mathscr{I}^+$, we obtain

$$\mu + \Sigma_{ab}h^{ab} \approx \sqrt{3/\Lambda}N^a\nabla_a\nu, \qquad \sqrt{3/\Lambda}D^b\Sigma_{ba} \approx -\big(\pounds_N J_b\big)P^b_a, \tag{2.23}$$

where, by (2.5), we used $D_aN = O(\Omega^2)$. Thus, if ν were constant in time at $\mathscr{I}^+$ (e.g. when the whole physical energy-momentum tensor fell off as $\Omega^4 T^a{}_b$, in which case ν itself would be zero), then by the first of (2.23) the Ω^4 order part of the physical energy-momentum tensor $\hat{T}^a{}_b$ would be asymptotically trace-free.

Taking into account (2.21), equation (2.19) can be rewritten into the form

$$\begin{aligned} 2\nabla^a K_{abcd} = &\varkappa\big(\nabla_d T_{cb} - \nabla_c T_{db}\big) + \frac{1}{3}\varkappa\big(g_{bd}\nabla_c T - g_{bc}\nabla_d T\big) + \varkappa\frac{1}{N}\Big(N_b\big(J_d N_c - J_c N_d\big) \\ &+\big(h_{bd}J_c - h_{bc}J_d\big) + 2\big(\Sigma_{bd}N_c - \Sigma_{bc}N_d\big) - \Sigma_{ef}h^{ef}\big(h_{bd}N_c - h_{bc}N_d\big)\Big). \end{aligned} \tag{2.24}$$

Recalling that the electric and magnetic parts of the rescaled Weyl tensor K_{abcd} are defined by $\mathcal{E}_{ab} := K_{acbd}N^cN^d$ and $\mathcal{B}_{ab} := \frac{1}{2}K_{acef}\varepsilon^{ef}{}_{bd}N^cN^d$, respectively, from (2.24) we obtain that

$$\begin{aligned} D^a\mathcal{E}_{ab} &= P^e_a\nabla_e\big(K^a{}_{cfd}N^cN^d\big)P^f_b = \big(\nabla^a K_{acfd}\big)N^cN^dP^f_b - N^aK_{acdf}\chi^{cd}P^f_b \\ &= -\frac{\varkappa}{N}J_b - \frac{1}{3}\varkappa D_b\nu - \frac{1}{2}\frac{\varkappa}{N}\nu D_bN - N^aK_{acdf}\chi^{cd}P^f_b + \frac{1}{2}\varkappa\Omega\Big(-\mu\big(D_b\ln N\big) \\ &\quad +\big(D^a\ln N\big)\Sigma_{ab} - \frac{2}{3}D_b\mu + 2J^a\chi_{ab} + \frac{1}{3}\big(D_b\Sigma_{cd}\big)h^{cd} + N^a\big(\nabla_aJ_c\big)P^c_b\Big) \\ &\approx -\varkappa\sqrt{\Lambda/3}J_b - \frac{1}{3}\varkappa D_b\nu \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} D^a\mathcal{B}_{ab} &= P^h_a\nabla_h\big(\frac{1}{2}K^a{}_{ecd}\varepsilon^{cd}{}_{gf}N^eN^f\big)P^g_b \\ &= \frac{1}{2}\big(\nabla^aK_{aecd}\big)\varepsilon^{cd}{}_{bf}N^eN^f - \frac{1}{2}K_{aecd}\varepsilon^{cd}{}_{fg}\chi^{af}N^eP^g_b - \big(D^a\ln N\big)\mathcal{B}_{ab} \\ &= \frac{1}{2}\varkappa\Omega\big(\chi_c{}^e\Sigma_{ed} - D_cJ_d\big)\varepsilon^{cd}{}_{af}N^f + \frac{1}{2}N^a\chi^{cd}K_{acef}\varepsilon^{ef}{}_{da}P^a_b - \big(D^a\ln N\big)\mathcal{B}_{ab} \approx 0, \end{aligned} \tag{2.26}$$

where we used that $D_bN \approx 0$ and $\chi_{cd} \approx 0$. (2.25) shows that, in addition to J_b, the gradient of the slow fall-off (i.e. Ω^3 order) part ν of the energy density also contributes to the divergence of the electric part of the rescaled Weyl curvature. An analogous term on the right hand side of equation (10) of [14] should also be present.

Finally, by the definition of the Weyl tensor, equations (2.10)-(2.12) and the analogous decomposition of the Riemann tensor,

$$R_{efgh}P^e_aP^f_bP^g_cP^h_d = \mathcal{R}_{abcd} + \chi_{ac}\chi_{bd} - \chi_{ad}\chi_{bc}, \tag{2.27}$$
$$R_{efgh}N^eP^f_bP^g_cP^h_d = D_c\chi_{db} - D_d\chi_{cb}, \tag{2.28}$$
$$R_{efgh}N^eN^gP^f_bP^h_d = P^a_bP^c_d\big(\pounds_N\chi_{ac}\big) - \chi_{be}\chi^e{}_d - D_ba_d + a_ba_d, \tag{2.29}$$

and of the scalar curvature,

$$R=\mathcal{R}+2h^{ab}\big(\pounds_N\chi_{ab}\big)+\chi^2-3\chi_{ab}\chi^{ab}+\frac{1}{N}D_aD^aN, \tag{2.30}$$

we can determine the explicit form of the electric and magnetic parts of the Weyl tensor themselves, $E_{ab}:=C_{acbd}N^cN^d$ and $B_{ab}:=\frac{1}{2}C_{acef}\varepsilon^{ef}{}_{bd}N^cN^d$, respectively. We obtain

$$\begin{aligned}E_{ab}=&\frac{1}{2}\Big(P^c_aP^d_b\big(\pounds_N\chi_{cd}\big)-\mathcal{R}_{ab}-\chi\chi_{ab}+\frac{1}{N}D_aD_bN\\ &-\frac{1}{3}h_{ab}\big(h^{cd}\pounds_N\chi_{cd}-\mathcal{R}-\chi^2+\frac{1}{N}D_cD^cN\big)\Big), &(2.31)\\ B_{ab}=&\big(D_c\chi_{d(a}\big)\varepsilon^{cd}{}_{b)}, &(2.32)\end{aligned}$$

where $\varepsilon_{abc}:=N^e\varepsilon_{eabc}$ is the induced volume 3-form on $\mathcal{H}_\Omega$. In the rest of the paper, we need the explicit form of $\mathcal{B}_{ab}=\Omega^{-1}B_{ab}$ at $\mathscr{I}^+$ only, for which, by (2.16), we obtain

$$\sqrt{\frac{3}{\Lambda}}\mathcal{B}_{ab}\approx D_c\big(-\mathcal{R}_{d(a}+\frac{1}{4}\mathcal{R}h_{d(a}\big)\varepsilon^{cd}{}_{b)}=:Y_{ab}, \tag{2.33}$$

where Y_{ab} is known as the Cotton–York tensor: Its vanishing is known to be equivalent to the *local* conformal flatness of $(\mathscr{I}^+,h_{ab})$.

If the physical energy-momentum tensor $\hat{T}^a{}_b$ falls off purely at order Ω^3, then by (2.21) this matter is a dust, and by the first of (2.23) the rescaled energy density $\nu=\Omega^{-3}\hat{T}_{ab}\hat{N}^a\hat{N}^b$ is asymptotically constant. In particular, in the dust filled FRW spacetime with positive Λ this ν is just the conserved first integral of one of the two Friedman equations.

On the other hand, if $\hat{T}^a{}_b=\Omega^4T^a{}_b$ (i.e. there is no Ω^3 order term in $\hat{T}^a{}_b$), then by the first of (2.23) $\hat{T}^a{}_b$ is asymptotically trace-free, which is a characteristic property of conformally invariant (e.g. Maxwell or Yang–Mills) fields. For example, in the radiation filled FRW spacetime with positive Λ theenergy-momentum tensor is trace-free, the energy density and isotropicpressure with respect to $\hat{N}^a$ fall off as Ω^4, and the momentum-density J_a is vanishing, being compatible with the homogeneity of theisotropic pressure and the energy density, just according to the second of (2.23). Thus, the existence of a (spacelike) $\mathscr{I}^+$ restricts the form of the matter fields near $\mathscr{I}^+$.

3 On the Penrose mass at $\mathscr{I}^+$

If $\mathcal{S}$ is a closed, orientable spacelike 2-surface in $\hat{M}$, then for any smooth symmetric spinor field ω^{AB} we can form the integral of the complex 2-form $\omega^{AB}\hat{R}_{ABcd}$ on $\mathcal{S}$, where $\hat{R}_{ABcd}$ is the anti-self-dual part of the physical spacetime curvature tensor. However, even in the de Sitter spacetime this integral diverges if we allow the 2-surface to tend to a cut of $\mathscr{I}^+$. Therefore, to get finite value, it seems reasonable to use in the charge integral the 'renormalized' curvature

$$\tilde{R}_{ABcd}:=\hat{R}_{ABcd}-\frac{1}{3}\Lambda\varepsilon_{A(C}\varepsilon_{D)B}\varepsilon_{C'D'}, \tag{3.1}$$

rather than $\hat{R}_{ABcd}$. (Clearly, $\tilde{R}_{ABcd}$ is the anti-self-dual part of the *renormalized* curvature tensor $\tilde{R}_{abcd}:=\hat{R}_{abcd}-\frac{1}{3}\Lambda(\hat{g}_{ac}\hat{g}_{bd}-\hat{g}_{ad}\hat{g}_{bc})$.) Then by the definition of $K^a{}_{bcd}$ and the Weyl tensor, Einstein's equation and the conformal rescaling formulae, the corresponding complex 2-form is

$$\begin{aligned}\omega^{AB}\tilde{R}_{ABcd} &= -\frac{1}{2}\omega^{AB}\hat{\varepsilon}^{A'B'}\Big(\hat{R}_{abcd}-\frac{1}{3}\Lambda\big(\hat{g}_{ac}\hat{g}_{bd}-\hat{g}_{ad}\hat{g}_{bc}\big)\Big)\\ &= -\frac{1}{2}\omega^{AB}\hat{\varepsilon}^{A'B'}\Big(\hat{C}_{abcd}-\varkappa\big(\hat{g}_{ac}\hat{T}_{bd}-\hat{g}_{ad}\hat{T}_{bc}\big)+\frac{2}{3}\varkappa\hat{T}\hat{g}_{ac}\hat{g}_{bd}\Big)\\ &= -\frac{1}{2}\omega^{AB}\varepsilon^{A'B'}\Big(K_{abcd}-\varkappa\big(g_{ac}T_{bd}-g_{ad}T_{bc}\big)+\frac{2}{3}\varkappa T g_{ac}g_{bd}\Big).\end{aligned}$$

Thus the cosmological constant has, in fact, been canceled, and the expression on the right is well defined and has a finite integral on any closed spacelike 2-surface in M, even on a cut of $\mathscr{I}^+$.

Hence, let $\mathcal{S}$ be a closed orientable 2-surface in $\mathscr{I}^+$, and denote its outward pointing g_{ab}-unit normal in $\mathscr{I}^+$ by V^a. Then, by the fall-off properties (2.21) of the various pieces of T_{ab} and the definition of $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$, we obtain

$$\begin{aligned}\tilde{\mathtt{A}}\big[\omega_{AB}\big] &:= \frac{\mathrm{i}}{\varkappa}\oint_{\mathcal{S}}\omega^{AB}\tilde{R}_{ABcd}\\ &= -\frac{\mathrm{i}}{\varkappa}\oint_{\mathcal{S}}\omega^{AB}\varepsilon^{A'B'}\Big(\big(N_a\mathcal{B}_{bc}-N_b\mathcal{B}_{ac}\big)V^c+\varepsilon_{abc}\big(\frac{1}{3}\varkappa\nu V^c+\mathcal{E}^c{}_dV^d\big)\Big)\mathrm{d}\mathcal{S}. \quad (3.2)\end{aligned}$$

Therefore, a general charge integral of the curvature is built from $\mathcal{E}_{ab}$, $\mathcal{B}_{ab}$ and the slow fall-off (i.e. Ω^3 order) part ν of the energy density.

If $D\subset\mathscr{I}^+$ is an open domain with compact closure such that $\partial D=\mathcal{S}$ and the spinor field ω_{AB} is well defined on D, then, by equations (2.25) and (2.26), the charge integral $\tilde{\mathtt{A}}[\omega_{AB}]$ can be rewritten as a 3-surface integral on D. In fact, if

$$Q_{ab}:=\frac{1}{2}\varepsilon_{abcd}\omega^{CD}\varepsilon^{C'D'}=-\mathrm{i}\omega_{AB}\varepsilon_{A'B'},$$

then

$$\tilde{\mathtt{A}}\big[\omega_{AB}\big]=\frac{2\mathrm{i}}{\varkappa}\int_D\Big\{D_eQ_{ab}\Big(N^a\big(\frac{1}{3}\varkappa\nu h^{be}+\mathcal{E}^{be}\big)-\varepsilon^{ab}{}_c\mathcal{B}^{ce}\Big)-\varkappa\sqrt{\frac{\Lambda}{3}}Q_{ab}N^aJ^b\Big\}\mathrm{d}\mathcal{H}_0. \quad (3.3)$$

Here $\mathrm{d}\mathcal{H}_0$ is the induced volume element on $\mathscr{I}^+$, and we extended the action of the intrinsic derivative operator D_e from purely spatial tensors to arbitrary ones on $\mathscr{I}^+$ by $D_eN^a=0$.

It is known that on a spacelike hypersurface that can be embedded with its first and second fundamental forms into some conformal Minkowski spacetimes the 3-surface twistor equation admits four (i.e. maximal number of) linearly independent solutions [23]. In particular, since $\mathscr{I}^+$ is extrinsically flat, this embeddability is equivalent to the local conformal flatness of $(\mathscr{I}^+,h_{ab})$. Thus, in the special case when its Cotton–York tensor is vanishing, $Y_{ab}=0$, the (already Riemannian) 3-surface twistor equation, $D_{(AB}\lambda_{C)}=0$, admits four linearly independent solutions. Here $D_{AB}:=\sqrt{2}N_B{}^{A'}D_{AA'}=D_{(AB)}$ is the unitary spinor form of the intrinsic Levi-Civita derivative operator. These solutions are globally defined on D if D is homeomorphic to the 3-ball. Then, a direct calculation shows that N^aQ_{ab} is a (complex) conformal Killing field on D if ω_{AB} is a linear combination of symmetrized products $\lambda_{(A}\mu_{B)}$ of solutions of the 3-surface twistor equation. (The result that N^aQ_{ab} is a conformal Killing vector was proven in the $\Lambda<0$ case by Kelly

[15] by considering the conformal boundary to be embedded in a conformal Minkowski spacetime as a hyperplane and using the solutions of the 1-valence twistor equation of that spacetime.)

Since, however, $\mathcal{B}_{ab} = \sqrt{\Lambda/3}\,Y_{ab}$ on $\mathscr{I}^+$, by (3.3) for such spinor fields it is only ν and J_a that contribute to the integral even if $\mathcal{E}_{ab} \neq 0$. In particular, for the Maxwell (or Yang–Mills) field it is only J_a that contributes to $\tilde{\mathtt{A}}[\lambda_{(A}\mu_{B)}]$ but its energy density does not; while in vacuum $\tilde{\mathtt{A}}[\lambda_{(A}\mu_{B)}]$ is zero even if $\mathcal{E}_{ab} \neq 0$. For example, in vacuum spacetimes with metric of Starobinskii form [24] and intrinsically conformally flat $\mathscr{I}^+$ the electric part of the rescaled Weyl tensor is *not* zero, but the charge integral with spinor fields solving the 3-surface twistor equation on a contractible D is vanishing. Such an exact solution is the vacuum Kasner solution with positive Λ (see [25], Sect. 13.3.3).

It is known that if λ_A is a solution of the 3-surface twistor equation on a spacelike hypersurface, then its restriction to a 2-surface in this hypersurface solves the 2-surface twistor equation [23]. Thus by the general results above, when $(\mathscr{I}^+, h_{ab})$ is intrinsically locally conformally flat, the matter field is conformally invariant and its local energy flow is vanishing asymptotically (i.e. $J_a \approx 0$), then the Penrose mass is zero even if the rescaled conformal electric curvature $\mathcal{E}_{ab}$ is not. Therefore, the kinematical twistor (and, in particular, the Penrose mass) does not have the rigidity property: Its vanishing does *not* imply the triviality of the spacetime geometry, i.e. actually that the past domain of dependence of D is (locally) isometric to the de Sitter spacetime.

4 Energy-momentum based on the Nester–Witten form

Another strategy to associate energy-momentum to 2-surfaces in $\mathscr{I}^+$ might be based on the use of the integral of the Nester–Witten 2-form. In fact, this formalism was successfully used to give a unified spinorial reformulation of the ADM and Bondi–Sachs energy-momenta in asymptotically flat spacetimes [26], and of the Abbott–Deser energy in asymptotically anti-de Sitter spacetimes [27]. Moreover, probably the simplest proof of the positivity of these energies is given in this formalism. Therefore, it seems natural to try to introduce energy-momentum in this manner in the presence of a positive cosmological constant, too.

4.1 The Nester–Witten form and the Sen–Witten identity

In the present subsection we work exclusively in the *physical* spacetime, but, for the sake of simplicity, *in this subsection*, we leave the 'hats' off of the quantities and objects.

Let Σ be a spacelike hypersurface in M which extends to a hypersurface-with-boundary in the unphysical spacetime intersecting $\mathscr{I}^+$ in a smooth 2-surface homeomorphic to S^2 (a cut). Let t^a denote its future pointing g_{ab}-unit timelike normal, $P^a_b := \delta^a_b - t^a t_b$ is the g_{ab}-orthogonal projection to Σ and $h_{ab} := P^c_a P^d_b g_{cd}$ is the induced metric. (Although we use the same symbols P^a_b and h_{ab}, and χ_{ab} below for the extrinsic curvature, they should not be confused with those introduced on the $\Omega =$ const hypersurfaces in section 2.)

For any pair λ_A, μ_A of spinor fields the general Nester–Witten form is defined by

$$u(\lambda, \bar{\mu})_{ab} := \frac{\mathrm{i}}{2}\big(\bar{\mu}_{A'}\nabla_{BB'}\lambda_A - \bar{\mu}_{B'}\nabla_{AA'}\lambda_B\big). \tag{4.1}$$

Then the components of the energy-momentum 4-vector associated with a closed orientable 2-surface $\mathcal{S}$, which in the present case is the cut of the conformal boundary, will be defined by its integral,

$$H[\lambda,\bar\mu] := \frac{2}{\varkappa}\oint_{\mathcal S} u(\lambda,\bar\mu)_{cd}, \tag{4.2}$$

for spinor fields λ^A and μ^A belonging to some appropriately chosen two dimensional subspace $\mathbf{S_A}\subset C^\infty(\mathcal S,\mathbb S^A)$ of the infinite dimensional space of the smooth spinor fields on $\mathcal S$. Thus, the boldface index is referring to this (still not specified) two dimensional space of spinor fields. Since a basis in this space is a pair of spinor fields, $\lambda^A_{\mathbf A}=(\lambda^A_{\mathbf 0},\lambda^A_{\mathbf 1})$, the boldface index can also be considered as a name index, too, taking numerical values: $\mathbf A=\mathbf 0,\mathbf 1$. It is a simple calculation to show that this integral, as a bilinear map $H:\mathbf{S_A}\times\bar{\mathbf S}_{\mathbf A'}\to\mathbb C$, is Hermitian in the sense that $H[\lambda,\bar\mu]=\overline{H[\mu,\bar\lambda]}$ for any $\lambda^A,\mu^A\in\mathbf{S_A}$, and overline denotes complex conjugation. However, by the so-called polarization formula,

$$H[\lambda,\bar\mu]=\frac12\Big(H\big[\lambda+\mu,\overline{\lambda+\mu}\big]+\mathrm iH\big[\lambda+\mathrm i\mu,\overline{\lambda+\mathrm i\mu}\big]-(1+\mathrm i)H\big[\lambda,\bar\lambda\big]-(1+\mathrm i)H\big[\mu,\bar\mu\big]\Big), \tag{4.3}$$

the bilinear form $H[\lambda,\bar\mu]$ is completely determined by the quadratic form $H[\alpha,\bar\alpha]$ on $\mathbf{S_A}$. Moreover, it would be enough to prove $H[\alpha,\bar\alpha]\geq0$ for any $\alpha^A\in\mathbf{S_A}$, because this would already imply that H as a Hermitian bilinear form is positive.

The standard Witten-type spinorial proof of the positivity of the total energy is based on the integrated form of the Sen–Witten identity on a spacelike hypersurface Σ whose boundary $\partial\Sigma$ at infinity is the 2-surface $\mathcal S$ in question (see e.g. [28]):

$$\oint_{\partial\Sigma}u(\alpha,\bar\alpha)_{cd}=\int_\Sigma B(\alpha)\mathrm d\Sigma,$$

where α_A is defined on Σ and

$$B(\alpha):=-2t^{AA'}\big(\mathcal D_{A'B}\alpha^B\big)\big(\mathcal D_{AB'}\bar\alpha^{B'}\big)-h^{ef}t^{AA'}\big(\mathcal D_e\alpha_A\big)\big(\mathcal D_f\bar\alpha_{A'}\big)-\frac12t^aG_{aBB'}\alpha^B\bar\alpha^{B'}.$$

Here $\mathcal D_a:=P^b_a\nabla_b$ is the derivative operator of the so-called Sen connection, and, in terms of the intrinsic Levi-Civita derivative operator D_e and the extrinsic curvature χ_{ab}, its action on the spinor field α^A is given by $\mathcal D_e\alpha^A=D_e\alpha^A-\chi_e{}^A{}_{A'}t^{A'}{}_B\alpha^B$. However, by Einstein's equations in the presence of a cosmological constant the last term on the right contains $\frac12\Lambda t_{AA'}\alpha^A\bar\alpha^{A'}$, whose integral on an infinite Σ is *a priori diverging* for spinor fields for which $t_{AA'}\alpha^A\bar\alpha^{A'}$ does not fall off appropriately (e.g. when the spinor fields tend to a nonzero asymptotic value), independently of the fall-off properties of the matter fields. Therefore, we must 'renormalize' the derivative operators in our integrals.

Thus, following [27], for any pair $(\alpha_A,\bar\beta_{A'})$ of spinor fields (or, equivalently, a Dirac spinor Ψ^α with Weyl spinor constituents α^A and $\bar\beta^{A'}$ and with $\alpha=A\oplus A'$) we define the renormalized spacetime connection by

$$\tilde\nabla_{AA'}\alpha_B:=\nabla_{AA'}\alpha_B+K\varepsilon_{AB}\bar\beta_{A'},\qquad \tilde\nabla_{AA'}\bar\beta_{B'}:=\nabla_{AA'}\bar\beta_{B'}+\tilde K\varepsilon_{A'B'}\alpha_A, \tag{4.4}$$

for some complex constants K and $\tilde K$. It is a straightforward calculation to show that the curvature of $\tilde\nabla_e$, acting on the bundle of Dirac spinors, is just the direct sum of

$\tilde{R}^A{}_{Bcd}$, given by (3.1), and its complex conjugate $\bar{\tilde{R}}^{A'}{}_{B'cd}$ precisely when $6K\bar{\tilde{K}} = -\Lambda$. The corresponding renormalized Sen connection on the spacelike hypersurface Σ is

$$\tilde{\mathcal{D}}_{AA'}\alpha_B := \mathcal{D}_{AA'}\alpha_B + KP^{DD'}_{AA'}\varepsilon_{DB}\bar{\beta}_{D'}, \qquad \tilde{\mathcal{D}}_{AA'}\bar{\beta}_{B'} := \mathcal{D}_{AA'}\bar{\beta}_{B'} + \tilde{K}P^{DD'}_{AA'}\varepsilon_{D'B'}\alpha_D. \quad (4.5)$$

Denoting by tilde the quantities built from the renormalized connection (4.4), the Nester–Witten form is

$$u(\alpha,\bar{\alpha})_{ab} = \tilde{u}(\alpha,\bar{\alpha})_{ab} + \mathrm{i}K\varepsilon_{AB}\bar{\alpha}_{(A'}\bar{\beta}_{B')}, \qquad u(\beta,\bar{\beta})_{ab} = \tilde{u}(\beta,\bar{\beta})_{ab} + \mathrm{i}\bar{\tilde{K}}\varepsilon_{AB}\bar{\alpha}_{(A'}\bar{\beta}_{B')}. \quad (4.6)$$

Note that here $\tilde{u}(\alpha,\bar{\alpha})_{cd}$ already depends on $\bar{\beta}_{A'}$, and $\tilde{u}(\beta,\bar{\beta})_{cd}$ on α_A, too. Then, a straightforward calculation yields that

$$\begin{aligned}\oint_{\partial\Sigma}\big(\tilde{u}(\alpha,\bar{\alpha})_{ab} + \tilde{u}(\beta,\bar{\beta})_{ab}\big) = \int_\Sigma\Big(&B(\alpha)+B(\beta)\\ &+3(K+\bar{\tilde{K}})t^{AA'}\big(\tilde{K}\alpha_A\bar{\alpha}_{A'} + \bar{K}\beta_A\bar{\beta}_{A'}\big)\\ &-2(K+\bar{\tilde{K}})t^{AA'}\varepsilon^{B'D'}\big(\bar{\beta}_{A'}\overline{\tilde{\mathcal{D}}}_{AB'}\bar{\alpha}_{D'} + \bar{\alpha}_{A'}\tilde{\mathcal{D}}_{AB'}\bar{\beta}_{D'}\big)\Big)\mathrm{d}\Sigma,\end{aligned} \quad (4.7)$$

where $\mathrm{d}\Sigma$ is the natural volume element on Σ, and $B(\alpha)+B(\beta)$ in terms of $\tilde{\mathcal{D}}_e$ is

$$\begin{aligned}B(\alpha)+B(\beta) =& -t^{AA'}h^{ef}\big((\tilde{\mathcal{D}}_e\alpha_A)(\overline{\tilde{\mathcal{D}}}_f\bar{\alpha}_{A'}) + (\overline{\tilde{\mathcal{D}}}_e\beta_A)(\tilde{\mathcal{D}}_f\bar{\beta}_{A'})\big)\\ &+\frac{1}{2}\varkappa t^aT_{ab}(\alpha^B\bar{\alpha}^{B'} + \beta^B\bar{\beta}^{B'})\\ &+\frac{1}{2}(\Lambda - 6K\bar{K})t_{AA'}\beta^A\bar{\beta}^{A'} + \frac{1}{2}(\Lambda - 6\tilde{K}\bar{\tilde{K}})t_{AA'}\alpha^A\bar{\alpha}^{A'}\\ &-2t^{AA'}\varepsilon^{BD}\varepsilon^{B'D'}\Big(\big(\tilde{\mathcal{D}}_{A'B}\alpha_D\big)\big(\overline{\tilde{\mathcal{D}}}_{AB'}\bar{\alpha}_{D'}\big) + \big(\overline{\tilde{\mathcal{D}}}_{A'B}\beta_D\big)\big(\tilde{\mathcal{D}}_{AB'}\bar{\beta}_{D'}\big)\Big)\\ &+4\Big(K\bar{\beta}_{A'}t^{A'A}\varepsilon^{B'D'}\big(\overline{\tilde{\mathcal{D}}}_{AB'}\bar{\alpha}_{D'}\big) + \bar{K}\beta_At^{AA'}\varepsilon^{BD}\big(\tilde{\mathcal{D}}_{A'B}\alpha_D\big)\\ &\quad+\tilde{K}\alpha_At^{AA'}\varepsilon^{BD}\big(\overline{\tilde{\mathcal{D}}}_{A'B}\beta_D\big) + \bar{\tilde{K}}\bar{\alpha}_{A'}t^{A'A}\varepsilon^{B'D'}\big(\tilde{\mathcal{D}}_{AB'}\bar{\beta}_{D'}\big)\Big),\end{aligned} \quad (4.8)$$

where we used Einstein's equations.

Next, following [27], we require the spinor fields α_A, β_A on Σ to solve Witten's gauge condition (with still unspecified boundary conditions) with the renormalized connection:

$$\varepsilon^{AB}\tilde{\mathcal{D}}_{A'A}\alpha_B = 0, \qquad \varepsilon^{A'B'}\tilde{\mathcal{D}}_{AA'}\bar{\beta}_{B'} = 0. \quad (4.9)$$

In this gauge the last three lines of (4.8) and the third line of (4.7) are vanishing, and hence the integral of the renormalized Nester–Witten forms is

$$\begin{aligned}\oint_{\partial\Sigma}\Big(\tilde{u}(\alpha,\bar{\alpha})_{ab} + \tilde{u}(\beta,\bar{\beta})_{ab}\Big) = \int_\Sigma\Big\{&-t^{AA'}h^{ef}\big((\tilde{\mathcal{D}}_e\alpha_A)(\overline{\tilde{\mathcal{D}}}_f\bar{\alpha}_{A'}) + (\overline{\tilde{\mathcal{D}}}_e\beta_A)(\tilde{\mathcal{D}}_f\bar{\beta}_{A'})\big)\\ &+\frac{1}{2}\varkappa t^aT_{ab}(\alpha^B\bar{\alpha}^{B'} + \beta^B\bar{\beta}^{B'})\\ &+\frac{1}{2}(\Lambda + 6\bar{K}\bar{\tilde{K}})t_{AA'}\beta^A\bar{\beta}^{A'} + \frac{1}{2}(\Lambda + 6K\tilde{K})t_{AA'}\alpha^A\bar{\alpha}^{A'}\Big\}\mathrm{d}\Sigma.\end{aligned} \quad (4.10)$$

To kill the last two (*a priori* diverging) terms of the integrand, we should require that $6K\tilde{K} = -\Lambda$, just the condition that we already obtained above. A particularly convenient choice (that we make) is that $\tilde{K} = K$ and $6K^2 = -\Lambda$. Then, if the spinor fields solve the renormalized Witten type gauge condition (4.9) and the matter fields satisfy the dominant energy condition, then the integral of the renormalized Nester-Witten form is non-negative, and not *a priori* diverging.

Since Λ is positive and hence K is imaginary, by (4.6) $u(\alpha,\bar{\alpha})_{ab}+u(\beta,\bar{\beta})_{ab} = \tilde{u}(\alpha,\bar{\alpha})_{ab}+\tilde{u}(\beta,\bar{\beta})_{ab}$ holds. Thus, under the same conditions, the left hand side of (4.10) would be finite even if it were built from the 'un-renormalized' connection. We use this observation in subsection 4.6.

Our aim is to find the boundary conditions for the spinor fields α_A and β_A on the cut of the conformal boundary such that (i.) the renormalized Witten type gauge conditions (4.9) admit a non-trivial solution on the spacelike hypersurface with the cut as its boundary, and (ii.) the resulting integrals could be interpreted as the components of a finite, real energy-momentum 4-vector, or at least could yield an invariant that could be interpreted as the mass.

The finiteness of the energy-momentum 4-vector can be ensured by the requirement that the integral of the various terms on the right hand side of (4.10) be finite. In particular, we need to specify the fall-off properties of the physical energy-momentum tensor, the induced metric and the extrinsic curvature on the hypersurface Σ. We carry out these investigations in a coordinate system near the asymptotic end of Σ that is analogous to the Bondi type coordinates in asymptotically flat spacetimes.

4.2 Bondi type coordinates near $\mathscr{I}^+$

Let $\mathcal{S} := \Sigma \cap \mathscr{I}^+ \approx S^2$, the cut of the future conformal boundary defined by the hypersurface Σ, and let $\mathcal{S}_u$, $u \in (-1,1)$, be a foliation of a neighbourhood of $\mathcal{S}$ in $\mathscr{I}^+$ by smooth topological 2-spheres such that $\mathcal{S}_0 = \mathcal{S}$ and u is increasing in the *inward* direction. We define the lapse n of this foliation in the usual way by $1 =: -nV^a\nabla_a u$, where V^a, as in section 3, is the *outward* pointing unit normal of the surfaces $\mathcal{S}_u$; and introduce the 'evolution vector field' by $(\partial/\partial u)^a := -nV^a$. Thus, in particular, its shift part will be chosen to be vanishing. Let $x^\mu = (x^2, x^3)$ be a local coordinate system on $\mathcal{S}_0 \approx S^2$ (e.g. the $(\zeta,\bar{\zeta})$ complex stereographic or the (θ,ϕ) angle coordinates), and extend them from $\mathcal{S}_0$ to the other surfaces of the foliation by $V^a\nabla_a x^\mu = 0$. Thus we obtained a local coordinate system (u, x^μ) on a neighbourhood of $\mathcal{S}$ in $\mathscr{I}^+$. Since the shift part of the evolution vector field was chosen to be vanishing, the metric induced from g_{ab} on the conformal boundary $\mathscr{I}^+$ ('boundary metric') takes the form $db^2 = -A^2du^2 + q_{\mu\nu}dx^\mu dx^\nu$, where $A = A(u, x^\mu)$ is a strictly positive function and $q_{\mu\nu} = q_{\mu\nu}(u, x^\rho)$ is negative definite. The freedom in the definition of this coordinate system is to choose a different foliation $\mathcal{S}_{\tilde{u}}$ in a neighbourhood of $\mathcal{S}$, e.g. by the level sets of a new function $\tilde{u} := Hu$, where H is some strictly positive function on this neighbourhood, and to choose different coordinates (x^2, x^3) on $\mathcal{S}$.

Next we complete (u, x^μ) to be a local coordinate system on a neighbourhood of the 'asymptotic end' of Σ in M. Thus, let us fix (u, x^μ) and let $\mathcal{N}_u$ denote the past directed, ingoing null hypersurface emanated from the 2-surface $\mathcal{S}_u$. In a neighbourhood of $\mathscr{I}^+$ this is smooth, generated by past directed ingoing null geodesics γ with future end points on $\mathcal{S}_u$ with coordinates x^μ, and yields an extension of u from $\mathscr{I}^+$ to a neighbourhood of $\mathscr{I}^+$ in M. Let us define $l^a := g^{ab}\nabla_b u$, which is a future directed null normal of the hypersurfaces $\mathcal{N}_u$ and, by $l^a\nabla_a l_b = l^a\nabla_b l_a = 0$, it is a tangent of the *affine parametrized* null geodesic generators γ. Let w denote the affine parameter measured from $\mathcal{S}_u$, and hence we write $l^a = -(\partial/\partial w)^a$. (Note that w is *decreasing* in the future direction.) Then the coordinates of a point p in a neighbourhood of $\mathcal{S}_0$ in M are defined to be u, w and x^μ if $p = \gamma(w)$ and the coordinates of the future end point of γ on $\mathscr{I}^+$ are (u, x^μ). The resulting coordinate system (u, w, x^μ) is analogous to that used to analyze asymptotically flat spacetimes near their future null infinity (see e.g. [29]).

Since by the definitions $\mathscr{I}^+ = \{\Omega = 0\} = \{w = 0\}$ holds and Ω is smooth, we can write that $\Omega = aw + bw^2 + O(w^3)$ for some functions a and b on $\mathscr{I}^+$, where a is positive. Substituting this expression into $\nabla_a\nabla_b\Omega \approx 0$ (see equation (2.3)), we obtain

$$\big(\nabla_a w\big)\big(\nabla_b a\big) + \big(\nabla_a a\big)\big(\nabla_b w\big) + 2b\big(\nabla_a w\big)\big(\nabla_b w\big) \approx 0.$$

Contracting this with $(\partial/\partial w)^a(\partial/\partial w)^b$ and using that $(\partial/\partial w)^a$ is the tangent of affine parametrized geodesics and that a does not depend on w, we find that $b = 0$.

However, it is only the *conformal class* of db^2 that is physically determined. Thus we can use the conformal gauge freedom $g_{ab} \mapsto \omega^2 g_{ab}$ mentioned in subsection 2.1 with $\omega = a^{-1}$ to yield $\Omega = w + O(w^3)$. Hence, with this choice Ω is an asymptotic affine parameter in the unphysical metric along the null geodesic generators γ even in the first two orders. Thus, we have fixed the conformal factor on $\mathscr{I}^+$, i.e. the remaining conformal gauge freedom is the rescaling of g_{ab} with conformal factors of the form $1 + \Omega^2\Theta$. We call this conformal gauge a *special conformal Bondi gauge*. In the rest of the paper we assume that in our unphysical spacetime (M, g_{ab}, Ω) the conformal gauge is such a special conformal Bondi gauge.

(Since the surfaces $\mathcal{S}_u$ in $\mathscr{I}^+$ are homeomorphic to 2-spheres, there exists a conformal factor $R = R(u, x^\rho)$ such that $q_{\mu\nu}(u, x^\rho) = R^2(u, x^\rho){}_0q_{\mu\nu}(u, x^\rho)$, where ${}_0q_{\mu\nu}(u, x^\rho)$ are the components of the unit sphere metric on $\mathcal{S}_u$ in the coordinates (x^2, x^3). However, note that although ${}_0q_{\mu\nu}dx^\mu dx^\nu$ is a unit sphere metric, its components ${}_0q_{\mu\nu}$ take some simple special form, e.g. $\mathrm{diag}(-1, -\sin^2\theta)$ in the angle coordinates, *only on a single surface*, e.g. on $\mathcal{S}_0$. In fact, (x^2, x^3) were specified freely only on $\mathcal{S}_0$, but their extension to the other surfaces was fixed essentially by the requirement of the vanishing of the $du\, dx^\mu$ components in the line element db^2. Hence, in general, the components ${}_0q_{\mu\nu}$ of the unit sphere metric depend on the coordinate u, too.)

In the coordinate system (u, w, x^μ),

$$\begin{aligned}
g_{01} &:= g\big(\frac{\partial}{\partial u}, \frac{\partial}{\partial w}\big) = -\big(\frac{\partial}{\partial u}\big)^a\nabla_a u = -1, \\
g_{11} &:= g\big(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\big) = g_{ab}l^a l^b = 0, \\
g_{1\mu} &:= g\big(\frac{\partial}{\partial w}, \frac{\partial}{\partial x^\mu}\big) = -\big(\frac{\partial}{\partial x^\mu}\big)^a\nabla_a u = 0.
\end{aligned}$$

Thus, in these coordinates, the form of the conformal metric is

$$ds^2 = g_{00}du^2 - 2du\, dw + 2g_{0\mu}du\, dx^\mu + g_{\mu\nu}dx^\mu\, dx^\nu. \tag{4.11}$$

Comparing its pull back to $\mathscr{I}^+ = \{\, w = 0 \,\}$ with db^2 we see that

$$g_{00} + A^2, \;\; g_{0\mu}, \;\; g_{\mu\nu} - q_{\mu\nu} = O(w). \tag{4.12}$$

The contravariant form of the unphysical metric is

$$g^{ab} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -g_{00} + g^{\nu\rho}g_{\nu 0}g_{\rho 0} & g^{\mu\rho}g_{\rho 0} \\ 0 & g^{\mu\rho}g_{\rho 0} & g^{\mu\nu} \end{pmatrix}, \tag{4.13}$$

where $g^{\mu\nu}$ is the inverse of the 2×2 matrix $g_{\mu\nu}$, i.e. defined by $g^{\mu\rho}g_{\rho\nu} := \delta^\mu_\nu$. Since $\Lambda/3 \approx (\nabla_a\Omega)(\nabla^a\Omega)$ holds by equation (2.4), the component g^{11} of the conformal metric is asymptotically constant: $g^{11} = \frac{1}{3}\Lambda + O(w^2)$. Comparing this with (4.13) and taking into account that $g_{\mu 0} = O(w)$ (see equation (4.12)) we find that

$$A^2 = \frac{1}{3}\Lambda, \qquad g_{00} = -\frac{1}{3}\Lambda + O(w^2). \tag{4.14}$$

Moreover, contracting $0 \approx \nabla_a\nabla_b\Omega \approx \nabla_a\nabla_b w$ with the various coordinate vectors, for the Christoffel symbols we obtain that $\Gamma^1_{ab} = O(w)$, where $a, b = 0, ..., 3$. (N.B.: $\Gamma^a_{11} = 0$ identically, because the generators of $\mathcal{N}_u$ are affine parametrized geodesics.) In particular,

$$\begin{aligned} O(w) &= 2\Gamma^1_{0\mu} = -\partial_\mu g_{00} - g^{11}\partial_1 g_{0\mu} + g^{1\rho}\bigl(-\partial_\rho g_{0\mu} + \partial_0 g_{\rho\mu} + \partial_\mu g_{0\rho}\bigr) \\ &= -\frac{1}{3}\Lambda\partial_1 g_{0\mu} + O(w), \end{aligned} \tag{4.15}$$

$$\begin{aligned} O(w) &= 2\Gamma^1_{\mu\nu} = \partial_0 g_{\mu\nu} - \partial_\mu g_{0\nu} - \partial_\nu g_{0\mu} - g^{11}\partial_1 g_{\mu\nu} + g^{1\rho}\bigl(-\partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu}\bigr) \\ &= \partial_0 g_{\mu\nu} - \frac{1}{3}\Lambda\partial_1 g_{\mu\nu} + O(w). \end{aligned} \tag{4.16}$$

The first implies that $\partial_1 g_{0\mu} = O(w)$. To evaluate the second, let us write $g_{\mu\nu} = q_{\mu\nu} + r_{\mu\nu}w + O(w^2)$, where $r_{\mu\nu} = r_{\mu\nu}(u, x^\rho)$ (see equation (4.12)). Substituting this into (4.16) we find that $\partial_0 q_{\mu\nu} = \frac{1}{3}\Lambda r_{\mu\nu}$. Thus, from five of the equations $\nabla_a\nabla_b\Omega \approx 0$ we obtain that

$$g_{0\mu} = O(w^2), \qquad g_{\mu\nu} = q_{\mu\nu} + \frac{3}{\Lambda}\bigl(\partial_0 q_{\mu\nu}\bigr)w + O(w^2). \tag{4.17}$$

Equations (4.14) and (4.17) provide a refinement of the fall-off properties (4.12). The second of (4.17) implies that $g^{\mu\nu} = q^{\mu\nu} + \frac{3}{\Lambda}(\partial_0 q^{\mu\nu})w + O(w^2)$, where $q^{\mu\nu}$ is the inverse of $q_{\mu\nu}$. The remaining components of $\nabla_a\nabla_b\Omega \approx 0$ do not yield any further restriction on the asymptotic form of the metric.

Introducing the new coordinate $r := 1/w$, the asymptotic form of the physical spacetime metric $d\hat{s}^2 = \Omega^{-2}ds^2$ is

$$\begin{aligned} d\hat{s}^2 = &-\frac{1}{3}\Lambda r^2 du^2 + 2du\,dr + r^2\Bigl(q_{\mu\nu} + \frac{3}{\Lambda}\bigl(\partial_0 q_{\mu\nu}\bigr)\frac{1}{r}\Bigr)dx^\mu dx^\nu \\ &+ O(1)du^2 + O(\frac{1}{r^2})du\,dr + O(1)du\,dx^\mu + O(1)dx^\mu\,dx^\nu. \end{aligned} \tag{4.18}$$

In particular, the metric of the de Sitter spacetime also has this form. To see this, first let us rewrite the line element of the Einstein universe, $ds^2_E = d\tau^2 - d\bar{r}^2 - \sin^2\bar{r}\,d\omega^2$, in the new coordinates

$$u := \sqrt{\frac{3}{\Lambda}}\bigl(\tau - \bar{r}\bigr), \qquad w := \sqrt{\frac{\Lambda}{3}}\bigl(\tau_0 - \tau\bigr).$$

(Here τ_0 is some constant and $d\omega^2 := d\theta^2 + \sin^2\theta d\phi^2 = 4(1+\zeta\bar\zeta)^{-2}d\zeta d\bar\zeta$, the line element of the unit sphere metric in the angle and the complex null coordinates, respectively, and $\bar r \in [0,\pi]$, $\tau \in \mathbb{R}$.) We obtain

$$
\begin{aligned}
ds^2_E =& -\frac{1}{3}\Lambda du^2 - 2du\,dw - \sin^2\Bigl(\sqrt{\frac{\Lambda}{3}}u - \tau_0 + \sqrt{\frac{3}{\Lambda}}w\Bigr)d\omega^2 \\
=& -\frac{1}{3}\Lambda du^2 - 2du\,dw - \sin^2\Bigl(\sqrt{\frac{\Lambda}{3}}u - \tau_0\Bigr)d\omega^2 - \frac{3}{\Lambda}\frac{\partial}{\partial u}\Bigl(\sin^2\Bigl(\sqrt{\frac{\Lambda}{3}}u - \tau_0\Bigr)d\omega^2\Bigr)w \\
&+ O(w^2),
\end{aligned}
$$

which has the expected asymptotic form near the $w=0$ hypersurface. Thus, to see that the de Sitter metric has indeed the form (4.18), it is already enough to recall that the de Sitter spacetime is conformal e.g. to the $-\pi/2 < \tau < \pi/2$ part of the Einstein universe. Its future conformal boundary $\mathscr{I}^+ = \{\tau = \pi/2\}$ coincides with the $w=0$ hypersurface precisely when $\tau_0 = \pi/2$, in which case $u = \sqrt{\frac{3}{\Lambda}}\frac{\pi}{2}$, 0 and $-\sqrt{\frac{3}{\Lambda}}\frac{\pi}{2}$ correspond, respectively, to the $\bar r = 0$ origin, the $\bar r = \pi/2$ maximal 2-surface and the $\bar r = \pi$ anti-podal point; while the conformal factor is $\Omega = \sqrt{\frac{\Lambda}{3}}\sin(\tau + \pi/2) = \sqrt{\frac{\Lambda}{3}}\sin(\pi - \sqrt{\frac{3}{\Lambda}}w) = w + O(w^3)$.

4.3 The Newman–Penrose tetrad

Let $\mathcal{S}_{u,w}$ denote the $w=$ const topological 2-sphere in the null hypersurface $\mathcal{N}_u$, and adapt a Newman–Penrose (NP) complex null tetrad $\{l^a, n^a, m^a, \bar m^a\}$ to these surfaces: First, recall that l^a is an outgoing null normal to $\mathcal{S}_{u,w}$, and let us choose n^a to be the future pointing ingoing null normal and normalized by $n^a l_a = 1$; and let us choose m^a to be a complex null tangent to $\mathcal{S}_{u,w}$, $\bar m^a$ to be its complex conjugate and they are normalized according to $m^a \bar m_a = -1$. This basis is fixed up to the change $m^a \mapsto e^{\mathrm{i}\alpha} m^a$ for any real function α of the coordinates. Since $l_a = \nabla_a u$, $n^a l_a = 1$, $m^a \nabla_a u = 0$ and $m^a \nabla_a w = 0$, the vectors of the tetrad have the form

$$
l^a = -\bigl(\frac{\partial}{\partial w}\bigr)^a, \qquad n^a = \bigl(\frac{\partial}{\partial u}\bigr)^a + B\bigl(\frac{\partial}{\partial w}\bigr)^a + C^\mu\bigl(\frac{\partial}{\partial x^\mu}\bigr)^a, \qquad m^a = D^\mu\bigl(\frac{\partial}{\partial x^\mu}\bigr)^a \tag{4.19}
$$

for some functions B, C^μ and D^μ of the coordinates. Here B is real and, if (x^2, x^3) are complex, then $C^2 = \bar C^3$ holds. Comparing $g^{ab} = l^a n^b + n^a l^b - m^a \bar m^b - \bar m^a m^b$ with (4.13) we see that

$$
B = -\frac{1}{6}\Lambda + O(w^2), \qquad C^\mu = -g^{\mu\rho}g_{\rho 0} = O(w^2), \qquad D^\mu \bar D^\nu + \bar D^\mu D^\nu = g^{\mu\nu}. \tag{4.20}
$$

Since the future pointing unit timelike normal of $\mathscr{I}^+$ is $N^a = -\sqrt{3/\Lambda}g^{ab}\nabla_b w$, its contraction with l_a is constant on the whole $\mathscr{I}^+$: $N^a l_a = \sqrt{3/\Lambda}$. This implies that on $\mathscr{I}^+$

$$
l_a = \sqrt{\frac{3}{\Lambda}}\bigl(N_a + V_a\bigr), \qquad n_a = \sqrt{\frac{\Lambda}{12}}\bigl(N_a - V_a\bigr). \tag{4.21}
$$

Thus, the real null normals l_a and n_a of the 2-surface $\mathcal{S}$ are boosted with respect to $\frac{1}{\sqrt{2}}(N_a \pm V_a)$, the ones built from the timelike and spacelike normals *symmetrically.*

Hence, *we adapt our normalized GHP spinor dyad* $\{o^A, \iota^A\}$ *to the un-boosted null normals* $\frac{1}{\sqrt{2}}(N_a \pm V_a)$, i.e. $o^A\bar{o}^{A'} = \sqrt{\Lambda/6}\, l^a$, $\iota^A\bar{\iota}^{A'} = \sqrt{6/\Lambda}\, n^a$, $o^A\bar{\iota}^{A'} = m^a$ and $\iota^A\bar{o}^{A'} = \bar{m}^a$. Also, *we define the GHP spin coefficients [30] in this un-boosted frame*, rather than in $\{l^a, n^a, m^a, \bar{m}^a\}$.

To find the asymptotic form of the spin coefficients in the *physical* spacetime, we conformally rescale the tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$. Since, however, neither l^a nor n^a is distinguished physically over the other, we rescale them, and the spinor dyad also, symmetrically:

$$l^a = \Omega^{-1}\hat{l}^a, \qquad n^a = \Omega^{-1}\hat{n}^a, \qquad m^a = \Omega^{-1}\hat{m}^a; \tag{4.22}$$

$$o^A = \Omega^{-\frac{1}{2}}\hat{o}^A, \quad \iota^A = \Omega^{-\frac{1}{2}}\hat{\iota}^A. \tag{4.23}$$

These imply the general formulae how the various GHP spin coefficients change under such a symmetric rescaling:

$$\begin{aligned}
\hat{\kappa} &= \Omega\kappa, & \hat{\kappa}' &= \Omega\kappa', \\
\hat{\sigma} &= \Omega\sigma, & \hat{\sigma}' &= \Omega\sigma', \\
\hat{\rho} &= \Omega\rho + o^A\bar{o}^{A'}\nabla_a\Omega, & \hat{\rho}' &= \Omega\rho' + \iota^A\bar{\iota}^{A'}\nabla_a\Omega, \\
\hat{\tau} &= \Omega\tau + o^A\bar{\iota}^{A'}\nabla_a\Omega, & \hat{\tau}' &= \Omega\tau' + \iota^A\bar{o}^{A'}\nabla_a\Omega, \\
\hat{\beta} &= \Omega\beta - \frac{1}{2}o^A\bar{\iota}^{A'}\nabla_a\Omega, & \hat{\beta}' &= \Omega\beta' - \frac{1}{2}\iota^A\bar{o}^{A'}\nabla_a\Omega, \\
\hat{\varepsilon} &= \Omega\varepsilon - \frac{1}{2}o^A\bar{o}^{A'}\nabla_a\Omega, & \hat{\varepsilon}' &= \Omega\varepsilon' - \frac{1}{2}\iota^A\bar{\iota}^{A'}\nabla_a\Omega.
\end{aligned}$$

Since l_a is a gradient and n_a is hypersurface orthogonal, certain GHP spin coefficients take special value: $\kappa = 0$, $\bar{\rho} = \rho$, $\bar{\rho}' = \rho'$, $\varepsilon + \bar{\varepsilon} = 0$ and $\tau = \beta - \bar{\beta}'$ hold. In addition, by (4.21) at $\mathscr{I}^+$

$$\frac{1}{\sqrt{2}}\Big(\nabla_a\big(o_B\bar{o}_{B'}\big) + \nabla_a\big(\iota_B\bar{\iota}_{B'}\big)\Big) = \nabla_a N_b = 0,$$

whose contraction with the various tetrad vectors yields that

$$\kappa', \ \ \tau', \ \ \tau = \beta - \bar{\beta}', \ \ \varepsilon' + \bar{\varepsilon}', \ \ \sigma' + \bar{\sigma}, \ \ \rho' + \rho = O(w).$$

Hence, by $\Omega = 1/r + O(1/r^3)$, the asymptotic form of the GHP spin coefficients in the physical spacetime is

$$\hat{\kappa} = 0, \qquad \hat{\kappa}' = O(\frac{1}{r^2}), \tag{4.24}$$

$$\hat{\sigma} = \frac{1}{r}\sigma^0 + O(\frac{1}{r^2}), \qquad \hat{\sigma}' = -\frac{1}{r}\bar{\sigma}^0 + O(\frac{1}{r^2}), \tag{4.25}$$

$$\hat{\rho} = -\sqrt{\frac{\Lambda}{6}} + \frac{1}{r}\rho^0 + O(\frac{1}{r^2}), \qquad \hat{\rho}' = -\sqrt{\frac{\Lambda}{6}} - \frac{1}{r}\rho^0 + O(\frac{1}{r^2}), \tag{4.26}$$

$$\hat{\tau} = O(\frac{1}{r^2}), \qquad \hat{\tau}' = O(\frac{1}{r^2}), \tag{4.27}$$

$$\hat{\beta} + \bar{\hat{\beta}}' = \frac{2}{r}\beta^0 + O(\frac{1}{r^2}), \qquad \hat{\beta} - \bar{\hat{\beta}}' = O(\frac{1}{r^2}), \tag{4.28}$$

$$\hat{\varepsilon} + \bar{\hat{\varepsilon}} = \sqrt{\frac{\Lambda}{6}} + O(\frac{1}{r^2}), \qquad \hat{\varepsilon} - \bar{\hat{\varepsilon}} = \frac{2\mathrm{i}}{r}\varepsilon^0 + O(\frac{1}{r^2}), \tag{4.29}$$

$$\hat{\varepsilon}' + \bar{\hat{\varepsilon}}' = \sqrt{\frac{\Lambda}{6}} + O(\frac{1}{r^2}), \qquad \hat{\varepsilon}' - \bar{\hat{\varepsilon}}' = \frac{2\mathrm{i}}{r}\varepsilon'^0 + O(\frac{1}{r^2}), \tag{4.30}$$

where σ^0 and β^0 are complex, while ρ^0, ε^0 and ε'^0 are real functions on $\mathscr{I}^+$. The first three of them are linked to the intrinsic geometry of the conformal boundary. In fact,

$$\sigma^0 = \frac{1}{\sqrt{2}}m^a\big(\nabla_a V_b\big)m^b, \qquad \rho^0 = \frac{1}{\sqrt{2}}\bar{m}^a\big(\nabla_a V_b\big)m^b, \qquad \beta^0 = -\frac{1}{2}m^a\big(\nabla_a m_b\big)\bar{m}^b \tag{4.31}$$

represent, respectively, the trace-free part of the extrinsic curvature of the 2-surfaces $\mathcal{S}_u$ in $\mathscr{I}^+$, the trace of this extrinsic curvature, and the rest of the connection 1-form of the intrinsic geometry of $\mathscr{I}^+$. On the other hand, on $\mathscr{I}^+$

$$\varepsilon^0 = \frac{\mathrm{i}}{2}\sqrt{\frac{\Lambda}{6}}l^a\big(\nabla_a m_b\big)\bar{m}^b, \qquad \varepsilon'^0 = \frac{\mathrm{i}}{2}\sqrt{\frac{6}{\Lambda}}n^a\big(\nabla_a \bar{m}_b\big)m^b, \tag{4.32}$$

which specify how the complex null vectors m^a and $\bar{m}^a$ are extended off the conformal boundary. Since, however, the complex null vectors m^a and $\bar{m}^a$ are fixed only up to the phase transformation $m^a \mapsto \exp(\mathrm{i}\alpha)m^a$ with an arbitrary smooth function $\alpha = \alpha(u, w, x^\mu)$, the functions ε^0 and ε'^0 can be chosen to be vanishing on $\mathscr{I}^+$. In fact, since on a neighbourhood of $\mathscr{I}^+$ they are given by (4.32) up to $O(\Omega)$ terms, under such a transformation they change according to $\varepsilon^0 \mapsto \varepsilon^0 + \frac{1}{2}\sqrt{\frac{\Lambda}{6}}l^a\nabla_a\alpha$ and $\varepsilon'^0 \mapsto \varepsilon'^0 - \frac{1}{2}\sqrt{\frac{6}{\Lambda}}n^a\nabla_a\alpha$. Thus, if α is chosen to be

$$\alpha\big(u, w, x^\mu\big) = 2\sqrt{\frac{6}{\Lambda}}\int_0^w \varepsilon^0\big(u, w', x^\mu\big)dw',$$

then the new ε^0 in the transformed frame is vanishing even on a neighbourhood of $\mathscr{I}^+$. The NP frame is still not fixed, phase transformations of the complex null vectors with w-independent phase are still allowed. Thus, if this phase in such a further transformation is chosen to be

$$\alpha\big(u, x^\mu\big) = 2\sqrt{\frac{\Lambda}{6}}\int_0^u \varepsilon'^0\big(u', 0, x^\mu\big)du',$$

then the new ε'^0 in the transformed frame is vanishing on $\mathscr{I}^+$. Therefore, the ε^0 and ε'^0 in (4.29) and (4.30), respectively, can be chosen to be vanishing. Note also that, by the first of (4.29), r is *not* an affine parameter along the null geodesic generators of the null hypersurfaces $\mathcal{N}_u$ in the physical spacetime.

4.4 The asymptotic properties of the geometry of Σ

To determine the asymptotic form of the solutions of the renormalized Witten equation, and also to find the appropriate function spaces in which the renormalized Witten equation can be proven to admit a solution, we need to know the detailed asymptotic structure of the spacelike hypersurface Σ. Thus, suppose that in a neighbourhood of $\mathscr{I}^+$ in M the hypersurface Σ is given by $u - U(w, x^\mu) = 0$ for some smooth function U of w and x^μ, where, by $\Sigma \cap \mathscr{I}^+ = \mathcal{S}_0$, $U(0, x^\mu) = 0$ holds. (More generally, a 1-parameter family Σ_t of spacelike hypersurfaces that intersect $\mathscr{I}^+$ in the 2-surfaces $\mathcal{S}_u$ with $u = t$ is given by the level sets $t := u - U(w, x^\mu) = \text{const.}$) Hence we can write $U(w, x^\mu) = Ww + O(w^2)$ for some smooth function W of the coordinates x^μ. Since Σ is spacelike (even at $\mathscr{I}^+$) and the hypersurfaces $\mathcal{N}_u$ are null, the u coordinate along Σ must be increasing with increasing w, and hence W must be strictly positive. Since the components of the normal T_a of Σ in the coordinate system (u, w, x^μ) are $(1, -\frac{\partial U}{\partial w}, -\frac{\partial U}{\partial x^\mu})$, its norm is $|T_e|^2 := g^{ab} T_a T_b = \frac{1}{3}\Lambda W^2 + 2W + O(w)$, its scalar product with the unit normal of the $\Omega = \text{const}$ hypersurfaces is $N^a T_a = \sqrt{\frac{3}{\Lambda}}(1 + \frac{1}{3}\Lambda W) + O(w)$, and the function W is completely determined by $|T_e|^2$ on $\mathscr{I}^+$. Note that Σ would be asymptotically null (e.g. the null hypersurface $\mathcal{N}_0$ itself) precisely when W were vanishing. Then it is straightforward to derive the asymptotic form of the induced physical metric $\hat{h}_{ab} = \Omega^{-2} h_{ab}$ on Σ. It is

$$d\hat{h}^2 = -\frac{1}{r^2}\Big(|T_e|^2 + O(\frac{1}{r})\Big)dr^2 + O(\frac{1}{r})dr\, dx^\mu + r^2\Big(q_{\mu\nu} + O(\frac{1}{r})\Big)dx^\mu\, dx^\nu.$$

If $|T_e|$ (and hence W, too) were constant and $q_{\mu\nu}$ were the unit sphere metric, then this would be just the asymptotic form of the standard hyperboloidal metric

$$d\hat{h}^2_H = -\frac{|T_e|^2}{|T_e|^2 + r^2} dr^2 - r^2 d\omega^2$$

with constant curvature (and curvature scalar $\hat{\mathcal{R}} = -6/|T_e|^2$). Therefore, the induced intrinsic metric on Σ is some 'deformed', or asymptotically hyperboloidal one, characterized asymptotically by the function W and the 2-metric $q_{\mu\nu}$, in which r is an asymptotic *areal* (rather than a radial distance) coordinate. The function W plays the role of the local boost parameters, characterizing the relative direction of the normal of Σ with respect to that of $\mathscr{I}^+$ at $\mathscr{I}^+$.

However, it seems useful to rewrite the induced metric in a slightly different, intrinsic coordinate system on Σ. Thus, let us foliate the asymptotic end of Σ by the level sets of the conformal factor, $\hat{\mathcal{S}}_\Omega := \Sigma \cap \{\Omega = \text{const}\}$. In general, for $\Omega > 0$, these surfaces do not coincide with any $\mathcal{S}_{u,w} := \mathcal{N}_u \cap \{w = \text{const}\}$, but in the $\Omega \to 0$ limit $\hat{\mathcal{S}}_\Omega \to \mathcal{S}_0 \subset \mathscr{I}^+$. Let $v_a := D_a\Omega / |D_e\Omega|$, the g_{ab}-unit normal to $\hat{\mathcal{S}}_\Omega$ which is tangent to Σ, where $|D_e\Omega|^2 := -g^{ab}(D_a\Omega)(D_b\Omega)$. This v^a points 'outward' to the conformal boundary, and the lapse $\tilde{n}$ of this foliation, defined by $1 =: -\tilde{n} v^a D_a\Omega$, is just $\tilde{n} = 1/|D_e\Omega|$.

Let us complete this v^a to be a frame field $\{v^a, M^a, \bar{M}^a\}$ on Σ. Here M^a and $\bar{M}^a$ are complex null tangents of the surfaces $\hat{\mathcal{S}}_\Omega$, orthogonal to v^a, and normalized with respect to g_{ab} by $M^a \bar{M}_a = -1$. A simple calculation yields that the g_{ab}-unit normal to Σ and the vectors of this frame field on Σ are given by

$$t^a = \frac{1}{|T_e|}\Big((1+\frac{1}{6}\Lambda W)l^a + Wn^a\Big) + O(w), \tag{4.33}$$
$$v^a = \frac{1}{|T_e|}\Big((1+\frac{1}{6}\Lambda W)l^a - Wn^a\Big) + O(w), \tag{4.34}$$
$$M^a = \exp(\mathrm{i}\alpha)\, m^a + O(w) \tag{4.35}$$

with an irrelevant phase α, which will be chosen to be zero. By means of the last two it is straightforward to give the explicit form of the projection $P^a_b = -v^a v_b - M^a\bar M_a - \bar M^a M_b = -v^a v_b - m^a\bar m_a - \bar m^a m_b + O(w)$. By (4.34), (4.19) and (4.20) the integral curves of v^a (with parameter $\tilde w$) in the coordinates (u,w,x^μ) are

$$u(\tilde w) = -\frac{W}{|T_e|}\tilde w + O(\tilde w^2), \qquad w(\tilde w) = -\frac{1}{|T_e|}\tilde w + O(\tilde w^2), \qquad x^\mu(\tilde w) = x^\mu(0) + O(\tilde w^2); \tag{4.36}$$

and their end points on $\mathscr{I}^+$ are at $\tilde w = 0$. Hence these integral curves define a diffeomorphism between $\mathcal{S}_0$ and the surfaces $\hat{\mathcal{S}}_\Omega$. Moreover, $\tilde w$ coincides with the affine parameter w in the first order up to a scale transformation (though this scale factor depends on the coordinate x^μ of the end point of the integral curves on $\mathscr{I}^+$). Thus, the coordinates x^μ of the end points of the integral curves can be extended from $\mathcal{S}_0$ to the whole asymptotic end of Σ by $v^a D_a\tilde x^\mu = 0$ with the initial condition $\tilde x^\mu \approx x^\mu$. By the third of (4.36) $\tilde x^\mu = x^\mu + O(\tilde w^2)$. Therefore, $(\Omega, \tilde x^\mu)$, or rather $(\tilde r, \tilde x^\mu)$ with $\tilde r := 1/\Omega$, form a coordinate system on the asymptotic end of Σ. The radial coordinate $\tilde r$ coincides with r in the first two orders: $\tilde r = r + O(r^{-1})$. The coordinate vector $(\partial/\partial\Omega)^a$ is just the lapse times of the unit normal of the surfaces, $-v^a/|D_e\Omega|$, with vanishing 'shift part'. But by (4.33)

$$\begin{aligned}|D_e\Omega|^2 &= -g^{ab}(D_a\Omega)(D_b\Omega) = -g^{11} + t^1t^1 + O(w^2)\\ &= -\frac{1}{3}\Lambda + \frac{1}{|T_e|^2}\big(1+\frac{1}{3}\Lambda W + O(w)\big)^2 + O(w^2) = \frac{1}{|T_e|^2} + O(w),\end{aligned} \tag{4.37}$$

i.e. the lapse is $\tilde n = |T_e| + O(w)$. Therefore, in these coordinates, $(\partial/\partial\Omega)^a = -|T_e|v^a + O(\Omega)$, and the asymptotic form of the induced physical metric $\hat h_{ab}$ is

$$d\hat h^2 = -\frac{1}{\tilde r^2}\Big(|T_e|^2 + O(\frac{1}{\tilde r})\Big)d\tilde r^2 + \tilde r^2\Big(R^2\,{}_0q_{\mu\nu} + O(\frac{1}{\tilde r})\Big)d\tilde x^\mu\, d\tilde x^\nu, \tag{4.38}$$

where R is the conformal factor such that ${}_0q_{\mu\nu} = R^{-2}q_{\mu\nu}$ is the unit sphere metric, and the coordinates $\tilde x^\mu$ can be chosen to be the familiar angle or the complex stereographic coordinates in which ${}_0q_{\mu\nu}$ takes the standard form of the unit sphere metric (see subsection 4.2). We use this form of the metric in appendix 5.2, and this form of the coordinate vector $(\partial/\partial\Omega)^a$ in subsections 4.5.2 and 4.6.1. In particular, the asymptotic form of the induced volume element on the hypersurface is $\mathrm{d}\hat\Sigma = \tilde r R^2|T_e|\,\mathrm{d}\mathcal{S}_0\,\mathrm{d}\tilde r$, where $\mathrm{d}\mathcal{S}_0$ is the area element on the unit sphere.

The most convenient way to calculate the extrinsic curvature of Σ, both in the conformal and in the physical spacetime, is the use of the family Σ_t of hypersurfaces and the coordinates (t,w,x^μ). We obtain that

$$\hat\chi_{ab} = \frac{1}{|T_e|}\big(1+\frac{1}{3}\Lambda W\big)\hat h_{ab} + O(\frac{1}{\tilde r}). \tag{4.39}$$

Thus the physical extrinsic curvature of Σ is asymptotically proportional to its intrinsic physical metric, just like in the case of spacelike hypersurfaces extending to the future null infinity of asymptotically flat spacetimes. Therefore, the leading terms in the asymptotic form of the metric $\hat{h}_{ab}$ and of the extrinsic curvature $\hat{\chi}_{ab}$ on the single hypersurface Σ are determined by two functions on the cut $\mathcal{S}=\mathscr{I}^+\cap\Sigma$: the boost 'parameter' W and the conformal factor R.

4.5 The boundary conditions from the Witten equation

4.5.1 The fall-off and the algebraic boundary conditions

The boundary conditions of the renormalized Witten equations (4.9), given explicitly by

$$\hat{\mathcal{D}}_{A'A}\hat{\alpha}^A+\frac{3}{2}K\bar{\hat{\beta}}_{A'}=0,\qquad \hat{\mathcal{D}}_{AA'}\bar{\hat{\beta}}^{A'}+\frac{3}{2}K\hat{\alpha}_A=0, \tag{4.40}$$

consist of two parts. (Here $K=\pm\mathrm{i}\sqrt{\Lambda/6}$, see the text following equation (4.10). Though the sign can be fixed without loss of generality, we leave this ambiguity in the formalism. All the sign ambiguities in what follows come from this ambiguity.) The first is an appropriate fall-off condition specifying how fast the spinor fields tend to their own asymptotic value at infinity, while the second is a condition on the asymptotic values of the spinor fields. In the present subsection we determine the first, and the part of the second that comes from the equations (4.40) themselves.

To find these conditions, we rewrite (4.40) in the unphysical spacetime. We associate zero conformal weight to the contravariant form of the spinor fields, i.e. $\alpha^A=\hat{\alpha}^A$, $\beta^A=\hat{\beta}^A$, and, for the sake of simplicity, we *a priori assume* that α^A and β^A are smooth on M. Hence we can write their components, defined in the unphysical spinor dyad $\varepsilon^A_{\underline{A}}:=\{o^A,\iota^A\}$ e.g. by $\alpha_{\underline{A}}:=\alpha_A\varepsilon^A_{\underline{A}}$, as

$$\alpha_{\underline{A}}=\alpha^{(0)}_{\underline{A}}+\Omega\alpha^{(1)}_{\underline{A}}+O(\Omega^2),\qquad \beta_{\underline{A}}=\beta^{(0)}_{\underline{A}}+\Omega\beta^{(1)}_{\underline{A}}+O(\Omega^2); \tag{4.41}$$

where the functions $\alpha^{(0)}_{\underline{A}},...,\beta^{(1)}_{\underline{A}}$ depend only on the coordinates $\tilde{x}^\mu$, i.e. they are functions on $\mathcal{S}=\Sigma\cap\mathscr{I}^+$. (The subsequent analysis shows that even a slightly less restrictive condition, viz. $\alpha_{\underline{A}}=\alpha^{(0)}_{\underline{A}}+\Omega^k\alpha^{(1)}_{\underline{A}}+O(\Omega^{k+1})$, $k>1/2$, would already be enough.) Since $\hat{\alpha}_A=\Omega^{-1}\alpha_A$ and $\hat{\beta}_A=\Omega^{-1}\beta_A$ hold, by (4.23) the components of the spinor fields in the physical spinor dyad $\hat{\varepsilon}^A_{\underline{A}}=\{\hat{o}^A,\hat{\iota}^A\}=\Omega^{\frac{1}{2}}\varepsilon^A_{\underline{A}}$, $\underline{A}=0,1$, have the asymptotic form

$$\hat{\alpha}_{\underline{A}}=\Omega^{-\frac{1}{2}}\Big(\alpha^{(0)}_{\underline{A}}+\Omega\alpha^{(1)}_{\underline{A}}+O(\Omega^2)\Big),\qquad \hat{\beta}_{\underline{A}}=\Omega^{-\frac{1}{2}}\Big(\beta^{(0)}_{\underline{A}}+\Omega\beta^{(1)}_{\underline{A}}+O(\Omega^2)\Big). \tag{4.42}$$

Thus, the components of the spinor fields in the physical spacetime *diverge* as $\sqrt{\tilde{r}}$.

In the unphysical spacetime (4.40) takes the form

$$0=\mathcal{D}_{A'A}\alpha^A+\frac{3}{2}\Omega^{-1}\Big(K\bar{\beta}_{A'}-(\nabla_{A'A}\Omega)\alpha^A\Big), \tag{4.43}$$

$$0=\mathcal{D}_{AA'}\bar{\beta}^{A'}+\frac{3}{2}\Omega^{-1}\Big(K\alpha_A-(\nabla_{AA'}\Omega)\bar{\beta}^{A'}\Big). \tag{4.44}$$

Let us recall that near $\mathscr{I}^+=\{\Omega=0\}$ the unit normal of the $\Omega=$ const hypersurfaces is $N_a=-\sqrt{3/\Lambda}(\nabla_a\Omega)+O(\Omega^2)$ (see equation (2.4)). Thus, multiplying these equations by Ω and evaluating at $\Omega=0$, we obtain

$$\pm\,\mathrm{i}\bar\beta_{A'}\approx-\sqrt{2}N_{A'A}\alpha^A,\qquad \pm\mathrm{i}\alpha_A\approx-\sqrt{2}N_{AA'}\bar\beta^{A'}. \tag{4.45}$$

These are not independent, one implies the other. Therefore, the pair (α^A,β^A) of spinor fields can solve (4.40) only if the asymptotic value of one of them determines the other at the conformal boundary according to (4.45). In terms of the spinor components (4.45) is equivalent to $\alpha^{(0)}_0=\pm\mathrm{i}\bar\beta^{(0)}_{1'}$ and $\alpha^{(0)}_1=\mp\mathrm{i}\bar\beta^{(0)}_{0'}$. Thus the two spinor fields are linked to each other, but they are still not specified at the conformal boundary.

4.5.2 The asymptotic structure of the solution

Since by (4.45) $K\bar\beta_{A'}-(\nabla_{A'A}\Omega)\alpha^A\approx0$, we may write

$$\Omega\bar\gamma_{A'}:=K\bar\beta_{A'}-(\nabla_{A'A}\Omega)\alpha^A$$

for some smooth spinor field $\bar\gamma_{A'}$ on M. In terms of α^A and $\bar\gamma^{A'}$ the renormalized Witten equations are

$$0=\mathcal{D}_{A'A}\alpha^A+\frac{3}{2}\bar\gamma_{A'}, \tag{4.46}$$

$$\begin{aligned}0=&\,\Omega\mathcal{D}_{AA'}\bar\gamma^{A'}+\bar\gamma^{A'}(D_{A'A}\Omega)+P^c_{AA'}\varepsilon^{A'B'}\big(\nabla_c\nabla_b\Omega\big)\alpha^B+(\nabla^{A'}{}_B\Omega)\mathcal{D}_{A'A}\alpha^B\\&+\frac{3}{2}\Omega^{-1}\Big(K^2+\frac{1}{2}(\nabla_b\Omega)(\nabla^b\Omega)\Big)\alpha_A-\frac{3}{2}\bar\gamma^{A'}\nabla_{A'A}\Omega.\end{aligned} \tag{4.47}$$

Using the 3+1 decomposition $\nabla_e\Omega=D_e\Omega+t_et^f\nabla_f\Omega$ and equations (2.3), (2.4) and (4.46), the second of these takes the form

$$0=(D^{A'}{}_B\Omega)\Big(\mathcal{D}_{A'A}\alpha^B+\frac{1}{2}\bar\gamma_{A'}\delta^B_A\Big)+\Omega\Big(\mathcal{D}_{AA'}\bar\gamma^{A'}+P^c_{AA'}\varepsilon^{A'B'}\big(\frac{1}{4}g_{bc}\Psi+\Psi_{bc}\big)\alpha^B+\frac{3}{2}\Phi\alpha_A\Big).$$

Evaluating this equation at $\Omega=0$, we find that

$$v^{A'}{}_B\mathcal{D}_{A'A}\alpha^B+\frac{1}{2}\bar\gamma_{A'}v^{A'}{}_A\approx0, \tag{4.48}$$

where $v_a:=D_a\Omega/|D_e\Omega|$ (see subsection 4.4). Let $\Pi^a_b:=\delta^a_b-t^at_b+v^av_b=P^a_b+v^av_b$, the orthogonal projection to the 2-surfaces $\hat{\mathcal{S}}_\Omega$, and define $\Delta_a:=\Pi^b_a\nabla_b$, the two-dimensional version of the Sen connection on the 2-surfaces. Then the 2+1 decomposition of the derivative $\mathcal{D}_a$ by Δ_a and the directional derivative $v^e\mathcal{D}_e$ yields that (4.48) has the form

$$\Delta_{A'A}\alpha^A-v^e(\mathcal{D}_e\alpha_A)v^A{}_{A'}+\frac{1}{2}\bar\gamma_{A'}\approx0. \tag{4.49}$$

Here we used that $v^{A'}{}_A\Delta_{A'B}=v^{A'}{}_B\Delta_{A'A}$ and $2v^{AA'}v_{AB'}=-\delta^{A'}_{B'}$. On the other hand, after a similar decomposition the first of the renormalized Witten equations, equation (4.46), yields

$$\Delta_{A'A}\alpha^A+v^e(\mathcal{D}_e\alpha_A)v^A{}_{A'}+\frac{3}{2}\bar\gamma_{A'}\approx0. \tag{4.50}$$

By (4.49) and (4.50)

$$\Delta_{A'A}\alpha^A+\bar\gamma_{A'}=O(\Omega),\qquad v^e\mathcal{D}_e\alpha^A+v^{AA'}\bar\gamma_{A'}=O(\Omega); \tag{4.51}$$

which imply, in particular, that

$$v^e\mathcal{D}_e\alpha^A \approx v^{AA'}\Delta_{A'B}\alpha^B. \tag{4.52}$$

Clearly, there is a similar relationship between the tangential and radial derivatives of the spinor field β^A, too. Hence, the radial and tangential derivatives of the spinor fields on the cut are linked together. To evaluate this, we rewrite it into its GHP form.

Contracting (4.52) with the vectors of the spinor dyad, using the asymptotic form of the spin coefficients (in the gauge $\varepsilon^0 = \varepsilon'^0 = 0$), equations (4.35) and (4.34), $v^a = -|T_e|^{-1}(\partial/\partial\Omega)^a + O(\Omega)$ and the expansion (4.41), we obtain that

$$\alpha^{(1)}_0 = -\sqrt{\frac{\Lambda}{6}}W\Big(\eth\alpha^{(0)}_1 - \rho^0\alpha^{(0)}_0\Big), \qquad \alpha^{(1)}_1 = \sqrt{\frac{6}{\Lambda}}\big(1+\frac{1}{6}\Lambda W\big)\Big(\eth'\alpha^{(0)}_0 + \rho^0\alpha^{(0)}_1\Big); \tag{4.53}$$

and there are analogous formulae for the expansion coefficients $\beta^{(1)}_0$ and $\beta^{(1)}_1$, too. Here $\eth$ and $\eth'$ are the standard GHP edth operators on $\mathcal{S}$ [30]. Thus, *in the solutions of the renormalized Witten equation the $\Omega = 1/\tilde r$ order terms in their asymptotic expansion are determined completely by their boundary value and the boost gauge defined by Σ (and represented by W) at $\mathscr{I}^+$*. In particular, on asymptotically null hypersurfaces $\alpha^{(1)}_0$ and $\beta^{(1)}_0$ would be vanishing. Therefore, with the definitions

$${}_0\alpha_A := \big(\alpha^{(0)}_{\underline{A}} + \Omega\alpha^{(1)}_{\underline{A}}\big)\varepsilon^{\underline{A}}_A, \qquad {}_0\beta_A := \big(\beta^{(0)}_{\underline{A}} + \Omega\beta^{(1)}_{\underline{A}}\big)\varepsilon^{\underline{A}}_A \tag{4.54}$$

the solution of the renormalized Witten equations has the asymptotic form

$$\hat\alpha^A = \alpha^A = {}_0\alpha^A + \hat\sigma^A, \qquad \bar{\hat\beta}^{A'} = \bar\beta^{A'} = {}_0\bar\beta^{A'} + \bar{\hat\pi}^{A'}, \tag{4.55}$$

where e.g. $\alpha^{(0)}_{\underline{A}}$ determines $\beta^{(0)}_{\underline{A}}$ through the algebraic boundary condition (4.45), the coefficients $\alpha^{(1)}_{\underline{A}}$ satisfy (4.53), $\beta^{(1)}_{\underline{A}}$ satisfy the analogous equation (and hence also determined by $\alpha^{(0)}_{\underline{A}}$), and the components of $\hat\sigma^A$ and $\bar{\hat\pi}^{A'}$ in the unphysical spin frame $\varepsilon^A_{\underline{A}}$ are of order Ω^2. (N.B.: The dual spin frame is $\varepsilon^{\underline{A}}_A = -\epsilon^{\underline{A}\underline{B}}\varepsilon^B_{\underline{B}}\varepsilon_{BA}$, where $\epsilon^{\underline{A}\underline{B}}$ is the anti-symmetric Levi-Civita symbol.)

4.5.3 Example: The de Sitter spacetime

In the positivity and rigidity proofs in subsections 4.6.2 and 4.6.3 we need to know some of the properties of the de Sitter spacetime. It is known that in this spacetime the differential equation

$$\hat\nabla_e\Psi^\alpha = \pm\frac{\mathrm{i}}{\sqrt2}\sqrt{\frac{\Lambda}{6}}\hat\gamma^\alpha_{e\beta}\Psi^\beta \tag{4.56}$$

is completely integrable (for either sign on the right), where $\Psi^\alpha = (\alpha^A, \bar\beta^{A'})$ (as a column vector) and the Dirac 'matrices' are given explicitly in terms of the metric spinor by

$$\gamma^\alpha_{e\beta} = \sqrt2\begin{pmatrix} 0 & \varepsilon_{E'B'}\delta^A_E \\ \varepsilon_{EB}\delta^{A'}_{E'} & 0\end{pmatrix}$$

(see e.g. [31], pp 221). Hence it admits four linearly independent solutions and these solutions can be specified by prescribing Ψ^α at any given point of the spacetime. In fact, its Weyl spinor constituents solve the 1–valence twistor equation such that the primary spinor part of one twistor is just the secondary spinor part of the other [10]; and the solutions of (4.56) also solve (4.40) on any spacelike hypersurface Σ. In flat spacetime ($\Lambda = 0$) its solutions are just the spinor constituents of the *translational* Killing vectors. Thus the solutions of (4.56) are the spinor constituents of what substitutes the translational Killing fields in de Sitter spacetime most naturally. (For a more detailed discussion of the geometry of the de Sitter spacetime and the twistor equation, see e.g. [22].)

To find the explicit solutions, let us rewrite (4.56) in the GHP formalism. In the coordinate system based on a spherically symmetric foliation of $\mathscr{I}^+$ and the GHP spin frame (up to phase transformation of the complex null vectors) of subsection 4.2 and 4.3, respectively, the only non-zero GHP spin coefficients are

$$\hat\rho = \sqrt{\frac{\Lambda}{6}}\sin\bigl(\sqrt{\frac{3}{\Lambda}}w\bigr)\cot\bigl(\sqrt{\frac{\Lambda}{3}}u - \frac{\pi}{2} + \sqrt{\frac{3}{\Lambda}}w\bigr) - \sqrt{\frac{\Lambda}{6}}\cos\bigl(\sqrt{\frac{3}{\Lambda}}w\bigr),$$
$$\hat\rho' = -\sqrt{\frac{\Lambda}{6}}\sin\bigl(\sqrt{\frac{3}{\Lambda}}w\bigr)\cot\bigl(\sqrt{\frac{\Lambda}{3}}u - \frac{\pi}{2} + \sqrt{\frac{3}{\Lambda}}w\bigr) - \sqrt{\frac{\Lambda}{6}}\cos\bigl(\sqrt{\frac{3}{\Lambda}}w\bigr),$$
$$\hat\beta = \bar{\hat\beta}' = -\frac{1}{2\sqrt2}\sqrt{\frac{\Lambda}{3}}\frac{\sin\bigl(\sqrt{\frac{3}{\Lambda}}w\bigr)}{\sin\bigl(\sqrt{\frac{\Lambda}{3}}u - \frac{\pi}{2} + \sqrt{\frac{3}{\Lambda}}w\bigr)}\zeta,$$
$$\hat\varepsilon = \hat\varepsilon' = \frac{1}{2}\sqrt{\frac{\Lambda}{6}}\cos\bigl(\sqrt{\frac{3}{\Lambda}}w\bigr).$$

Then, contracting (4.56) with $\hat o^E \bar{\hat o}^{E'}$ we obtain the so-called radial equations (i.e. the parts of (4.56) tangential to the null geodesic generators of the null hypersurfaces $\mathcal{N}_u$), whose solution is given by

$$\hat\alpha_0 = \frac{\alpha_0^{(0)}}{\sqrt{\sin\bigl(\sqrt{\frac{3}{\Lambda}}w\bigr)}}, \qquad \bar{\hat\beta}_{0'} = \frac{\bar\beta_{0'}^{(0)}}{\sqrt{\sin\bigl(\sqrt{\frac{3}{\Lambda}}w\bigr)}},$$
$$\hat\alpha_1 = \frac{1}{\sqrt{\sin\bigl(\sqrt{\frac{3}{\Lambda}}w\bigr)}}\Bigl(\mp \mathrm{i}\bar\beta_{0'}^{(0)}\cos\bigl(\sqrt{\frac{3}{\Lambda}}w\bigr) + \sin\bigl(\sqrt{\frac{3}{\Lambda}}w\bigr)A_1\Bigr),$$
$$\bar{\hat\beta}_{1'} = \frac{1}{\sqrt{\sin\bigl(\sqrt{\frac{3}{\Lambda}}w\bigr)}}\Bigl(\mp \mathrm{i}\alpha_0^{(0)}\cos\bigl(\sqrt{\frac{3}{\Lambda}}w\bigr) + \sin\bigl(\sqrt{\frac{3}{\Lambda}}w\bigr)\bar B_{1'}\Bigr).$$

Here $\alpha_0^{(0)}$, $\bar\beta_{0'}^{(0)}$, A_1 and $\bar B_{1'}$ are still to be determined functions of u and x^μ. Thus, in particular, this solution is compatible with both the general fall off properties (4.41) and the algebraic boundary conditions (4.45).

The contraction of (4.56) with $\hat o^E \bar{\hat\iota}^{E'}$ and with $\hat\iota^E \bar{\hat o}^{E'}$ give the so-called surface equations, i.e. the ones tangential to the $u =$ const, $w =$ const 2-spheres. Substituting the solution of the radial equations here we obtain, in particular, that

$${}_0\eth\alpha_0^{(0)} = 0, \qquad {}_0\eth'\bar\beta_{0'}^{(0)} = 0, \qquad {}_0\eth' A_1 = 0, \qquad {}_0\eth\bar B_{1'} = 0,$$

where ${}_0\eth$ and ${}_0\eth'$ denote the standard edth operators on the *unit sphere* [31]. The solution of these equations is well known to be

$$\alpha^{(0)}_0=\sum_m a_m\,{}_{\frac12}Y_{\frac12 m},\quad \bar\beta^{(0)}_{0'}=\sum_m b_m\,{}_{-\frac12}Y_{\frac12 m},\quad A_1=\sum_m A_m\,{}_{-\frac12}Y_{\frac12 m},\quad \bar B_{1'}=\sum_m B_m\,{}_{\frac12}Y_{\frac12 m},$$

where ${}_{\pm\frac12}Y_{\frac12 m}$ are the $\pm\frac12$ spin weighted spherical harmonics, $m=-\frac12,\frac12$, and the coefficients a_m, b_m, A_m and B_m are still not specified functions of u. Substituting these into the remaining four surface equations and using how the edth operators act on the spin weighted spherical harmonics we find that A_m and B_m are determined by a_m and b_m according to

$$\begin{aligned}\sin\bigl(\frac{\pi}{2}+\sqrt{\frac{\Lambda}{3}}u\bigr)\,A_m&=a_m\mp\mathrm{i}\cos\bigl(\frac{\pi}{2}+\sqrt{\frac{\Lambda}{3}}u\bigr)\,b_m,\\ \sin\bigl(\frac{\pi}{2}+\sqrt{\frac{\Lambda}{3}}u\bigr)\,B_m&=-b_m\mp\mathrm{i}\cos\bigl(\frac{\pi}{2}+\sqrt{\frac{\Lambda}{3}}u\bigr)\,a_m;\end{aligned}$$

but a_m and b_m remain independent. It might be worth noting that these are just the conditions (4.53), in which we substitute $W=0$ (since the hypersurface on which the spinor components are expanded is null), the radius of the $u=$ const cut is $R=\sin(\frac{\pi}{2}-\sqrt{\frac{\Lambda}{3}}u)$ (see the line element of the Einstein universe in subsection 4.2) and $\rho^0=\frac{1}{\sqrt2}\cot(\sqrt{\frac{\Lambda}{3}}u-\frac{\pi}{2})$. The u-dependence of a_m and b_m is determined by the so-called evolution equations, obtained by contracting (4.56) with $\hat\iota^E\bar{\hat\iota}^{E'}$, and the whole solution is completely determined by the value of the four functions a_m and b_m e.g. at $u=0$. Therefore, the solution $(\alpha^A,\bar\beta^{A'})$ of equation (4.56) is completely determined by its spinor components $\alpha_0=\alpha_A o^A$ and $\bar\beta_{0'}=\bar\beta_{A'}\bar o^{A'}$ on one $u=$ const cut of the conformal boundary.

Let $\mathcal{S}_u$ denote the $u=$ const cut. This can be considered as the intersection of some spherically symmetric spacelike hypersurface Σ and the conformal infinity, and hence we should ask how the solutions α^A and β^A on $\mathcal{S}_u$ could be recovered *purely* in terms of the geometry of $\mathcal{S}_u$. Since on $\mathscr{I}^+$ α^A determines β^A algebraically, it is enough to consider α^A. For its structure we obtained $\alpha_0=\sum_m a_m\,{}_{\frac12}Y_{\frac12 m}$ and $\alpha_1=\mp\mathrm{i}\sum_m b_m\,{}_{-\frac12}Y_{\frac12 m}$. These are the general solution of

$${}_0\eth\alpha_0=0,\qquad {}_0\eth'\alpha_1=0,\tag{4.57}$$

which are the *2-surface twistor equations* on the spherically symmetric $\mathcal{S}_u$.

4.6 The positivity and rigidity of H

In this subsection we begin the proof of existence for solutions of the Witten equation. We show that the requirement of the finiteness of the functional $H[\alpha,\bar\alpha]+H[\beta,\bar\beta]$ on the solutions of the renormalized Witten equations (4.40) yields that the spinor fields must solve the 2-surface twistor equations on the conformal boundary, and with these boundary values they have a unique solution, controlled by the boundary value of e.g. α^A, provided the matter fields satisfy the dominant energy condition. This implies that $H[\alpha,\bar\alpha]+H[\beta,\bar\beta]$ is non-negative for such boundary values (positivity), and that it is vanishing precisely when the domain of dependence of the spacelike hypersurface Σ is locally isometric to the de Sitter spacetime (rigidity).

4.6.1 Boundary conditions from the finiteness of H: The 2-surface twistor equations

Since K in (4.6) is imaginary, the finiteness of $H[\alpha,\bar\alpha]+H[\beta,\bar\beta]$ defined in the physical spacetime is ensured by the finiteness of the integral on the right of (4.10). Since we associated zero conformal weight to the (contravariant form of the) spinor fields, moreover under the conformal rescaling of the spacetime metric the volume element changes according to $\mathrm{d}\hat\Sigma=\Omega^{-3}\mathrm{d}\Sigma$, the integral of the second term on the right hand side of (4.10) is finite precisely when this term falls off *slightly faster* than Ω^2, i.e. when

$$o\big(\Omega^2\big)=\hat t_a\hat T^a{}_b\big(\hat\alpha^B\bar{\hat\alpha}^{B'}+\hat\beta^B\bar{\hat\beta}^{B'}\big)=\Omega^2 t_aT^a{}_b\big(\alpha^B\bar\alpha^{B'}+\beta^B\bar\beta^{B'}\big). \tag{4.58}$$

Hence the rescaled energy-momentum tensor $T^a{}_b$ must tend to zero at the conformal boundary, i.e. the physical energy-momentum tensor must fall off as $\hat T^a{}_b=o(\Omega^3)$, *slightly faster* than Ω^3. Therefore, assuming the smoothness of the rescaled energy-momentum tensor on M, we must require $\hat T^a{}_b=O(\Omega^4)$ (rather than only $\hat T^a{}_b=\Omega^3T^a{}_b$). Thus, in particular, dust-like matter cannot be present near the conformal boundary.

To determine the condition of the finiteness of the integral of the first term on the right of (4.10), we rewrite the derivative $\tilde{\mathcal{D}}_e\hat\alpha_A$ into a more familiar form. It is known that

$$\sqrt{2}\hat t_F{}^{E'}\hat{\mathcal{D}}_{EE'}\hat\alpha_A=\hat{\mathcal{D}}_{(EF}\hat\alpha_{A)}+\sqrt{2}\hat t_F{}^{E'}\frac{2}{3}P^{DD'}_{EE'}\hat\varepsilon_{DA}\hat{\mathcal{D}}_{D'C}\hat\alpha^C \tag{4.59}$$

is the complete algebraically irreducible, $\hat t^{AA'}$–orthogonal decomposition of the derivative. Here $\hat{\mathcal{D}}_{AB}:=\sqrt{2}\hat t_B{}^{A'}\hat{\mathcal{D}}_{AA'}=\hat{\mathcal{D}}_{(AB)}$ is the unitary spinor form of $\hat{\mathcal{D}}_{AA'}$, and the totally symmetric part $\hat{\mathcal{D}}_{(AB}\hat\alpha_{C)}$ defines the 3-surface twistor operator [23]. Substituting this decomposition into the explicit expression of $\tilde{\mathcal{D}}_e\hat\alpha_A$ given by (4.5) and using the renormalized Witten equations (4.40), we find that *it is precisely the 3-surface twistor operator acting on* $\hat\alpha_A$, i.e.

$$\begin{aligned}H\big[\alpha,\bar\alpha\big]+H\big[\beta,\bar\beta\big]=\int_\Sigma\Big\{\frac{4}{\varkappa}\hat t^{AA'}\hat t^{BB'}\hat t^{CC'}\Big((\hat{\mathcal{D}}_{(AB}\hat\alpha_{C)})(\hat{\mathcal{D}}_{(A'B'}\bar{\hat\alpha}_{C')})\\ +(\hat{\mathcal{D}}_{(AB}\hat\beta_{C)})(\hat{\mathcal{D}}_{(A'B'}\bar{\hat\beta}_{C')})\Big)+\hat t_a\hat T^a{}_b(\hat\alpha^B\bar{\hat\alpha}^{B'}+\hat\beta^B\bar{\hat\beta}^{B'})\Big\}\mathrm{d}\hat\Sigma.\end{aligned} \tag{4.60}$$

It has exactly the same structure that the components of the ADM, Bondi–Sachs and Abbott–Deser energy-momenta and the total mass of closed universes (with $\Lambda\geq0$) have in their spinorial form [26, 28, 27, 9, 10]: It is the sum of the square of the L_2-norm of the 3-surface twistor derivative of the spinor field satisfying the gauge condition and the integral of the energy-momentum of the matter fields.

Returning to the question of the finiteness of the integral of the first term on the right of (4.10), (4.60) shows that $\hat{\mathcal{D}}_{(AB}\hat\alpha_{C)}$ and $\hat{\mathcal{D}}_{(AB}\hat\beta_{C)}$ *must be square integrable* on Σ in the physical spacetime. Since the 3-surface twistor operator is conformally covariant, viz. $\hat{\mathcal{D}}_{(AB}(\Omega^{-1}\alpha_{C)})=\Omega^{-1}\mathcal{D}_{(AB}\alpha_{C)}$, this condition is equivalent to the

$$t^{AA'}t^{BB'}t^{CC'}\big(\mathcal{D}_{(AB}\alpha_{C)}\big)\big(\mathcal{D}_{(A'B'}\bar\alpha_{C')}\big)=o(\Omega) \tag{4.61}$$

fall-off in the unphysical spacetime, and to an analogous one for β_A. By (4.33) this is equivalent to

$$o^A o^B o^C\big(\mathcal{D}_{AB}\alpha_C\big),\ o^A o^B \iota^C\big(\mathcal{D}_{(AB}\alpha_{C)}\big),\ o^A \iota^B \iota^C\big(\mathcal{D}_{(AB}\alpha_{C)}\big),\ \iota^A \iota^B \iota^C\big(\mathcal{D}_{AB}\alpha_C\big) = o(\Omega^{\frac{1}{2}}).$$

Then, by (4.33) and the asymptotic value of the GHP spin coefficients in the conformal spacetime, we obtain that

$$\begin{aligned} o^A o^B o^C\big(\mathcal{D}_{AB}\alpha_C\big) &= o^A o^B \sqrt{2} t_B{}^{A'}\big(\nabla_{AA'}\alpha_C\big)o^C = -\sqrt{\frac{\Lambda}{3}}\frac{W}{|T_e|}m^a\big(\nabla_a\alpha_C\big)o^C + O(\Omega) \\ &= -\sqrt{\frac{\Lambda}{3}}\frac{W}{|T_e|}\Big(\eth\alpha_0^{(0)} + \sigma^0\alpha_1^{(0)}\Big) + O(\Omega), \qquad (4.62) \\ \iota^A \iota^B \iota^C\big(\mathcal{D}_{AB}\alpha_C\big) &= \sqrt{\frac{12}{\Lambda}}\frac{1}{|T_e|}\big(1 + \frac{1}{6}\Lambda W\big)\Big(\eth'\alpha_1^{(0)} - \sigma^0\alpha_0^{(0)}\Big) + O(\Omega). \qquad (4.63) \end{aligned}$$

Using (4.33)-(4.35), $|T_e|^2 = \frac{1}{3}\Lambda W^2 + 2W + O(\Omega)$ and $v^a = -|T_e|^{-1}(\partial/\partial\Omega)^a + O(\Omega)$, we find that

$$\begin{aligned} &3\ o^A o^B \iota^C\big(\mathcal{D}_{(AB}\alpha_{C)}\big) = o^A o^B \sqrt{2} t_B{}^{A'} P^e_a\big(\nabla_e\alpha_C\big)\iota^C + 2 o^A \iota^B \sqrt{2} t_B{}^{A'} P^e_a\big(\nabla_e\alpha_C\big)o^C \\ &= -\sqrt{\frac{\Lambda}{3}}\frac{W}{|T_e|}m^a\big(\nabla_a\alpha_C\big)\iota^C + \frac{2\sqrt{2}}{|T_e|}\sqrt{\frac{6}{\Lambda}}\big(1 + \frac{1}{6}\Lambda W\big)o^A\bar{o}^{A'}P^b_a\big(\nabla_b\alpha_C\big)o^C + O(\Omega) \\ &= -\sqrt{\frac{\Lambda}{3}}\frac{W}{|T_e|}\big(\eth\alpha_1^{(0)} - \rho^0\alpha_0^{(0)}\big) + 2\sqrt{2}\frac{W}{|T_e|^2}\big(1 + \frac{1}{6}\Lambda W\big)v^a\big(\mathcal{D}_a\alpha_C\big)o^C + O(\Omega) \\ &= -\frac{W}{|T_e|}\Big(\sqrt{\frac{\Lambda}{3}}\big(\eth\alpha_1^{(0)} - \rho^0\alpha_0^{(0)}\big) + \frac{2\sqrt{2}}{|T_e|^2}\big(1 + \frac{1}{6}\Lambda W\big)\alpha_0^{(1)}\Big) \\ &\quad + \frac{2\sqrt{2}W}{|T_e|^3}\big(1 + \frac{1}{6}\Lambda W\big)\Big(\sqrt{\frac{6}{\Lambda}}\big(1 + \frac{1}{6}\Lambda W\big)\big(\kappa\alpha_1 - \varepsilon\alpha_0\big) - \sqrt{\frac{\Lambda}{6}}W\big(\tau\alpha_1 + \varepsilon'\alpha_0\big)\Big) + O(\Omega). \end{aligned}$$

Then by (4.53) and the asymptotic form of the GHP spin coefficients this, and the analogous expression for $o^A \iota^B \iota^C(\mathcal{D}_{(AB}\alpha_{C)})$, yield that

$$o^A o^B \iota^C\big(\mathcal{D}_{(AB}\alpha_{C)}\big) = O(\Omega), \qquad o^A \iota^B \iota^C\big(\mathcal{D}_{(AB}\alpha_{C)}\big) = O(\Omega). \qquad (4.64)$$

Therefore, by (4.64) $o^A o^B \iota^C(\mathcal{D}_{(AB}\alpha_{C)})$ and $o^A \iota^B \iota^C(\mathcal{D}_{(AB}\alpha_{C)})$ fall off appropriately, but by (4.62) and (4.63) the condition (4.61) is satisfied precisely when

$$\eth\alpha_0^{(0)} + \sigma^0\alpha_1^{(0)} = 0, \qquad \eth'\alpha_1^{(0)} - \sigma^0\alpha_0^{(0)} = 0 \qquad (4.65)$$

also hold, i.e. *if the spinor field α_A on the cut $\mathcal{S} = \Sigma \cap \mathscr{I}^+$ solves the 2-surface twistor equations* of Penrose [12]. In particular, by (4.57) the solution of (4.56) in the de Sitter spacetime also satisfies this condition.

4.6.2 Positivity

Assuming that the matter fields satisfy the dominant energy condition, the proof of the non-negativity of $H[\alpha, \bar{\alpha}] + H[\beta, \bar{\beta}]$ reduces to the proof of the existence of solutions of (4.40) with the boundary values satisfying (4.45) and (4.65). To prove this existence, let us use the decomposition (4.54)-(4.55) of the spinor fields in (4.40). Then the homogeneous renormalized Witten equations take the form of the system

$$\hat{\mathcal{D}}_{A'A}\hat{\sigma}^A+\frac{3}{2}K\bar{\hat{\pi}}_{A'}=-\Big(\hat{\mathcal{D}}_{A'A}\,{}_0\hat{\alpha}^A+\frac{3}{2}K\,{}_0\bar{\hat{\beta}}_{A'}\Big)=:\bar{\hat{\omega}}_{A'}, \tag{4.66}$$

$$\hat{\mathcal{D}}_{AA'}\bar{\hat{\pi}}^{A'}+\frac{3}{2}K\hat{\sigma}_A=-\Big(\hat{\mathcal{D}}_{AA'}\,{}_0\bar{\hat{\beta}}^{A'}+\frac{3}{2}K\,{}_0\hat{\alpha}_A\Big)=:\hat{\rho}_A \tag{4.67}$$

of inhomogeneous equations. The advantage of using $(\hat{\sigma}^A,\bar{\hat{\pi}}^{A'})$ instead of $(\hat{\alpha}^A,\bar{\hat{\beta}}^{A'})$ is that the spinor fields satisfying the homogeneous boundary condition form a vector space, while those satisfying an inhomogeneous one do not. Thus, the techniques of linear functional analysis can be applied to them more easily. Moreover, the spinor fields on the right hand side of (4.66) and (4.67) are fixed by the boundary conditions, and hence what we should prove is only the existence of the spinor fields $(\hat{\sigma}^A,\bar{\hat{\pi}}^{A'})$ with appropriate fall-off. Also, we will need the uniqueness of this solution.

First we show that the *homogeneous* equations corresponding to (4.66)-(4.67), i.e.

$$\hat{\mathcal{D}}_{A'A}\hat{\sigma}^A+\frac{3}{2}K\bar{\hat{\pi}}_{A'}=0,\qquad \hat{\mathcal{D}}_{AA'}\bar{\hat{\pi}}^{A'}+\frac{3}{2}K\hat{\sigma}_A=0, \tag{4.68}$$

do not admit any non-trivial smooth solution with the $\hat{\sigma}^A,\hat{\pi}^A=o(\Omega^{3/2})$ fall-off, provided the matter fields satisfy the dominant energy condition on Σ. (Note that, by the results of subsection 4.6.1, the fall-off $\hat{\sigma}^A,\hat{\pi}^A=o(\Omega^{3/2})$ is needed to ensure the finiteness of $H[\alpha,\bar{\alpha}]+H[\beta,\bar{\beta}]$. This fall-off condition is equivalent to their square integrability, see below.) Suppose, on the contrary, that $(\hat{\sigma}^A,\bar{\hat{\pi}}^{A'})$ is such a solution, and let us apply the Sen–Witten type identity (4.10) to this solution. Then by the dominant energy condition we have that

$$\begin{aligned}0\leq\int_\Sigma\Big\{&-\frac{2}{\varkappa}\hat{t}_{AA'}\hat{h}^{ef}\big((\tilde{\mathcal{D}}_e\hat{\sigma}^A)(\bar{\tilde{\mathcal{D}}}_f\bar{\hat{\sigma}}^{A'})+(\bar{\tilde{\mathcal{D}}}_e\hat{\pi}^A)(\tilde{\mathcal{D}}_f\bar{\hat{\pi}}^{A'})\big)\\&+\hat{t}_a\hat{T}^a{}_b(\hat{\sigma}^B\bar{\hat{\sigma}}^{B'}+\hat{\pi}^B\bar{\hat{\pi}}^{B'})\Big\}\mathrm{d}\hat{\Sigma}=H\big[\hat{\sigma},\bar{\hat{\sigma}}\big]+H\big[\hat{\pi},\bar{\hat{\pi}}\big].\end{aligned} \tag{4.69}$$

We show that the right hand side of this inequality is also zero. The GHP form of $H[\hat{\sigma},\bar{\hat{\sigma}}]$ is $2/\varkappa$ times the $\Omega\to0$ limit of

$$\begin{aligned}&\oint_{\hat{\mathcal{S}}_\Omega}\Big(\bar{\hat{\sigma}}_{1'}\big(\hat{ð}'\hat{\sigma}_0+\hat{\rho}\hat{\sigma}_1\big)-\bar{\hat{\lambda}}_{0'}\big(\hat{ð}\hat{\sigma}_1+\hat{\rho}'\hat{\sigma}_0\big)\Big)\mathrm{d}\hat{\mathcal{S}} \qquad (4.70)\\ =&\oint_{\hat{\mathcal{S}}_\Omega}\Big(\bar{\hat{\sigma}}_{1'}\big(\hat{ð}'\hat{\sigma}_0-\sqrt{\frac{\Lambda}{6}}\hat{\sigma}_1+\hat{\sigma}_1O(\Omega)\big)-\bar{\hat{\sigma}}_{0'}\big(\hat{ð}\hat{\sigma}_1-\sqrt{\frac{\Lambda}{6}}\hat{\sigma}_0+\hat{\sigma}_0O(\Omega)\big)\Big)\Omega^{-2}\mathrm{d}\mathcal{S},\end{aligned}$$

where we used the asymptotic form (4.26) of the GHP convergences. Thus, if $\hat{\sigma}^A=\Omega^l\,{}_0\sigma^A$ with $l>3/2$ and some bounded spinor field ${}_0\sigma^A$ near $\mathscr{I}^+$, then $\hat{\sigma}_0$ and $\hat{\sigma}_1$ fall off *faster* than Ω. This, together with the same argument for $\hat{\pi}^A$, imply that the right hand side of (4.69) is indeed zero. Hence, the integrand of the middle term of (4.69) is vanishing, i.e.

$$\hat{\mathcal{D}}_e\hat{\sigma}^A+KP^{AA'}_e\bar{\hat{\pi}}_{A'}=0,\qquad \hat{\mathcal{D}}_e\bar{\hat{\pi}}^{A'}+KP^{A'A}_e\hat{\sigma}_A=0, \tag{4.71}$$

$$\hat{t}_a\hat{T}^a{}_b\big(\hat{\sigma}^B\bar{\hat{\sigma}}^{B'}+\hat{\pi}^B\bar{\hat{\pi}}^{B'}\big)=0. \tag{4.72}$$

Now we show that the spinor fields $\hat{\sigma}^A$ and $\hat{\pi}^A$ cannot be proportional to each other on any open subset of Σ. Thus, suppose, on the contrary, that $\hat{\pi}^A = F\hat{\sigma}^A$ for some smooth complex function F on some open subset $U \subset \Sigma$. Then by (4.71)

$$0 = \hat{D}_e\big(\hat{\sigma}_A\hat{\pi}^A\big) = -\big(\hat{\mathcal{D}}_e\hat{\sigma}^A\big)\hat{\pi}_A + \hat{\sigma}_A\big(\hat{\mathcal{D}}_e\hat{\pi}^A\big) = K\big(1 + F\bar{F}\big)P_e^{AA'}\hat{\sigma}_A\bar{\hat{\sigma}}_{A'}.$$

Since Σ is spacelike and $\hat{\sigma}_A\bar{\hat{\sigma}}_{A'}$ is null, this implies the vanishing of $\hat{\sigma}_A$ on U. However, by (4.71) $\hat{\sigma}^A$ solves the eigenvalue equation $\hat{\mathcal{D}}^{AA'}\hat{\mathcal{D}}_{A'B}\hat{\sigma}^B = \frac{3}{8}\Lambda\hat{\sigma}^A$, and hence, by an appropriate modification of the proof of Aronszajn's theorem for its eigenspinors, the spinor field $\hat{\sigma}_A$ cannot be vanishing on any open set $U \subset \Sigma$. (For the details, see the appendix of [10].) Therefore, $\hat{\sigma}^A$ and $\hat{\pi}^A$ cannot be proportional to each other on any open subset of Σ, and hence $\hat{\sigma}^A\bar{\hat{\sigma}}^{A'} + \hat{\pi}^A\bar{\hat{\pi}}^{A'}$ is future pointing and *timelike* on an open dense subset of Σ. But then by (4.72) and the dominant energy condition it follows that $\hat{T}_{ab} = 0$ on Σ.

Evaluating the integrability condition of the system (4.71) and using that $\hat{\sigma}^A$ and $\hat{\pi}^A$ can be proportional with each other only on closed subsets of Σ with empty interior, we find that the curvature $\hat{R}_{ABcd}$ of the spacetime at the points of Σ is that of the de Sitter spacetime (see [10]). Foliating the domain of dependence of Σ by smooth Cauchy surfaces Σ_s by Lie dragging Σ along its own timelike $\hat{g}_{ab}$-unit normals, by the Bianchi identities (written in their 3+1 form with respect to this foliation by Friedrich [32]) we obtain that the geometry of the domain of dependence is locally isometric to the de Sitter spacetime. (Note that these Cauchy surfaces Σ_s for the domain of dependence of Σ are *not* the hypersurfaces Σ_t of subsection 4.4. In fact, while all the Σ_s cut $\mathscr{I}^+$ in the *same* 2-surface $\mathcal{S}$, the surfaces $\Sigma_t \cap \mathscr{I}^+$ foliate a neighbourhood of $\mathcal{S}$ in $\mathscr{I}^+$.)

Finally, since the domain of dependence of Σ is locally de Sitter, the spinor fields $\hat{\sigma}^A$ and $\bar{\hat{\pi}}^{A'}$ provide a correct initial condition for (4.56) on Σ. However, by the results of subsection 4.5.3, its solution is completely determined by the value of the solution e.g. at a point of $\Sigma \cap \mathscr{I}^+$. Since both $\hat{\sigma}^A$ and $\hat{\pi}^A$ are vanishing there, the whole solution on Σ must be vanishing. Hence, the differential operator

$$\tilde{\mathcal{D}} : C^\infty\big(\Sigma, \mathbb{D}^\alpha\big) \cap L_2\big(\Sigma, \mathbb{D}^\alpha\big) \to C^\infty\big(\Sigma, \mathbb{D}^\alpha\big) : \Phi^\alpha \mapsto \tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta \tag{4.73}$$

is an *injective* linear map on the space of the smooth square integrable Dirac spinor fields, where $\Phi^\alpha := (\hat{\sigma}^A, \bar{\hat{\pi}}^{A'})$ (as a column vector) and

$$\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta := \big(\hat{\mathcal{D}}^A{}_{B'}\bar{\hat{\pi}}^{B'} + \frac{3}{2}K\hat{\sigma}^A, \hat{\mathcal{D}}^{A'}{}_B\hat{\sigma}^B + \frac{3}{2}K\bar{\hat{\pi}}^{A'}\big). \tag{4.74}$$

Therefore, if the system (4.66)-(4.67) admits a smooth solution, then that is unique in $C^\infty(\Sigma, \mathbb{D}^\alpha)$.

To show the existence of a solution of the *inhomogeneous* (4.66)-(4.67), and also that non-smooth solutions of (4.68) do not exist either, we should reformulate the problem in appropriate function spaces and use certain functional analytic techniques. We start this here and defer the details to the appendix. If $(\hat{\sigma}^A, \bar{\hat{\pi}}^{A'})$ were a solution of (4.66)-(4.67) with *differentiable* extension of the corresponding $\hat{\alpha}^A$ and $\hat{\beta}^A$ to the conformal boundary, then $(\hat{\sigma}^A, \bar{\hat{\pi}}^{A'}) = O(\Omega^2)$ would hold (see equation (4.55)). Hence, for the Dirac spinor $\Phi^\alpha = (\hat{\sigma}^A, \bar{\hat{\pi}}^{A'})$ we would have that

$$|\Phi^\alpha|^2 := \sqrt{2}\hat{t}_{AA'}\big(\hat{\sigma}^A\bar{\hat{\sigma}}^{A'} + \hat{\pi}^A\bar{\hat{\pi}}^{A'}\big) = O(\Omega^3),$$
$$|\hat{\mathcal{D}}_e\Phi^\alpha|^2 := -\hat{h}^{ef}\sqrt{2}\hat{t}_{AA'}\Big((\hat{\mathcal{D}}_e\hat{\sigma}^A)(\hat{\mathcal{D}}_f\bar{\hat{\sigma}}^{A'}) + (\hat{\mathcal{D}}_e\hat{\pi}^A)(\hat{\mathcal{D}}_f\bar{\hat{\pi}}^{A'})\Big) = O(\Omega^3).$$

(In the second of these, we used how $\hat{\mathcal{D}}_e$ is related to the Sen derivative operator $\mathcal{D}_e$ in the conformal geometry, and that $\hat{\sigma}^A = \Omega^2{}_0\sigma^A$ and $\hat{\pi}^A = \Omega^2{}_0\pi^A$ hold for some smooth ${}_0\sigma^A$ and ${}_0\pi^A$ on M.) Thus, both $\Omega^{-2\delta}|\Phi^\alpha|^2$ and $\Omega^{-2\delta}|\hat{\mathcal{D}}_e\Phi^\alpha|^2$ would be integrable on Σ for the *same* $\delta < \frac{1}{2}$, and hence

$$\big(\|\Phi^\alpha\|_{1,\delta}\big)^2 := \int_\Sigma \Omega^{-2\delta}\big(|\Phi^\alpha|^2 + |\hat{\mathcal{D}}_e\Phi^\alpha|^2\big)\mathrm{d}\hat{\Sigma} < \infty. \tag{4.75}$$

However, the norm that (4.75) defines is *not* the weighted Sobolev norm (for the latter see e.g. [33]). This is the classical Sobolev norm[2] with respect to the *weighted volume element* $\Omega^{-2\delta}\mathrm{d}\hat{\Sigma}$. The weighted Sobolev spaces do not appear to be the natural function spaces on the asymptotically hyperboloidal Σ because the fields and their derivatives have the *same* fall-off properties. However, this fall-off rate cannot be arbitrary: The investigations in subsection 4.6.1 show that the spinor fields $\hat{\sigma}^A$ and $\hat{\pi}^A$ *must* fall off *faster* than $\Omega^{\frac{3}{2}}$, i.e. they must be square integrable (with $\delta = 0$), otherwise $H[\alpha, \bar{\alpha}] + H[\beta, \bar{\beta}]$ would not be finite. The spaces $H_{s,\delta}(\Sigma, \mathbb{D}^\alpha)$ (or simply $H_{s,\delta}$) of the Dirac spinor fields for which the norm has the structure (4.75) with the number of derivatives $s = 0, 1, 2, ...$ will be discussed in Appendix 5.1.

Now we show that the Dirac spinor $(\hat{\rho}^A, \bar{\hat{\omega}}^{A'})$ belongs to the weighted Lebesgue spaces $L_2^\delta = H_{0,\delta}$ for $\delta < \frac{1}{2}$. Since ${}_0\hat{\alpha}^A = {}_0\alpha^A$ and ${}_0\bar{\hat{\beta}}^{A'} = {}_0\bar{\beta}^{A'}$ satisfy the algebraic boundary condition (4.45) and since they were constructed from the solution of (4.49) and (4.50), by equation (4.51) (with the notations of subsection 4.5.2), we have that

$$\hat{\mathcal{D}}_{A'A}\,{}_0\hat{\alpha}^A + \frac{3}{2}\,K\,{}_0\bar{\hat{\beta}}_{A'} = \mathcal{D}_{A'A}\,{}_0\alpha^A + \frac{3}{2}\,{}_0\bar{\gamma}_{A'} = \Delta_{A'A}\,{}_0\alpha^A - v_{A'A}v^e\mathcal{D}_e\,{}_0\alpha^A + \frac{3}{2}\,{}_0\bar{\gamma}_{A'} = O(\Omega),$$

$$\begin{aligned} K\hat{\mathcal{D}}_{AA'}\,{}_0\bar{\hat{\beta}}^{A'} + \frac{3}{2}K^2\,{}_0\hat{\alpha}_A &= |D_e\Omega|\,v_B{}^{A'}\big(\mathcal{D}_{A'A}\,{}_0\alpha^B + \frac{1}{2}\,{}_0\bar{\gamma}_{A'}\delta^B_A\big) + O(\Omega) \\ &= |D_e\Omega|\,\Big(v_A{}^{A'}\Delta_{A'B}\,{}_0\alpha^B - \frac{1}{2}v^e(\mathcal{D}_e\,{}_0\alpha_A) - \frac{1}{2}v_{AA'}\,{}_0\bar{\gamma}^{A'}\Big) + O(\Omega) = O(\Omega). \end{aligned}$$

However, this fall-off means, in fact, that $(\hat{\rho}^A, \bar{\hat{\omega}}^{A'}) \in L_2^\delta$ for any $\delta < \frac{1}{2}$. Hence, it seems natural to expect that (4.74) defines a bounded linear operator $\tilde{\mathcal{D}}$ from $H_{1,\delta}$ into L_2^δ with $\delta < \frac{1}{2}$, and we need to show only that $(\hat{\rho}^A, \bar{\hat{\omega}}^{A'}) \in \operatorname{Im}\tilde{\mathcal{D}}$ and that $\ker\tilde{\mathcal{D}} = \emptyset$. The former would imply the existence, the latter the uniqueness of the solution of the renormalized Witten equation (even among the non-smooth solutions). We complete the proof in the Appendix.

In fact, in Appendix 5.3 we show that the extension of $\tilde{\mathcal{D}}$ from the space of the square integrable smooth Dirac spinor fields to the first Sobolev space of the spinor fields, i.e. $\tilde{\mathcal{D}} : H_{1,0} \to L_2$, is a topological vector space *isomorphism* if the hypersurface Σ is chosen such that its 'boost parameter function' W satisfies $\frac{1}{3}\Lambda W < 1$ (Theorem 5.4). Thus, in particular, $\ker\tilde{\mathcal{D}} \subset H_{1,0}$ is empty, i.e. the homogeneous equations (4.68) do not have even non-smooth square integrable solutions with square integrable first derivative. Also, $(\hat{\rho}^A, \bar{\hat{\omega}}^{A'}) \in \operatorname{Im}\tilde{\mathcal{D}}$ holds, and hence (4.40) has a unique square integrable smooth solution. However, since $(\hat{\rho}^A, \bar{\hat{\omega}}^{A'})$ is not only square integrable but belongs to

[2]Strictly speaking, the dimensionally correct norm would be the square root of the integral of $\Omega^{-2\delta}(|\Phi^\alpha|^2 + L^2|\hat{\mathcal{D}}_e\Phi^\alpha|^2)$, where L is a positive constant with *length* physical dimension, e.g. $L = 1/\sqrt{\Lambda}$. Since, however, it is the *topology* of the Banach spaces that the norm defines that has significance (but not the norm itself), we adopt the standard (but physically incorrect) definition of the Sobolev norms. This yields formally incorrect sums of quantities with different physical dimension in certain estimates.

the weighted Lebesgue spaces $L^\delta_2 = H_{0,\delta}$ for $\delta < \frac{1}{2}$, moreover the solution of (4.66)-(4.67) is unique, the solution is not only square integrable, but belongs to $H_{1,\delta}$ for any $\delta < \frac{1}{2}$. Therefore, the renormalized Witten equation (4.40) with the algebraic boundary condition (4.45) has a unique solution on such a Σ, proving that $H[\alpha,\bar\alpha] + H[\beta,\bar\beta]$ is finite and non-negative. Since $H[\alpha,\bar\alpha] + H[\beta,\bar\beta]$ depends only on the cut, its finiteness and non-negativity are independent of the choice of Σ.

Finally, it could be worth noting that the positivity proof can be extended to hypersurfaces with more than one asymptotic end; and also with inner boundaries representing future marginally trapped surfaces, where the spinor fields are subject to the chiral boundary conditions of [27].

4.6.3 Rigidity

It is easy to see that $H[\alpha,\bar\alpha] + H[\beta,\bar\beta]$ is vanishing for any smooth cut of the conformal boundary of the de Sitter spacetime. In fact, we saw in subsection 4.5.3 that in de Sitter spacetime equation (4.56) admits four linearly independent solutions. Then the restriction of the Weyl spinor constituents $\hat\alpha_A$ and $\hat\beta_A$ of such a solution to any smooth spacelike hypersurface Σ extending to the conformal boundary $\mathscr{I}^+$ solve the 3-surface twistor equation on Σ. Therefore, by (4.60), $H[\alpha,\bar\alpha] + H[\beta,\bar\beta]$ is vanishing.

Now we show that the converse of this statement is, in some sense, also true: If $H[\alpha,\bar\alpha] + H[\beta,\bar\beta]$ is zero, then the domain of dependence of the spacelike hypersurface Σ is isometric to an open neighbourhood of a piece of the conformal boundary of the de Sitter spacetime. Thus, the vanishing of $H[\alpha,\bar\alpha] + H[\beta,\bar\beta]$ is equivalent to the local de Sitter nature of the spacetime near its future conformal boundary. The present proof is an adaptation of the proof of an analogous statement in [10], and its logic is essentially the same that we followed in proving the non-existence of smooth solutions of the homogeneous equations (4.68) in the previous subsection. Thus here we only sketch the key points of the proof.

Thus, let us suppose that $H[\alpha,\bar\alpha] + H[\beta,\bar\beta] = 0$ for some solution $(\hat\alpha_A, \bar{\hat\beta}_{A'})$ of (4.40). Then both $\hat\alpha_A$ and $\hat\beta_A$ solve the 3-surface twistor equation, $\hat{\mathcal{D}}_{(AB}\hat\alpha_{C)} = 0$ and $\hat{\mathcal{D}}_{(AB}\hat\beta_{C)} = 0$, too. Hence, by the Witten equations (4.40) and the decomposition (4.59), these satisfy

$$\hat{\mathcal{D}}_e\hat\alpha^A + KP^{AA'}_e\bar{\hat\beta}_{A'} = 0, \qquad \hat{\mathcal{D}}_e\bar{\hat\beta}^{A'} + KP^{AA'}\hat\alpha_A = 0. \tag{4.76}$$

Like in the previous subsection, their solutions $\hat\alpha^A$ and $\hat\beta^A$ cannot be proportional with each other on any open subset of Σ, and hence, by the dominant energy condition, (4.60) gives that $\hat T_{ab} = 0$ on Σ. Also, the integrability conditions of (4.76) yields that the domain of dependence of Σ is locally isometric to the de Sitter spacetime.

4.7 The total energy-momentum

4.7.1 The structure of the 2-surface twistor space

In subsection 4.6.1 we saw that the functional $H[\alpha,\bar\alpha]+H[\beta,\bar\beta]$ of the solutions $(\hat\alpha^A, \bar{\hat\beta}^{A'})$ of the renormalized Witten equation can be finite only if the spinor fields solve the 2-surface twistor equations on $\mathcal{S} = \Sigma \cap \mathscr{I}^+$, i.e.:

$$-\mathcal{T}^+(\alpha) := \eth'\alpha_1 - \bar\sigma^0\alpha_0 = 0, \qquad \mathcal{T}^-(\alpha) := \eth\alpha_0 + \sigma^0\alpha_1 = 0. \tag{4.77}$$

It is known that on topological 2-spheres the 2-surface twistor equations admit at least four, and in the generic case precisely four linearly independent solutions [34]. However, examples are known for topological 2-spheres on which the 2-surface twistor equations admit five independent solutions [35]. We will show that the 2-surface twistor space, i.e. the space $\ker\mathcal{T} := \ker(\mathcal{T}^- \oplus \mathcal{T}^+)$ of the solutions of the 2-surface twistor equations on $\mathcal{S} \subset \mathscr{I}^+$, is even dimensional.

First, let us observe that the *constant* normal N_a of $\mathscr{I}^+$ yields a non-trivial *extra structure* on the kernel of a number of differential operators. Indeed, for any spinor field λ_A on $\mathcal{S}$ let us form the $\mathbb{C}$-anti-linear map

$$\nu : \lambda_A \mapsto \nu(\lambda)_A := \sqrt{2}N_A{}^{A'}\bar{\lambda}_{A'}, \tag{4.78}$$

i.e. in terms of spinor components $\nu : (\lambda_0, \lambda_1) \mapsto (-\bar{\lambda}_{1'}, \bar{\lambda}_{0'})$. Then the algebraic boundary condition (4.45) is simply $\beta_A = \pm \mathrm{i}\nu(\alpha)_A$. Then it is a simple calculation to check that this map yields the $\mathbb{C}$-anti-linear *isomorphisms*

$$\ker\mathcal{T} \to \ker\mathcal{T}, \quad \ker\Delta \to \ker\Delta, \quad \ker\mathcal{H}^+ \to \ker\mathcal{H}^-, \quad \ker\mathcal{C}^+ \to \ker\mathcal{C}^-. \tag{4.79}$$

Here $\Delta := \Delta^+ \oplus \Delta^-$, $\mathcal{H}^\pm := \Delta^\pm \oplus \mathcal{T}^\pm$ and $\mathcal{C}^\pm := \Delta^\pm \oplus \mathcal{T}^\mp$; and where

$$\Delta^+(\lambda) := \eth'\lambda_0 + \rho^0\lambda_1, \qquad -\Delta^-(\lambda) := \eth\lambda_1 - \rho^0\lambda_0. \tag{4.80}$$

Thus, $\ker\Delta$ is the kernel of the Dirac operator built from the 2-dimensional Sen connection $\Delta_a := \Pi^b_a\nabla_b$ on $\mathcal{S}$; $\ker\mathcal{H}^\pm$ is the space of the holomorphic/anti-holomorphic spinor fields of Dougan and Mason [36]; while, with the $\sigma^0 = 0$ substitution, $\ker\mathcal{C}^\pm$ is the space of Bramson's spinors [37] at the future/past null infinity of asymptotically flat spacetimes (where the relevant shears fall off *faster* then the divergences). (For a more detailed discussion of these operators, see the appendix of [38].)

In particular, ν takes solutions of the 2-surface twistor equation into solutions. Clearly, $\nu(\alpha)_A$ is not proportional to α_A (and hence it is linearly independent of α_A) because $\nu(\alpha)_A\alpha^A = -\sqrt{2}N_{AA'}\alpha^A\bar{\alpha}^{A'}$, which is zero only if α_A itself is vanishing. Moreover, the spinor fields $\nu(\alpha)_A$ and α_A are orthogonal to each other with respect to $N^{AA'}$, and $\nu^2 = -\mathrm{Id}$ holds. Hence, each solution α_A has a naturally determined linearly independent counterpart $\nu(\alpha)_A$ and the $\alpha_A \leftrightarrow \nu(\alpha)_A$ correspondence is one-to-one. Therefore, on 2-surfaces in $\mathscr{I}^+$, $\ker\mathcal{T}$ is necessarily even dimensional, i.e. no odd number of 'extra' solutions can exist. We can form the quotient $\ker\mathcal{T}/\nu$, which, since ν is an isomorphism, can be identified with the space of the complex 2-planes $[\alpha_A]$ of $\ker\mathcal{T}$ spanned by α_A and $\nu(\alpha)_A$ for any $\alpha_A \in \ker\mathcal{T}$. This $\ker\mathcal{T}/\nu$ is at least two, and generically is precisely two complex dimensional. We assume that there are no 'extra' solutions of the 2-surface twistor equations on $\mathcal{S}$, and hence that $\ker\mathcal{T}$ is precisely four, and hence that $\ker\mathcal{T}/\nu$ is precisely two dimensional.

Let us define $G(\ker\mathcal{T}, \nu)$ to be the set of automorphisms Φ of $\ker\mathcal{T}$ for which $\nu \circ \Phi = \Phi \circ \nu$. Clearly, this is a subgroup of $GL(\ker\mathcal{T})$, which can be called the symmetry group of the 2-surface twistor space. Let $\mathbf{S_A} \subset \ker\mathcal{T}$ be a two dimensional subspace which is not invariant under ν, and hence for which $\ker\mathcal{T} = \mathbf{S_A} \oplus \nu(\mathbf{S_A})$ holds. (This $\mathbf{S_A}$ is a representative of $\ker\mathcal{T}/\nu$, and obviously it is not canonically defined.) Let us fix a basis in $\mathbf{S_A}$, say $\{\alpha^A_{\mathbf{A}}\}$, $\mathbf{A} = \mathbf{0}, \mathbf{1}$. (Thus, note that the *boldface* name indices refer to a basis in the abstract solution space, while the *underlined* name indices to a frame *field* on $\mathcal{S}$. Note also that since ν contains complex conjugation and it is only the abstract index that is converted by $N^A{}_{A'}$ to an unprimed one, the boldface index in $\nu(\alpha_{\mathbf{A}'})^A$ is, in fact, primed.) Then $\{\alpha^A_{\mathbf{A}}, \nu(\alpha_{\mathbf{A}'})^A\}$ is a basis in $\ker \mathcal{T}$, and in this basis ν takes the form of a 4×4 complex matrix with 2×2 blocks 0, $-\delta^{\mathbf{A}}_{\mathbf{B}}$, and, in the second row, $\delta^{\mathbf{A}'}_{\mathbf{B}'}$ and 0. (Here 0 is the zero matrix.) Thus, since ν is *anti*-linear, its action in this basis is the matrix multiplication with this matrix, following the complex conjugation. Hence, in this basis, $\Phi \in G(\ker \mathcal{T}, \nu)$ is a matrix of the form

$$\Phi = \begin{pmatrix} A^{\mathbf{A}}{}_{\mathbf{B}} & B^{\mathbf{A}}{}_{\mathbf{B}'} \\ -\bar{B}^{\mathbf{A}'}{}_{\mathbf{B}} & \bar{A}^{\mathbf{A}'}{}_{\mathbf{B}'} \end{pmatrix}, \tag{4.81}$$

where A and B are 2×2 complex matrices and A is nonsingular. Clearly, these matrices can be factorized in a unique way according to

$$\Phi = \begin{pmatrix} A^{\mathbf{A}}{}_{\mathbf{B}} & 0 \\ 0 & \bar{A}^{\mathbf{A}'}{}_{\mathbf{B}'} \end{pmatrix} \begin{pmatrix} \delta^{\mathbf{B}}_{\mathbf{C}} & C^{\mathbf{B}}{}_{\mathbf{C}'} \\ -\bar{C}^{\mathbf{B}'}{}_{\mathbf{C}} & \delta^{\mathbf{B}'}_{\mathbf{C}'} \end{pmatrix}; \tag{4.82}$$

and the matrices of the form $\mathrm{diag}(A, \bar{A})$ form a subgroup in $G(\ker \mathcal{T}, \nu)$, which is isomorphic to $GL(1,\mathbb{C}) \times SL(2,\mathbb{C})$. The determinant of these factors is $|\det(A)|^2$ and $1 + \mathrm{Tr}(C\bar{C}) + |\det(C)|^2 \geq 0$, respectively. Hence the latter should be required to be positive. Therefore, the symmetry group $G(\ker \mathcal{T}, \nu)$ factorized by the multiplicative group of the determinants $\det(\Phi)$ is a 15 parameter subgroup of $SL(4,\mathbb{C})$. This subgroup turns out to be isomorphic to $SL(2,\mathbb{H})$, the spin group of $SO(1,5)$.

To see this, first let us determine its Lie algebra. Let us denote the standard $SL(2,\mathbb{C})$ Pauli matrices (divided by $\sqrt{2}$) by $\sigma_{\underline{a}}^{\mathbf{A}\mathbf{B}'}$, $\underline{a} = 0, ..., 3$; and, for $\mathbf{i} = 1, 2, 3$, let $\sigma_{\mathbf{i}}{}^{\mathbf{A}}{}_{\mathbf{B}} := \sqrt{2}\sigma_{\mathbf{i}}^{\mathbf{A}\mathbf{A}'}\sigma^0_{\mathbf{A}'\mathbf{B}}$, which are the standard $SU(2)$ Pauli matrices (also divided by $\sqrt{2}$). Then the basis of the Lie algebra of $G(\ker \mathcal{T}, \nu)$ corresponding to the factorization (4.82) is

$$d := \frac{1}{2}\begin{pmatrix} \delta^{\mathbf{A}}_{\mathbf{B}} & 0 \\ 0 & \delta^{\mathbf{A}'}_{\mathbf{B}'} \end{pmatrix}, \qquad a_0 := \frac{\mathrm{i}}{2}\begin{pmatrix} \delta^{\mathbf{A}}_{\mathbf{B}} & 0 \\ 0 & -\delta^{\mathbf{A}'}_{\mathbf{B}'} \end{pmatrix},$$
$$a_{\mathbf{i}} := \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma_{\mathbf{i}}{}^{\mathbf{A}}{}_{\mathbf{B}} & 0 \\ 0 & \sigma_{\mathbf{i}}{}^{\mathbf{A}'}{}_{\mathbf{B}'} \end{pmatrix}, \qquad \tilde{a}_{\mathbf{i}} := \frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} \sigma_{\mathbf{i}}{}^{\mathbf{A}}{}_{\mathbf{B}} & 0 \\ 0 & -\sigma_{\mathbf{i}}{}^{\mathbf{A}'}{}_{\mathbf{B}'} \end{pmatrix},$$
$$c_{\underline{a}} := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_{\underline{a}}^{\mathbf{A}}{}_{\mathbf{B}'} \\ -\sigma_{\underline{a}}^{\mathbf{A}'}{}_{\mathbf{B}} & 0 \end{pmatrix}, \qquad \tilde{c}_{\underline{a}} := \frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_{\underline{a}}^{\mathbf{A}}{}_{\mathbf{B}'} \\ \sigma_{\underline{a}}^{\mathbf{A}'}{}_{\mathbf{B}} & 0 \end{pmatrix}.$$

Clearly, d commutes with all the basis elements and generates the change of the determinant of Φ. The Lie algebra elements a_0, $a_{\mathbf{i}}$, $\tilde{a}_{\mathbf{i}}$, $c_{\underline{a}}$ and $\tilde{c}_{\underline{a}}$ form a 15 real dimensional Lie algebra with the Lie products

$$\begin{aligned}
&[a_0, a_{\mathbf{i}}] = [a_0, \tilde{a}_{\mathbf{i}}] = 0, \quad [a_0, c_{\underline{a}}] = \tilde{c}_{\underline{a}}, \quad [a_0, \tilde{c}_{\underline{a}}] = -c_{\underline{a}}, \\
&[a_{\mathbf{i}}, a_{\mathbf{j}}] = \epsilon_{\mathbf{ij}}{}^{\mathbf{k}}\tilde{a}_{\mathbf{k}}, \quad [a_{\mathbf{i}}, \tilde{a}_{\mathbf{j}}] = -\epsilon_{\mathbf{ij}}{}^{\mathbf{k}}a_{\mathbf{k}}, \\
&[a_{\mathbf{i}}, c_0] = c_{\mathbf{i}}, \quad [a_{\mathbf{i}}, c_{\mathbf{j}}] = -\eta_{\mathbf{ij}}c_0, \quad [a_{\mathbf{i}}, \tilde{c}_0] = \tilde{c}_{\mathbf{i}}, \quad [a_{\mathbf{i}}, \tilde{c}_{\mathbf{j}}] = -\eta_{\mathbf{ij}}\tilde{c}_0, \\
&[\tilde{a}_{\mathbf{i}}, \tilde{a}_{\mathbf{j}}] = -\epsilon_{\mathbf{ij}}{}^{\mathbf{k}}\tilde{a}_{\mathbf{k}}, \quad [\tilde{a}_{\mathbf{i}}, c_0] = [\tilde{a}_{\mathbf{i}}, \tilde{c}_0] = 0, \quad [\tilde{a}_{\mathbf{i}}, c_{\mathbf{j}}] = -\epsilon_{\mathbf{ij}}{}^{\mathbf{k}}c_{\mathbf{k}}, \quad [\tilde{a}_{\mathbf{i}}, \tilde{c}_{\mathbf{j}}] = -\epsilon_{\mathbf{ij}}{}^{\mathbf{k}}\tilde{c}_{\mathbf{k}}, \\
&[c_0, c_{\mathbf{i}}] = -a_{\mathbf{i}}, \quad [c_0, \tilde{c}_0] = -a_0, \quad [c_0, \tilde{c}_{\mathbf{i}}] = 0, \\
&[c_{\mathbf{i}}, c_{\mathbf{j}}] = -\epsilon_{\mathbf{ij}}{}^{\mathbf{k}}\tilde{a}_{\mathbf{k}}, \quad [c_{\mathbf{i}}, \tilde{c}_0] = 0, \quad [c_{\mathbf{i}}, \tilde{c}_{\mathbf{j}}] = -\eta_{\mathbf{ij}}a_0, \\
&[\tilde{c}_0, \tilde{c}_{\mathbf{i}}] = -a_{\mathbf{i}}, \quad [\tilde{c}_{\mathbf{i}}, \tilde{c}_{\mathbf{j}}] = -\epsilon_{\mathbf{ij}}{}^{\mathbf{k}}\tilde{a}_{\mathbf{k}}.
\end{aligned}$$

(Note that we lower the small boldface indices by the *negative definite* $\eta_{\mathbf{ij}} = -\delta_{\mathbf{ij}}$, the capital boldface indices by the anti-symmetric Levi-Civita symbol $\epsilon_{\mathbf{AB}}$; and we raise them by their inverses.)

To see the structure of this Lie algebra, consider its non-trivial subalgebras. First, $\{\tilde{a}_{\bf i}, a_{\bf i}\}$ spans the Lorentz Lie algebra $so(1,3) \simeq sl(2,\mathbb{C})$, and both $\{\tilde{a}_{\bf i}, c_{\bf i}\}$ and $\{\tilde{a}_{\bf i}, \tilde{c}_{\bf i}\}$ (which are isomorphic with each other) span $so(4) \simeq su(2) \oplus su(2)$. The subalgebras spanned by $\{\tilde{a}_{\bf i}, a_{\bf i}, c_{\bf i}, c_0\}$ and by $\{\tilde{a}_{\bf i}, a_{\bf i}, \tilde{c}_{\bf i}, \tilde{c}_0\}$ are also isomorphic, and since they contain $so(1,3)$ and $so(4)$, it is natural to expect them to be just the de Sitter algebra $so(1,4) \simeq sp(1,1)$. Similarly, $\{\tilde{a}_{\bf i}, c_{\bf i}, \tilde{c}_{\bf i}, a_0\}$ contains two (overlapping) copies of $so(4)$, hence this can be expected to be just $so(5) \simeq sp(2)$. It is straightforward (e.g. by explicit calculations) to show that these are, in fact, isomorphic to the $so(1,4)$ and $so(5)$ Lie algebras, respectively. In addition, $\{a_0, c_0, \tilde{c}_0\}$ spans $so(1,2) \simeq sl(2,\mathbb{R})$. Therefore, the Lie algebra of the symmetry group $G(\ker\mathcal{T},\nu)$ should be $so(1,5) \oplus \mathbb{R}$.

Since $so(1,5)$ is isomorphic to $sl(2,\mathbb{H})$, the Lie algebra of the special linear group in two dimensions over the quaternions [39], $G(\ker\mathcal{T},\nu)$ factorized by the determinants is locally isomorphic to $SL(2,\mathbb{H})$. In fact, the explicit form (4.81) of the symmetry group $G(\ker\mathcal{T},\nu)$ is just the complex realization of $GL(2,\mathbb{H})$ [40], i.e. the quotient of $G(\ker\mathcal{T},\nu)$ and the multiplicative group $(0,\infty)$ of the determinants $\det(\Phi)$ is *precisely* $SL(2,\mathbb{H})$ (which in its actual complex form is also denoted by $SU^*(4)$ [40]), the spin group of $SO(1,5)$.

4.7.2 The general form of the total energy-momentum

Since the solution $(\hat{\alpha}^A, \bar{\hat{\beta}}^{A'})$ of (4.40) is completely controlled e.g. by the boundary value of α^A on the cut of the conformal boundary, the functional H yields a well defined positive definite quadratic form on the space $\ker\mathcal{T}$ by

$$H^*: \ker\mathcal{T} \to [0,\infty) : \alpha^A \mapsto \frac{1}{2}\Bigl(H\bigl[\alpha,\bar{\alpha}\bigr] + H\bigl[\nu(\alpha),\overline{\nu(\alpha)}\bigr]\Bigr). \tag{4.83}$$

Then the polarization formula (4.3) (applied to H^*) makes it possible to extend H^* to be a positive Hermitian bilinear form on $\ker\mathcal{T}$. This gives, for any $\lambda^A, \mu^A \in \ker\mathcal{T}$, that

$$H^*\bigl[\lambda,\bar{\mu}\bigr] = \frac{1}{2}\Bigl(H\bigl[\lambda,\bar{\mu}\bigr] + H\bigl[\nu(\mu),\overline{\nu(\lambda)}\bigr]\Bigr), \tag{4.84}$$

by means of which it is easy to see that

$$H^*\bigl[\lambda,\overline{\nu(\mu)}\bigr] = -H^*\bigl[\mu,\overline{\nu(\lambda)}\bigr], \qquad H^*\bigl[\nu(\lambda),\overline{\nu(\mu)}\bigr] = \overline{H^*\bigl[\lambda,\bar{\mu}\bigr]} \tag{4.85}$$

hold. In particular, by the first of these $H^*[\lambda,\overline{\nu(\lambda)}] = 0$ for any $\lambda^A \in \ker\mathcal{T}$.

Let us fix a basis $\{\alpha^A_{\bf A}, \nu(\alpha_{\bf A'})^A\}$ in $\ker\mathcal{T}$ (see subsection 4.7.1). Then by (4.85) H^* in this basis is a 4×4 complex matrix

$$H^* = \begin{pmatrix} \mathtt{P} & \mathtt{Q} \\ -\bar{\mathtt{Q}} & \bar{\mathtt{P}} \end{pmatrix}, \tag{4.86}$$

where $\mathtt{P} := H^*[\alpha_{\bf A}, \bar{\alpha}_{\bf B'}]$ (or, rather $\mathtt{P}_{\bf AB'}$) is a 2×2 Hermitian, while $\mathtt{Q} := H^*[\alpha_{\bf A}, \overline{\nu(\alpha_{\bf B'})}] = H^*[\alpha_{\bf A}, \bar{\nu}(\alpha_{\bf B})]$ is a 2×2 complex anti-symmetric matrix. By the positivity and rigidity results of subsection 4.6 the matrix $\mathtt{P}$ is positive, and one of its diagonal elements (i.e. the product $\mathtt{P}_{\bf 00'}\mathtt{P}_{\bf 11'}$) is vanishing if and only if the domain of dependence of the hypersurface Σ is locally isometric to the de Sitter spacetime, in which case the whole H^* is vanishing. By the anti-symmetry of $\mathtt{Q}$ it can also be written as $Q\epsilon_{\bf AB}$, where $Q := H^*[\alpha_{\bf 0}, \bar{\nu}(\alpha_{\bf 1})] \in \mathbb{C}$.

Since the choice for $\mathbf{S_A}$ is *not* canonical, neither the basis nor $\mathtt{P}_{\mathbf{AB'}}$ is canonically defined. Under the action of a basis transformation with $\Phi \in G(\ker \mathcal{T}, \nu)$ given by (4.81), the components $\mathtt{P}_{\mathbf{AB'}}$ and Q change as

$$\begin{aligned}
&\mathtt{P}_{\mathbf{AA'}} \mapsto \mathtt{P}_{\mathbf{BB'}}\big(A^{\mathbf{B}}{}_{\mathbf{A}}\bar{A}^{\mathbf{B'}}{}_{\mathbf{A'}} + B^{\mathbf{B}}{}_{\mathbf{A'}}\bar{B}^{\mathbf{B'}}{}_{\mathbf{A}}\big) - Q\,\epsilon_{\mathbf{CD}}A^{\mathbf{C}}{}_{\mathbf{A}}B^{\mathbf{D}}{}_{\mathbf{A'}} - \bar{Q}\,\epsilon_{\mathbf{C'D'}}\bar{A}^{\mathbf{C'}}{}_{\mathbf{A'}}\bar{B}^{\mathbf{D'}}{}_{\mathbf{A}},\\
&Q \mapsto Q\,\det(A) + \bar{Q}\,\det(B) + \mathtt{P}_{\mathbf{AA'}}A^{\mathbf{A}}{}_{\mathbf{C}}\bar{B}^{\mathbf{A'}}{}_{\mathbf{D}}\epsilon^{\mathbf{CD}}.
\end{aligned}$$

Hence, in the lack of any *further* extra structure on $\ker \mathcal{T}$, it does not seem to be able to extract $\mathtt{P}_{\mathbf{AB'}}$ as the energy-momentum 4-vector in a *canonical* way from H^*, even though it shares the positivity and rigidity properties. Such an extra structure would be needed to reduce $G(\ker \mathcal{T}, \nu)$ to $GL(1,\mathbb{C}) \times SL(2,\mathbb{C}) = \{\mathrm{diag}(A,\bar{A})\}$. In this case we would have a well defined energy-momentum $\mathtt{P}_{\mathbf{AB'}}$. However, even if we had such a reduction, still we would not have any natural symplectic metric to define mass as the length of this energy-momentum. Still we would have to rule out the factor $GL(1,\mathbb{C})$.

A simple calculation shows that

$$\det(H^*) = \big(\det(\mathtt{P}_{\mathbf{AB'}}) - Q\bar{Q}\big)^2, \tag{4.87}$$

and hence it might be tempting to introduce the concept of mass by $\det(H^*)$, even if $\mathtt{P}_{\mathbf{AB'}}$ cannot be defined in a canonical way. Since H^* is positive definite, by the rigidity property $\det(H^*) = 0$ (i.e. $\det(\mathtt{P}_{\mathbf{AB'}}) = Q\bar{Q}$) is equivalent to the vanishing of H^*, i.e. to the local de Sitter nature of the domain of dependence of the spacelike hypersurface Σ in the spacetime. Unfortunately, however, $\det(\Phi)$ is not one, and hence this determinant is still *not* an invariant, it is only its sign (positive or zero) that is invariant. We would need an extra structure on the 2-surface twistor space, e.g. a geometrically defined *volume 4-form*, by means of which the symmetry group could be reduced to $SL(2,\mathbb{H})$.

Nevertheless, still H^* is a well defined observable with useful properties, but in general asymptotically de Sitter spacetimes this cannot be *interpreted* as energy-momentum in a natural way. In the next subsection we consider special cases when additional extra structures are present on the 2-surface twistor space by means of which the analog of the Bondi mass can be defined by (4.87).

4.7.3 Further extra structures on $\ker \mathcal{T}$ on non-contorted cuts

The 2-surface twistor space can also be considered as the space of the pairs $\mathtt{Z}^\alpha := (\lambda^A, \mathrm{i}\Delta_{A'B}\lambda^B)$, $\lambda^A \in \ker \mathcal{T}$; and $\pi_{A'} := \mathrm{i}\Delta_{A'B}\lambda^B$ is called the secondary part of the 2-surface twistor $\mathtt{Z}^\alpha$ (see [12]). For any four twistors $\mathtt{Z}^\alpha_i = (\lambda^A_i, \pi^i_{A'})$, $i = 1, ..., 4$, one can define

$$\varepsilon := \frac{1}{4}\varepsilon_{\alpha\beta\gamma\delta}\mathtt{Z}^\alpha_1\mathtt{Z}^\beta_2\mathtt{Z}^\gamma_3\mathtt{Z}^\delta_4 := \epsilon^{ij}{}_{kl}\lambda^0_i\lambda^1_j\pi^k_{0'}\pi^l_{1'} = \epsilon^{ijkl}\lambda^0_i\lambda^1_j\big(\eth\lambda^0_k\big)\big(\eth'\lambda^1_l\big),$$

where ϵ^{ijkl} is the Levi-Civita alternating symbol, and we used the GHP form of the secondary part of the twistors. In general, this ε is a complex valued *function* on the cut. If, however, ε were *constant* on the cut, then this $\varepsilon_{\alpha\beta\gamma\delta}$ would define a volume 4-form, a further *extra structure*, on the 2-surface twistor space $\ker \mathcal{T}$. The presence of this volume form would reduce the symmetry group to its volume-preserving subgroup, i.e. to $SL(2,\mathbb{H})$. Then we could define the mass in an invariant way by (4.87), where the basis $\{\alpha^A_{\mathbf{A}}, \nu(\alpha_{\mathbf{A'}})^A\}$ would be chosen such that the components of $\varepsilon_{\alpha\beta\gamma\delta}$ are those of the

Levi-Civita symbol. It is still not known what are the necessary and sufficient conditions on the geometrical properties of the cut that could ensure the existence of such a volume 4-form.

If a Hermitian metric exists on $\ker\mathcal{T}$, then it defines a geometrically given volume 4-form (see [12]). To introduce this, for any two twistors $\mathtt{Z}^\alpha=(\lambda^A,\pi_{A'})$ and $\mathtt{W}^\alpha=(\mu^A,\rho_{A'})$ let us define

$$\mathtt{h}_{\alpha\beta'}\mathtt{Z}^\alpha\bar{\mathtt{W}}^{\beta'}:=\lambda^A\bar\rho_A+\pi_{A'}\bar\mu^{A'}=:h(\lambda,\bar\mu).$$

In general, this is *not* constant on the cut, but when it is, then it defines a (conformally invariant) Hermitian metric on $\ker\mathcal{T}$ with signature $(+,+,-,-)$. By definition, the group of the linear transformations of $\ker\mathcal{T}$ preserving this metric is $SU(2,2)$, the spin group of $SO(2,4)$. Hence, in the presence of such a Hermitian metric, the symmetry group of the 2-surface twistor space is $SL(2,\mathbb{H})\cap SU(2,2)\simeq SP(1,1)$, just the spin group of the de Sitter group $SO(1,4)=SO(1,5)\cap SO(2,4)$. Thus, the de Sitter group (in its spinor representation) emerges as the (reduced) symmetry group of the 2-surface twistor space. The 2-surfaces for which such a Hermitian scalar product $\mathtt{h}_{\alpha\alpha'}$ exists are called *non-contorted*, and these are known to be just the 2-surfaces which can be embedded, at least locally, into a conform Minkowski spacetime with their first and second fundamental forms [41, 42, 43]. For example, the cuts of an *intrinsically locally conformally flat* conformal boundary are all non-contorted (see also [44]). Thus, we have a large class of radiative spacetimes in which the cuts are non-contorted.

Thus, suppose that the cut $\mathcal{S}$ is non-contorted and hence the Hermitian metric $\mathtt{h}_{\alpha\alpha'}$ exists on $\ker\mathcal{T}$. Then a straightforward calculation yields that

$$h(\nu(\lambda),\overline{\nu(\mu)})=\overline{h(\lambda,\bar\mu)},\qquad h(\lambda,\overline{\nu(\mu)})=-h(\mu,\overline{\nu(\lambda)});\tag{4.88}$$

i.e. the Hermitian metric is anti-invariant under the action of ν. (It might be worth noting that these equations already ensure the existence of the infinity twistor on $\ker\mathcal{T}$ [22].) Thus, in particular, $h(\lambda,\overline{\nu(\lambda)})=0$ for any λ^A; i.e. λ^A and $\nu(\lambda)^A$ are orthogonal to each other with respect to h, too. The properties (4.88) make it possible to choose the basis $\{\alpha^A_{\mathbf{A}},\nu(\alpha_{\mathbf{A}'})^A\}$ in $\ker\mathcal{T}$ in a more specific way. Namely, let us choose $\alpha^A_{\mathbf{0}}$ such that $h(\alpha_{\mathbf{0}},\bar\alpha_{\mathbf{0}'})=1$. Then by (4.88) $\nu(\alpha_{\mathbf{0}'})^A$ is h-orthogonal to $\alpha^A_{\mathbf{0}}$; and its norm is also 1. Thus the spinors in the 2-plane $[\alpha^A_{\mathbf{0}}]$ have positive norm. Then let us choose $\alpha^A_{\mathbf{1}}$ to be h-orthogonal to the 2-plane $[\alpha^A_{\mathbf{0}}]$. Because of the signature of $\mathtt{h}_{\alpha\alpha'}$ its norm is negative, and we choose it to be -1. Then by (4.88) the norm of $\nu(\alpha_{\mathbf{1}'})^A$ is -1, and it is h-orthogonal not only to $\alpha^A_{\mathbf{1}}$, but to the whole 2-plane $[\alpha^A_{\mathbf{0}}]$, too. Hence, the resulting basis $\{\alpha^A_{\mathbf{A}},\nu(\alpha_{\mathbf{A}'})^A\}$ is h–orthonormal. Thus, in this basis, h and its contravariant form h^{-1} have the form $\mathrm{diag}(1,-1,1,-1)$. Hence, the invariants of H^* are given by the trace of the first four powers of $h^{-1}H^*$. These are $2(\mathtt{P}_{\mathbf{00}'}-\mathtt{P}_{\mathbf{11}'})$, $4(\det(\mathtt{P}_{\mathbf{AB}'})-Q\bar Q)+2(\mathtt{P}_{\mathbf{00}'}-\mathtt{P}_{\mathbf{11}'})^2$, $6(\mathtt{P}_{\mathbf{00}'}-\mathtt{P}_{\mathbf{11}'})(\det(\mathtt{P}_{\mathbf{AB}'})-Q\bar Q)+2(\mathtt{P}_{\mathbf{00}'}-\mathtt{P}_{\mathbf{11}'})^3$ and $2(2(\det(\mathtt{P}_{\mathbf{AB}'})-Q\bar Q)+(\mathtt{P}_{\mathbf{00}'}-\mathtt{P}_{\mathbf{11}'})^2)^2$, respectively. Therefore, H^* has two independent invariants, both of them real:

$$\mathtt{M}^2:=\det\bigl(\mathtt{P}_{\mathbf{AB}'}\bigr)-Q\bar Q,\qquad \mathtt{N}:=\mathtt{P}_{\mathbf{00}'}-\mathtt{P}_{\mathbf{11}'}.\tag{4.89}$$

Clearly, $\mathtt{M}$ is analogous to the Bondi mass of asymptotically flat spacetimes, but the meaning of the other invariant, the difference $\mathtt{N}$ of the value of the functional H^* on the positive and negative norm elements of the subspace $\mathbf{S_A}$, is still unclear. Further investigation of these quantities, viz. their group theoretical properties, their alternative expressions, the analog of 'mass-loss', etc. will be given in a separate paper.

5 Appendix: The Sen–Witten operator in weighted function spaces

The aim of this appendix is to introduce and develop the necessary functional analytic tools by means of which we can prove the existence and uniqueness of the solution of the renormalized Witten equation (4.66)-(4.67) on *asymptotically hyperboloidal* hypersurfaces in a rigorous way. The key ideas and statements are motivated by those of [33] developed for *asymptotically flat* Riemannian manifolds. First we introduce the appropriate weighted function spaces and state two of their properties. Then we prove a number of estimates for the renormalized Sen–Witten operator $\tilde{\mathcal{D}}$, by means of which finally we prove that $\tilde{\mathcal{D}}$ is an *isomorphism*. In this appendix all the quantities and objects are in the *physical* spacetime, but, for the sake of simplicity, we leave the 'hats' off of them.

5.1 The weighted function spaces on asymptotically hyperboloidal hypersurfaces

Following the general ideas of [33] we define the weighted Lebesgue spaces of pairs of Weyl spinor fields, i.e. of Dirac spinor fields, on the asymptotically hyperboloidal hypersurfaces discussed in subsection 4.4. (The following concepts can be generalized in a natural way to cross-sections of Hermitian vector bundles over asymptotically hyperboloidal n-manifolds, even with more than one asymptotic end.) Thus, we assume that all the geometric structures of those hypersurfaces Σ are present and are smooth. In particular, h_{ab} is an asymptotically hyperboloidal (negative definite) metric and χ_{ab} is the extrinsic curvature with the asymptotic form (4.38) and (4.39), respectively. We also consider the future pointing unit timelike normal (and hence the positive definite Hermitian metric on the spinor spaces), the conformal factor Ω (or radial coordinate $1/\Omega$) and the foliation $\mathcal{S}_\Omega$ etc. to be given. Clearly, the conformal factor can be assumed to be one on some 'large enough' compact subset $K \subset \Sigma$ with smooth boundary $\partial K \approx S^2$ and strictly monotonically decreasing on $\Sigma - K$. $\mathcal{D}_e$ will denote the covariant derivative operator acting on the Dirac spinor fields, determined by the Sen connection.

For $\delta \in \mathbb{R}$ and a measurable Dirac spinor field $\Phi^\alpha = (\sigma^A, \bar{\pi}^{A'})$ on Σ we define

$$\big(\|\Phi^\alpha\|_\delta\big)^2 := \int_\Sigma \Omega^{-2\delta}\sqrt{2}t_{AA'}\big(\sigma^A\bar{\sigma}^{A'} + \pi^A\bar{\pi}^{A'}\big)\mathrm{d}\Sigma \tag{5.1}$$

and let $L_2^\delta(\Sigma, \mathbb{D}^\alpha)$, or shortly L_2^δ, denote the space of the spinor fields Φ^α for which $\|\Phi^\alpha\|_\delta < \infty$. This space is a Banach space with the norm $\|\,.\,\|_\delta$, which is, in fact, a Hilbert space with the obvious Hermitian scalar product: $\langle\Phi^\alpha, \Psi^\alpha\rangle := \int_\Sigma \Omega^{-2\delta}\sqrt{2}t_{AA'}(\sigma^A\bar{\alpha}^{A'} + \bar{\pi}^{A'}\beta^A)\mathrm{d}\Sigma$ for any $\Phi^\alpha = (\sigma^A, \bar{\pi}^{A'})$ and $\Psi^\alpha = (\alpha^A, \bar{\beta}^{A'})$.

With the convention $\|\Phi^\alpha\|_{0,\delta} := \|\Phi^\alpha\|_\delta$, for any $s = 0, 1, 2, \ldots$ let us define[3]

$$\big(\|\Phi^\alpha\|_{s,\delta}\big)^2 := \sum_{k=0}^{s}\int_\Sigma \Omega^{-2\delta}|\mathcal{D}_{e_k}\ldots\mathcal{D}_{e_1}\Phi^\alpha|^2\mathrm{d}\Sigma \tag{5.2}$$

for any measurable spinor field Φ^α with measurable $\mathcal{D}_e$–derivatives (in the weak sense) up to order s, where

[3] See the footnote to equation (4.75) in subsection 4.6.2.

$$|\mathcal{D}_{e_k}...\mathcal{D}_{e_1}\Phi^\alpha|^2 := \sqrt{2}t_{AA'}(-h^{e_1f_1})...(-h^{e_kf_k})\Big((\mathcal{D}_{e_k}...\mathcal{D}_{e_1}\sigma^A)(\mathcal{D}_{f_k}...\mathcal{D}_{f_1}\bar\sigma^{A'}) + \\ + (\mathcal{D}_{e_k}...\mathcal{D}_{e_1}\pi^A)(\mathcal{D}_{f_k}...\mathcal{D}_{f_1}\bar\pi^{A'})\Big) \qquad (5.3)$$

is the positive definite pointwise norm of the kth derivative. Then the space $H_{s,\delta}(\Sigma,\mathbb{D}^\alpha)$, or shortly $H_{s,\delta}$, is defined to be the space of those spinor fields Φ^α for which $\|\Phi^\alpha\|_{s,\delta} < \infty$. This is a Hilbert space with the obvious scalar product. Note that the spaces $H_{s,\delta}$ are *not* the familiar weighted Sobolev spaces (see e.g. [33]), rather these are the classical Sobolev spaces with the overall weighted volume element. The concept of this kind of spaces is motivated by the observation that the fall-off rate of both the solution of the renormalized Witten equation and its derivatives are the same (see subsection 4.6.2). By definition $H_{0,\delta} = L^\delta_2$, and $H_{s,0}$ is just the classical Sobolev space $H_s(\Sigma,\mathbb{D}^\alpha)$.

We also need to define the space $C^s_\delta(\Sigma,\mathbb{D}^\alpha)$ (or simply C^s_δ) of the C^s Dirac spinor fields Φ^α on Σ for which[4]

$$\|\Phi^\alpha\|_{C^s_\delta} := \sup\Big\{\sum_{k=0}^s \Omega^{-\delta}|\mathcal{D}_{e_k}...\mathcal{D}_{e_1}\Phi^\alpha|(p)\,|\,p\in\Sigma\Big\} < \infty. \qquad (5.4)$$

C^s_δ is a Banach space with the norm $\|\,.\,\|_{C^s_\delta}$. The concept of these spaces is motivated by the weighted function spaces of C^s tensor fields of [33], but note that the norms are different and hence the spaces C^s_δ are different from those of [33]. Here the weight functions in front of the different order terms are the *same*. The significance of these spaces is that they give a control on the fall-off properties of the spinor fields. In particular, Φ^α is continuous and $\Phi^\alpha = O(\Omega^k)$ (in the sense that $\Omega^{-k}\Phi^\alpha$ can be extended to the conformally compactified Σ as a continuous spinor field) precisely when $\Phi^\alpha \in C^0_\delta$ with $\delta = k - \frac{1}{2}$.

It follows immediately from the definitions that $H_{s,\delta} \subset H_{s',\delta'}$ if $\delta' \le \delta$ and $s' \le s$. The next lemma is analogous to the Rellich lemma for the classical Sobolev spaces over compact domains:

Lemma 5.1. *If $\delta' < \delta$ and $s' < s$, then the injection $i : H_{s,\delta} \to H_{s',\delta'}$ is compact.*

Proof. The statement is the adaptation of Lemma 2.1 of [33] to the actual spaces, and the proof is similar to that. □

Also, as a simple consequence of the definitions, $C^{s'}_{\delta'} \subset H_{s,\delta}$ holds if $s' \ge s$ and $\delta' > \delta+1$. (For tensor fields on n dimensional Σ the latter condition would be $\delta' > \delta+(n-1)/2$, while on asymptotically flat Σ it is known to be $\delta' > \delta + n/2$.) In particular, the spinor fields with $o(\Omega^{\frac{3}{2}})$ fall-off, i.e. the elements of $C^0_{\delta'}$ with $\delta' > 1$, are square integrable, while those with $O(\Omega^2)$ fall-off belong to L^δ_2 with $\delta < \frac{1}{2}$. The next lemma states that, for appropriate indices, the inclusion holds in the opposite direction, too. This statement is analogous to the classical Sobolev lemma, and is the adaptation of Lemma 2.4 of [33] to the present case:

Lemma 5.2. *If $s \ge s' + 2$ and $\delta > \delta' - 1$, then $H_{s,\delta} \subset C^{s'}_{\delta'}$.*

Proof. The proof can be based on the proof of Lemma 2.3 and Lemma 2.4 of [33], and on the classical Sobolev lemma $H_s \subset C^{s'}$, $s > s' + n/2$, for compact domains and its extension from the n dimensional asymptotically Euclidean to asymptotically hyperboloidal

[4]See the footnote to equation (4.75) in subsection 4.6.2.

geometries. The only deviation from the asymptotically flat case is that in Lemma 2.3 of [33] the map $T_\varepsilon : H_{s,\delta} \to H_{s,\delta'}$ on n dimensional asymptotically hyperboloidal Σ is a topological vector space isomorphism for $\delta > \delta' - (n-1)/2$, rather than for $\delta > \delta' - n/2$. $\square$

5.2 The basic estimates for $\tilde{\mathcal{D}}$

Since the Sen–Witten operator, given explicitly by $\mathcal{D}^\alpha{}_\beta \Phi^\beta = (\mathcal{D}^A{}_{B'}\bar\pi^{B'}, \mathcal{D}^{A'}{}_B\sigma^B)$, is elliptic, $\tilde{\mathcal{D}}$, given explicitly by $\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta := \mathcal{D}^\alpha{}_\beta\Phi^\beta + \frac{3}{2}K\Phi^\alpha$, is also elliptic. If $\Phi^\alpha \in H_{1,\delta}$ (with arbitrary $\delta \in \mathbb{R}$), then

$$\begin{aligned}\Big(\|\mathcal{D}^\alpha{}_\beta\Phi^\beta\|_{0,\delta}\Big)^2 &= \int_\Sigma \Omega^{-2\delta}\Big(|\mathcal{D}_{AA'}\bar\pi^{A'}|^2 + |\mathcal{D}_{A'A}\sigma^A|^2\Big)\mathrm{d}\Sigma \\ &\leq \int_\Sigma \Omega^{-2\delta}\Big(|\mathcal{D}_{AA'}\bar\pi^{A'}|^2 + \frac{3}{2}|\mathcal{D}_{(A'B'}\bar\pi_{C')}|^2 + |\mathcal{D}_{A'A}\sigma^A|^2 + \frac{3}{2}|\mathcal{D}_{(AB}\sigma_{C)}|^2\Big)\mathrm{d}\Sigma \\ &= \frac{3}{2}\int_\Sigma \Omega^{-2\delta}\Big(|\mathcal{D}_e\bar\pi^{A'}|^2 + |\mathcal{D}_e\sigma^A|^2\Big)\mathrm{d}\Sigma \leq \frac{3}{2}\Big(\|\Phi^\alpha\|_{1,\delta}\Big)^2\end{aligned}$$

holds, i.e. $\mathcal{D} : H_{1,\delta} \to L_2^\delta$ is a *bounded* linear operator. (Here we used the orthogonal decomposition (4.59) of the $\mathcal{D}_e$-derivative of the spinor fields.) Then

$$\begin{aligned}\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta\|_{0,\delta} &= \|\mathcal{D}^\alpha{}_\beta\Phi^\beta + \frac{3}{2}K\Phi^\alpha\|_{0,\delta} \leq \|\mathcal{D}^\alpha{}_\beta\Phi^\beta\|_{0,\delta} + \frac{3}{2}\sqrt{\frac{\Lambda}{6}}\|\Phi^\alpha\|_{0,\delta} \\ &\leq \sqrt{\frac{3}{2}}\|\Phi^\alpha\|_{1,\delta} + \frac{3}{2}\sqrt{\frac{\Lambda}{6}}\|\Phi^\alpha\|_{0,\delta} \leq \sqrt{\frac{3}{2}}\Big(1 + \sqrt{\frac{\Lambda}{4}}\Big)\|\Phi^\alpha\|_{1,\delta};\end{aligned}$$

i.e. the renormalized Sen–Witten operator

$$\tilde{\mathcal{D}} : H_{1,\delta} \to L_2^\delta \tag{5.5}$$

is also bounded, and hence *continuous*, for any $\delta \in \mathbb{R}$.

Next we prove a number of lemmas that we need in the proof of the isomorphism theorem for $\tilde{\mathcal{D}}$ in the next subsection. The first of these is the so-called fundamental elliptic estimate:

Lemma 5.3. *Let the dominant energy condition hold on Σ and let $B \subset \Sigma$ be an open set with compact closure and smooth boundary. Then*

$$\|\Phi_A^\alpha\|_{1,0} \leq \sqrt{2}\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi_A^\beta\|_{0,0} + \sqrt{1 + \frac{3}{4}\Lambda}\|\Phi_A^\alpha\|_{0,0} \tag{5.6}$$

for any $\Phi_A^\alpha \in H_{1,0}$, $\mathrm{supp}\,(\Phi_A^\alpha) \subset \Sigma - B$*. Also,*

$$\|\Phi_B^\alpha\|_{H_1(B)} \leq \sqrt{2}\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi_B^\beta\|_{L_2(B)} + \sqrt{1 + \frac{3}{4}\Lambda}\|\Phi_B^\alpha\|_{L_2(B)} \tag{5.7}$$

for any $\Phi_B^\alpha \in H_{1,0}$, $\mathrm{supp}\,(\Phi_B^\alpha) \subset B$.

Scholium: The index A of Φ_A^α and B of Φ_B^α are not spinor indices. The former indicates that the spinor field is localized in the asymptotic region on Σ, while the latter is localized in a bounded domain in Σ. (See also the first paragraph of the proof of Lemma 5.6 below.) $H_s(B)$, $s = 0, 1, 2, ...$, denotes the classical Sobolev spaces on the domain B with *compact* closure.

Proof. For any smooth spinor field σ^A on Σ we have the Sen–Witten identity

$$|\mathcal{D}_e\sigma^A|^2 = 2|\mathcal{D}_{A'A}\sigma^A|^2 - \frac{1}{\sqrt{2}}t_a\big(\varkappa T^a{}_b + \Lambda\delta^a_b\big)\sigma^B\bar\sigma^{B'} - \sqrt{2}D_{AA'}\Big(t^{AB'}\bar\sigma^{A'}\mathcal{D}_{B'B}\sigma^B - t^{BA'}\bar\sigma^{B'}\mathcal{D}_{BB'}\sigma^A\Big). \tag{5.8}$$

If σ^A is square integrable, then the integral of the total divergence at infinity is vanishing (see its 2-surface integral form (4.70) and the argumentation in the second paragraph of subsection 4.6.2), and if $\mathrm{supp}\,(\sigma^A) \subset \Sigma - B$ or $\mathrm{supp}\,(\sigma^A) \subset B$, then the boundary integral on ∂B is also vanishing. Hence, by $\Lambda > 0$ and the dominant energy condition this and the analogous argument for π^A in $\Phi^\alpha = (\sigma^A, \bar\pi^{A'})$ we obtain

$$\big(\|\Phi^\alpha\|_{1,0}\big)^2 \le 2\big(\|\mathcal{D}^\alpha{}_\beta\Phi^\beta\|_{0,0}\big)^2 + \big(\|\Phi^\alpha\|_{0,0}\big)^2 = 2\big(\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta - \frac{3}{2}K\Phi^\alpha\|_{0,0}\big)^2 + \big(\|\Phi^\alpha\|_{0,0}\big)^2 \le \Big(\sqrt{2}\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta\|_{0,0} + \sqrt{1+\frac{3}{4}\Lambda}\|\Phi^\alpha\|_{0,0}\Big)^2. \tag{5.9}$$

Thus, recalling that for $\Phi^\alpha = \Phi^\alpha_B$ the $H_{0,0}$ and $H_{1,0}$ norms coincide with the $L_2(B)$ and $H_1(B)$ norms, respectively, the inequalities hold for smooth Φ^α_A and Φ^α_B, respectively.

Finally, let Φ^α ($= \Phi^\alpha_A$ or Φ^α_B) be an arbitrary element of $H_{1,0}$. Since the smooth spinor fields form a dense subspace in $H_{1,0}$, there exists a sequence $\{\Phi^\alpha_i\}$, $i \in \mathbb{N}$, of smooth spinor fields in $H_{1,0}$ which converges to Φ^α strongly. Applying the estimate (5.9) to the smooth spinor fields Φ^α_i and recalling that the norms $\|\,.\,\|_{s,\delta} : H_{s,\delta} \to [0,\infty)$ are continuous, the inequalities follow. □

N.B.: The statement holds true even for $\Lambda \le 0$; and in the $\Lambda < 0$ case the cosmological constant term gives additional contribution to the constant in front of $\|\Phi^\alpha\|_{0,0}$. Note also that although in this appendix we assume that the dominant energy condition holds, estimates of the form (5.6), (5.7) could be derived from the ellipticity of $\tilde{\mathcal{D}}$ alone, without the use of the dominant energy condition. In this case the forthcoming statements that depend on the fundamental elliptic estimate would hold independently of the dominant energy condition. The price that we had to pay for the simplicity of the proof of this estimate is the requirement of the dominant energy condition.

Another consequence of the Sen–Witten identity is given by the next lemma:

Lemma 5.4. *Let $B \subset \Sigma$ be an open subset with compact closure and smooth boundary. Then*

$$\Big(\|\tilde{\mathcal{D}}_e\Phi^\alpha_A\|_{0,0}\Big)^2 \le 2\Big(\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A\|_{0,0}\Big)^2 + 4\sqrt{\frac{2}{3}\Lambda}\,\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A\|_{0,0}\,\|\Phi^\alpha_A\|_{0,0} \tag{5.10}$$

holds for any $\Phi^\alpha_A \in H_{1,0}$, $\mathrm{supp}\,(\Phi^\alpha_A) \subset \Sigma - B$.

Proof. By the Sen–Witten identity for $\Phi^\alpha_A = (\sigma^A, \bar\pi^{A'})$ and the form (4.8) of the boundary term

$$2\Big(\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A\|_{0,0}\Big)^2 = \Big(\|\tilde{\mathcal{D}}_e\Phi^\alpha_A\|_{0,0}\Big)^2 + \frac{\varkappa}{\sqrt{2}}\int_{\Sigma-B} t_aT^a{}_b\big(\sigma^B\bar\sigma^{B'} + \pi^B\bar\pi^{B'}\big)\mathrm{d}\Sigma$$
$$+4K\int_{\Sigma-B}\sqrt{2}t_{AA'}\,\Big(\big(\bar{\tilde{\mathcal{D}}}{}^A{}_{B'}\bar\sigma^{B'}\big)\bar\pi^{A'} - \big(\tilde{\mathcal{D}}^{A'}{}_B\sigma^B\big)\pi^A + \big(\bar{\tilde{\mathcal{D}}}{}^{A'}{}_B\pi^B\big)\sigma^A - \big(\tilde{\mathcal{D}}^A{}_{B'}\bar\pi^{B'}\big)\bar\sigma^{A'}\Big)\mathrm{d}\Sigma$$

holds, where we used that the integral of the boundary terms vanishes. Then by $K=-\bar{K}$ and the dominant energy condition it follows that

$$2\Big(\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A\|_{0,0}\Big)^2\geq\Big(\|\tilde{\mathcal{D}}_e\Phi^\alpha_A\|_{0,0}\Big)^2+4\Big(\langle\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A,K\Phi^\alpha_A\rangle+\overline{\langle\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A,K\Phi^\alpha_A\rangle}\Big),$$

where $\langle\,,\rangle$ denotes the natural L_2 scalar product of the Dirac spinors. Then by the Cauchy–Schwarz inequality this yields (5.10). □

The inequality in our next lemma is analogous to the Hardy inequality:

Lemma 5.5. *Let $\delta\in\mathbb{R}$ and let Σ be an asymptotically hyperboloidal hypersurface for which the 'boost parameter function' W satisfies $\frac{1}{3}\Lambda W<2\delta+1$. Then there exists an open set $B\subset\Sigma$ with compact closure and smooth boundary, and a positive constant c such that*

$$c\|\Phi^\alpha_A\|_{0,\delta}\leq\|\tilde{\mathcal{D}}_e\Phi^\alpha_A\|_{0,\delta}\tag{5.11}$$

for any $\Phi^\alpha_A\in H_{1,\delta}$, $\text{supp}\,(\Phi^\alpha_A)\subset\Sigma-B$.

Proof. Let $\Phi^\alpha=(\sigma^A,\bar{\pi}^{A'})\in H_{1,\delta}$ and $l\in\mathbb{R}$. Then

$$\begin{aligned}|\tilde{\mathcal{D}}_e\Phi^\alpha|^2:=&-h^{ef}\sqrt{2}t_{AA'}\Big(\big(\tilde{\mathcal{D}}_e\sigma^A\big)\big(\overline{\tilde{\mathcal{D}}}_f\bar{\sigma}^{A'}\big)+\big(\tilde{\mathcal{D}}_e\bar{\pi}^{A'}\big)\big(\overline{\tilde{\mathcal{D}}}_f\pi^A\big)\Big)\\
=&-h^{ef}\sqrt{2}t_{AA'}\Big\{\Big(\Omega^{-l}\tilde{\mathcal{D}}_e\big(\Omega^l\sigma^A\big)-l\sigma^A\Omega^{-1}D_e\Omega\Big)\Big(\Omega^{-l}\overline{\tilde{\mathcal{D}}}_f\big(\Omega^l\bar{\sigma}^{A'}\big)-l\bar{\sigma}^{A'}\Omega^{-1}D_f\Omega\Big)\\
&+\Big(\Omega^{-l}\tilde{\mathcal{D}}_e\big(\Omega^l\bar{\pi}^{A'}\big)-l\bar{\pi}^{A'}\Omega^{-1}D_e\Omega\Big)\Big(\Omega^{-l}\overline{\tilde{\mathcal{D}}}_f\big(\Omega^l\pi^A\big)-l\pi^A\Omega^{-1}D_f\Omega\Big)\Big\}\\
=&\,\Omega^{-2l}|\tilde{\mathcal{D}}_e\big(\Omega^l\Phi^\alpha\big)|^2+l^2\Omega^{-2}|D_e\Omega|^2|\Phi^\alpha|^2\\
&+l\Omega^{-2l-1}\sqrt{2}\big(D^e\Omega\big)\mathcal{D}_e\Big(\Omega^{2l}\big(\sigma^A\bar{\sigma}^{A'}+\pi^A\bar{\pi}^{A'}\big)\Big)t_{AA'},\end{aligned}$$

where in the last step we used the expression (4.5) of $\tilde{\mathcal{D}}_e$ in terms of $\mathcal{D}_e$ and K. Multiplying this by $\Omega^{-2\delta}$, choosing Φ^α to be Φ^α_A (i.e. in $H_{1,\delta}$ and such that $\text{supp}\,(\Phi^\alpha_A)\subset\Sigma-B$), integrating on $\Sigma-B$ and using that the boundary terms both on ∂B and at infinity give zero, we obtain

$$\begin{aligned}\int_{\Sigma-B}\Omega^{-2\delta}|\tilde{\mathcal{D}}_e\Phi^\alpha_A|^2\mathrm{d}\Sigma\geq -l\int_{\Sigma-B}\Omega^{-2\delta}\Big(&(2\delta+1+l)\Omega^{-2}|D_e\Omega|^2|\Phi^\alpha_A|^2\\
&+\Omega^{-1}\big(D_eD^e\Omega\big)|\Phi^\alpha_A|^2+\Omega^{-1}|D_e\Omega|\sqrt{2}v^a\chi_{ab}\big(\sigma^B\bar{\sigma}^{B'}+\pi^B\bar{\pi}^{B'}\big)\Big)\mathrm{d}\Sigma.\end{aligned}$$

If, however, B is chosen to be large enough, then by (4.37) and (4.39)

$$|D_e\Omega|=\frac{\Omega}{|T_e|}+O(\Omega^2),\qquad D_eD^e\Omega=\frac{\Omega}{|T_e|^2}+O(\Omega^2),\qquad v^a\chi_{ab}=\frac{1}{|T_e|}\big(1+\frac{1}{3}\Lambda W\big)v_b+O(\Omega)$$

hold on $\Sigma-B$ in which the corrections to the leading terms are already small, and hence the sign of these expressions on $\Sigma-B$ is the sign of their leading term. Taking into account these asymptotic expressions, and that since $\text{supp}\,(\Phi^\alpha_A)\subset\Sigma-B$ the left hand side is $(\|\tilde{\mathcal{D}}_e\Phi^\alpha_A\|_{0,\delta})^2$, we obtain

$$\big(\|\tilde{\mathcal{D}}_e\Phi^\alpha_A\|_{0,\delta}\big)^2 \geq -l\int_{\Sigma-B}\big(\frac{1}{|T_e|^2}+O(\Omega)\big)\Big(\ (2\delta+2+l)|\Phi^\alpha_A|^2 + (1+\frac{1}{3}\Lambda W)\sqrt{2}v_a\big(\sigma^A\bar\sigma^{A'}+\pi^A\bar\pi^{A'}\big)\Big)\mathrm{d}\Sigma.$$

Since $\frac{1}{3}\Lambda W < 2\delta+1$, we can always choose l to be *negative* such that $\frac{1}{3}\Lambda W - 1 - 2\delta < l < 0$. Moreover, since $(\sigma^A\bar\sigma^{A'}+\pi^A\bar\pi^{A'})$ is future pointing and non-spacelike, $|\Phi^\alpha_A|^2 = \sqrt{2}t_{AA'}(\sigma^A\bar\sigma^{A'}+\pi^A\bar\pi^{A'}) \geq \sqrt{2}|v_a(\sigma^A\bar\sigma^{A'}+\pi^A\bar\pi^{A'})|$ holds, and hence we find that

$$\big(\|\tilde{\mathcal{D}}_e\Phi^\alpha_A\|_{0,\delta}\big)^2 \geq |l|\big(2\delta+1-|l|-\frac{1}{3}\Lambda\max_{\mathcal{S}}\{W\}\big)\inf_{\Sigma-B}\big\{\frac{1}{|T_e|^2}+O(\Omega)\big\}\big(\|\Phi^\alpha_A\|_{0,\delta}\big)^2;$$

i.e. for some positive constant c and for all Φ^α_A the inequality $\|\tilde{\mathcal{D}}_e\Phi^\alpha_A\|_{0,\delta} \geq c\|\Phi^\alpha_A\|_{0,\delta}$ holds. □

Corollary 5.1. *Under the conditions of Lemma 5.5 with* $\delta = 0$

$$\Big(\sqrt{c^2+8|K|^2}-2\sqrt{2}|K|\Big)\|\Phi^\alpha_A\|_{0,0} \leq \sqrt{2}\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A\|_{0,0}$$

holds for any $\Phi^\alpha_A \in H_{1,0}$, $\mathrm{supp}\,(\Phi^\alpha_A) \subset \Sigma - B$.

Proof. Combining the inequalities of Lemma 5.5 and Lemma 5.4 we find that

$$c^2\big(\|\Phi^\alpha_A\|_{0,0}\big)^2 \leq \big(\|\tilde{\mathcal{D}}_e\Phi^\alpha_A\|_{0,0}\big)^2 \leq 2\Big(\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A\|_{0,0}+2|K|\|\Phi^\alpha_A\|_{0,0}\Big)^2 - 8|K|^2\big(\|\Phi^\alpha_A\|_{0,0}\big)^2,$$

which is just the inequality that we wanted to prove. □

Hence, on $\Sigma - B$, the L_2-norm of $\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A$ is bounded from below by the L_2-norm of Φ^α_A itself.

Corollary 5.2. *Under the conditions of Lemma 5.5 with* $\delta = 0$ *there is a positive constant* $\tilde c$ *such that*

$$\|\Phi^\alpha_A\|_{1,0} \leq \tilde c\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A\|_{0,0}$$

holds for any $\Phi^\alpha_A \in H_{1,0}$, $\mathrm{supp}\,(\Phi^\alpha_A) \subset \Sigma - B$.

Proof. By Corollary 5.1 there is a positive constant c_1 such that $\|\Phi^\alpha_A\|_{0,0} \leq c_1\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A\|_{0,0}$. Combining this with the inequality of Lemma 5.4, we obtain

$$\big(\|\tilde{\mathcal{D}}_e\Phi^\alpha_A\|_{0,0}\big)^2 \leq 2(1+4|K|c_1)\big(\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A\|_{0,0}\big)^2.$$

Clearly, for any $\delta \in \mathbb{R}$ the $H_{s,\delta}$-norms defined with $\mathcal{D}_e$ and with $\tilde{\mathcal{D}}_e$ are equivalent. In fact, in particular,

$$\begin{aligned}
\big(\|\Phi^\alpha\|_{1,\delta}\big)^2 &= \big(\|\Phi^\alpha\|_{0,\delta}\big)^2 + \big(\|\mathcal{D}_e\Phi^\alpha\|_{0,\delta}\big)^2\\
&\leq \big(\|\Phi^\alpha\|_{0,\delta}\big)^2 + \big(\|\tilde{\mathcal{D}}_e\Phi^\alpha\|_{0,\delta} + \frac{\sqrt{3}}{2}|K|\|\Phi^\alpha\|_{0,\delta}\big)^2\\
&\leq (1+\frac{3}{4}|K|^2)\Big(\big(\|\Phi^\alpha\|_{0,\delta}\big)^2 + \big(\|\tilde{\mathcal{D}}_e\Phi^\alpha\|_{0,\delta}\big)^2\Big) + \sqrt{3}|K|\|\tilde{\mathcal{D}}_e\Phi^\alpha\|_{0,\delta}\,\|\Phi^\alpha\|_{0,\delta}\\
&\leq (1+\frac{\sqrt{3}}{2}|K|)^2\Big(\big(\|\Phi^\alpha\|_{0,\delta}\big)^2 + \big(\|\tilde{\mathcal{D}}_e\Phi^\alpha\|_{0,\delta}\big)^2\Big).
\end{aligned}$$

Hence

$$\big(\|\Phi^\alpha_A\|_{1,0}\big)^2 \le (1+\frac{\sqrt{3}}{2}|K|)^2\Big(\big(\|\Phi^\alpha_A\|_{0,0}\big)^2+\big(\|\tilde{\mathcal{D}}_e\Phi^\alpha_A\|_{0,0}\big)^2\Big)$$
$$\le (1+\frac{\sqrt{3}}{2}|K|)^2\big(c_1^2+8|K|c_1+2\big)\big(\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A\|_{0,0}\big)^2;$$

i.e. the inequality of the corollary holds with $\tilde{c}^2=(1+\frac{\sqrt{3}}{2}|K|)^2(c_1^2+8|K|c_1+2)$. □

The next lemma and its corollary are the adaptation of Theorem 6.2 of [33] to the present situation:

Lemma 5.6. *Let Σ be an asymptotically hyperboloidal hypersurface for which the 'boost parameter function' W satisfies $\frac{1}{3}\Lambda W<1$. Then, for any $\delta'\le 0$, there exists a positive constant C such that*

$$\|\Phi^\alpha\|_{1,0}\le C\Big(\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta\|_{0,0}+\|\Phi^\alpha\|_{0,\delta'}\Big) \tag{5.12}$$

holds for any $\Phi^\alpha\in H_{1,0}$.

Proof. Let $0<a<1$, and $B_a:=\{p\in\Sigma\,|\,\Omega(p)>a\}$. Clearly, its closure, $\overline{B_a}$, is compact with smooth boundary. Let $\beta:\Sigma\to[0,1]$ be a smooth function such that $\beta(p)=1$ for $p\in B_a$ and $\beta(p)=0$ for $p\in\Sigma-B_{\frac{1}{2}a}$; i.e. in particular supp$\,(\beta)\subset\overline{B_{\frac{1}{2}a}}$. For any $\Phi^\alpha\in H_{1,0}$ let us define$\Phi^\alpha_B:=\beta\Phi^\alpha$ and $\Phi^\alpha_A:=(1-\beta)\Phi^\alpha$, thebounded (or localized) and the asymptotic part of Φ^α, respectively.Clearly, supp$\,(\Phi^\alpha_B)\subset\overline{B_{\frac{1}{2}a}}$, supp$\,(\Phi^\alpha_A)\subset\Sigma-B_a$ and

$$\|\Phi^\alpha\|_{1,0}\le\|\Phi^\alpha_B\|_{1,0}+\|\Phi^\alpha_A\|_{1,0} \tag{5.13}$$

hold. We derive the estimate (5.12) for Φ^α_B and Φ^α_A separately, which by (5.13) yield (5.12) for Φ^α.

Since supp$\,(\Phi^\alpha_B)\subset\overline{B_{\frac{1}{2}a}}$, by (5.7) in Lemma 5.3 there is a positive constant c' such that

$$\|\Phi^\alpha_B\|_{H_1(B_{\frac{1}{2}a})}\le c'\Big(\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_B\|_{L_2(B_{\frac{1}{2}a})}+\|\Phi^\alpha_B\|_{L_2(B_{\frac{1}{2}a})}\Big). \tag{5.14}$$

Since supp$\,(\Phi^\alpha_B)\subset\overline{B_{\frac{1}{2}a}}$, the norm on the left hand side is in fact $\|\Phi^\alpha_B\|_{1,0}$. In the first term on the right we can write

$$\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_B=\tilde{\mathcal{D}}^\alpha{}_\beta\big(\beta\Phi^\beta\big)=\beta\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta+\Big(\bar{\pi}^{B'}D_{B'}{}^A\beta,\sigma^BD_B{}^{A'}\beta\Big),$$

and a straightforward calculation shows that the square of the pointwise norm of its second term is

$$|\big(\bar{\pi}^{B'}D_{B'}{}^A\beta,\sigma^BD_B{}^{A'}\beta\big)|^2=\frac{1}{2}|D_e\beta|^2\,|\Phi^\alpha|^2.$$

Thus the first term on the right in (5.14) can be estimated as

$$\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_B\|_{L_2(B_{\frac{1}{2}a})}\le\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta\|_{L_2(B_{\frac{1}{2}a})}+\frac{1}{\sqrt{2}}\Big(\int_{B_{\frac{1}{2}a}}|D_e\beta|^2|\Phi^\alpha|^2\mathrm{d}\Sigma\Big)^{\frac{1}{2}}$$
$$\le\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta\|_{0,0}+\tilde{C}\|\Phi^\alpha\|_{L_2(B_{\frac{1}{2}a})}.$$

Here, in the first step we used the triangle inequality and that $\beta \leq 1$, and then, in the second, we used the notation $\sqrt{2}\tilde{C} := \sup\{|D_e\beta|(p)\,|\,p \in B_{\frac{1}{2}a}\}$, the Cauchy–Schwarz inequality and $\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta\|_{L_2(B_{\frac{1}{2}a})} \leq \|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta\|_{0,0} < \infty$. Combining this inequality with (5.14) we find that

$$\|\Phi^\alpha_B\|_{1,0} \leq c'\Bigl(\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta\|_{0,0} + (1+\tilde{C})\|\Phi^\alpha_B\|_{L_2(B_{\frac{1}{2}a})}\Bigr). \tag{5.15}$$

Since $\Omega^{-2\delta'} \leq 1$ for $\delta' \leq 0$, $\|\Phi^\alpha\|_{0,\delta'} \leq \|\Phi^\alpha\|_{0,0} \leq \|\Phi^\alpha\|_{1,0} < \infty$ holds; and hence the second norm on the right hand side can be estimated as

$$\begin{aligned}\bigl(\|\Phi^\alpha_B\|_{L_2(B_{\frac{1}{2}a})}\bigr)^2 \leq \bigl(\|\Phi^\alpha\|_{L_2(B_{\frac{1}{2}a})}\bigr)^2 &= \int_{B_{\frac{1}{2}a}} \Omega^{2\delta'}\Omega^{-2\delta'}|\Phi^\alpha|^2 \mathrm{d}\Sigma \\ &\leq \sup\{\Omega^{2\delta'}(p)\,|\,p \in B_{\frac{1}{2}a}\}\bigl(\|\Phi^\alpha\|_{0,\delta'}\bigr)^2.\end{aligned} \tag{5.16}$$

Combining this estimate with (5.15) we obtain that there exists a positive constant C_1 such that

$$\|\Phi^\alpha_B\|_{1,0} \leq C_1\Bigl(\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta\|_{0,0} + \|\Phi^\alpha\|_{0,\delta'}\Bigr) \tag{5.17}$$

holds.

By Lemma 5.5 and its Corollary 5.2 there exist a small enough $a \in (0,1)$ and a positive constant $\tilde{c}$ such that

$$\|\Phi^\alpha_A\|_{1,0} \leq \tilde{c}\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A\|_{0,0} \tag{5.18}$$

holds, where $\mathrm{supp}\,(\Phi^\alpha_A) \subset \Sigma - B_{\frac{1}{2}a}$. Since $\Phi^\alpha_A := (1-\beta)\Phi^\alpha$,

$$\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A = (1-\beta)\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta - \Bigl(\bar{\pi}^{B'}D_{B'}{}^A\beta, \sigma^B D_B{}^{A'}\beta\Bigr)$$

follows, and hence $|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A| \leq |\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta| + \frac{1}{\sqrt{2}}|D_e\beta|\,|\Phi^\alpha|$. Using $\mathrm{supp}\,(\beta) \subset \overline{B_{\frac{1}{2}a}}$ and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\bigl(\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_A\|_{0,0}\bigr)^2 &\leq \int_\Sigma |\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta|^2\mathrm{d}\Sigma + \int_{B_{\frac{1}{2}a}}\Bigl(|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta| 2\tilde{C}|\Phi^\alpha| + \tilde{C}^2|\Phi^\alpha|^2\Bigr)\mathrm{d}\Sigma \\ &\leq \bigl(\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta\|_{0,0}\bigr)^2 + 2\tilde{C}\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta\|_{0,0}\,\|\Phi^\alpha\|_{L_2(B_{\frac{1}{2}a})} + \tilde{C}^2\bigl(\|\Phi^\alpha\|_{L_2(B_{\frac{1}{2}a})}\bigr)^2,\end{aligned}$$

where $\tilde{C}$ has been defined above. But by (5.16) and (5.18) this implies that there exists a positive constant C_2 such that

$$\|\Phi^\alpha_A\|_{1,0} \leq C_2\Bigl(\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta\|_{0,0} + \|\Phi^\alpha\|_{0,\delta'}\Bigr) \tag{5.19}$$

holds. Finally, (5.13), (5.17) and (5.19) yield the estimate (5.12). □

Corollary 5.3. *Under the conditions of Lemma 5.6 there exists a positive constant C' such that*

$$\|\Phi^\alpha\|_{1,0} \leq C'\|\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta\|_{0,0} \tag{5.20}$$

for any $\Phi^\alpha \in (\ker\tilde{\mathcal{D}})^\perp \cap H_{1,0}$, where $(\ker\tilde{\mathcal{D}})^\perp$ denotes the orthogonal complement of $\ker\tilde{\mathcal{D}}$ in L_2.

Proof. Suppose, on the contrary, that for any $i \in \mathbb{N}$ there exists a spinor field $\Phi^\alpha_i \in (\ker \tilde{\mathcal{D}})^\perp \cap H_{1,0}$ for which $\|\Phi^\alpha_i\|_{1,0} > i\|\tilde{\mathcal{D}}^\alpha{}_\beta \Phi^\beta_i\|_{0,0}$. Then $\hat{\Phi}^\alpha_i := (\|\Phi^\alpha_i\|_{1,0})^{-1}\Phi^\alpha_i$ defines a sequence in $(\ker \tilde{\mathcal{D}})^\perp \cap H_{1,0}$ such that $\|\hat{\Phi}^\alpha_i\|_{1,0} = 1$, and hence

$$1 > i\|\tilde{\mathcal{D}}^\alpha{}_\beta \hat{\Phi}^\beta_i\|_{0,0}.$$

Thus, in particular, the sequence $\{\tilde{\mathcal{D}}^\alpha{}_\beta \hat{\Phi}^\beta_i\}$, $i \in \mathbb{N}$, is Cauchy in L_2 and converges to zero. Since $\{\hat{\Phi}^\alpha_i\}$, $i \in \mathbb{N}$, is bounded in $H_{1,0}$ and by Lemma 5.1 the injection $H_{1,0} \to H_{0,\delta'}$ is compact for $\delta' < 0$, there is a subsequence of $\{\hat{\Phi}^\alpha_i\}$, for the sake of simplicity $\{\hat{\Phi}^\alpha_i\}$ itself, which is Cauchy in $H_{0,\delta'}$. Applying (5.12) to $\hat{\Phi}^\alpha_i - \hat{\Phi}^\alpha_j$ we obtain

$$\|\hat{\Phi}^\alpha_i - \hat{\Phi}^\alpha_j\|_{1,0} \leq C\Big(\|\tilde{\mathcal{D}}^\alpha{}_\beta \hat{\Phi}^\beta_i - \tilde{\mathcal{D}}^\alpha{}_\beta \hat{\Phi}^\beta_j\|_{0,0} + \|\hat{\Phi}^\alpha_i - \hat{\Phi}^\alpha_j\|_{0,\delta'}\Big);$$

i.e. $\{\hat{\Phi}^\alpha_i\}$, $i \in \mathbb{N}$, is Cauchy in $H_{1,0}$. Hence it converges strongly in $H_{1,0}$ to some $\hat{\Phi}^\alpha$. Since the norm $\|\,.\,\|_{1,0} : H_{1,0} \to [0,\infty)$ is continuous, $\|\hat{\Phi}^\alpha\|_{1,0} = \lim_{i\to\infty} \|\hat{\Phi}^\alpha_i\|_{1,0} = 1$, and by the continuity of the scalar product $\hat{\Phi}^\alpha \in (\ker \tilde{\mathcal{D}})^\perp$ also holds. Thus $\hat{\Phi}^\alpha$ is a non-zero vector which does not belong to the kernel of $\tilde{\mathcal{D}}$. However, by the continuity of $\tilde{\mathcal{D}} : H_{1,0} \to L_2$ and of the norm we have that $\|\tilde{\mathcal{D}}^\alpha{}_\beta \hat{\Phi}^\beta\|_{0,0} = \lim_{i\to\infty} \|\tilde{\mathcal{D}}^\alpha{}_\beta \hat{\Phi}^\beta_i\|_{0,0} = 0$, which would yield that $\hat{\Phi}^\alpha \in \ker \tilde{\mathcal{D}}$. □

Our last lemma is the so-called elliptic regularity estimate, which is just Lemma A.2 of [9] applied now to Dirac spinors on the asymptotically hyperboloidal Σ:

Lemma 5.7. *There exist positive constants C_1 and C_2 such that for any $\Phi^\alpha \in H_{s,0}$, $s \geq 1$, for which $\mathcal{D}^\alpha{}_\beta \Phi^\beta \in H_{s,0}$ is also true, the inequality*

$$\|\Phi^\alpha\|_{s+1,0} \leq C_1 \|\mathcal{D}^\alpha{}_\beta \Phi^\beta\|_{s,0} + C_2 \|\Phi^\alpha\|_{s,0}$$

holds.

Proof. The proof is a straightforward modification of that of Lemma A.2 of [9]. The only essential difference between the two proofs is that in the present case we should use orthonormal dual bases $\{e^a_{\mathbf{i}}, \vartheta^{\mathbf{i}}_a\}$, $\mathbf{i} = 1,2,3$, for which the connection 1-form $\gamma^{\mathbf{i}}_{e\mathbf{j}} := \vartheta^{\mathbf{i}}_a \mathcal{D}_e e^a_{\mathbf{j}} = \vartheta^{\mathbf{i}}_a D_e e^a_{\mathbf{j}}$ is bounded on the whole Σ. Since Σ is asymptotically hyperboloidal with bounded intrinsic curvature (see subsection 4.4), such a frame field always exists. Also, though in the present case Σ is not compact, its intrinsic and extrinsic curvatures and their finitely many derivatives (up to order $(s-1)$) are also bounded. □

5.3 The isomorphism theorem for $\tilde{\mathcal{D}}$

In subsection 4.6.2 we showed that $\tilde{\mathcal{D}}^\alpha{}_\beta \Phi^\beta = 0$ does not have any smooth square integrable solution. First we show that it does not have any solution even in $H_{1,0}$.

Proposition 5.1. $\ker \tilde{\mathcal{D}} \subset H_{1,0}$ *is empty.*

Proof. If $\Phi^\alpha \in \ker \tilde{\mathcal{D}}$ such that $\Phi^\alpha \in H_{1,0}$, then $\mathcal{D}^\alpha{}_\beta \Phi^\beta = -\frac{3}{2} K \Phi^\alpha \in H_{1,0}$. Thus by the elliptic regularity estimate, Lemma 5.7, it follows that Φ^α belongs to $H_{2,0}$ too, which implies that it belongs to $H_{3,0}$, ... etc; i.e. it belongs to $H_{s,0}$ for any $s \in \mathbb{N}$. But by the Sobolev lemma, Lemma 5.2, this implies that Φ^α is smooth. However, in subsection 4.6.2 we showed that $\tilde{\mathcal{D}}^\alpha{}_\beta \Phi^\beta = 0$ does not have any *smooth* square integrable solution, and hence $\ker \tilde{\mathcal{D}} = \emptyset$. □

We also need the following two propositions:

Proposition 5.2. *Under the conditions of Lemma 5.6* $\mathrm{Im}\,\tilde{\mathcal{D}}\subset L_2$ *is a closed subspace.*

Proof. Let $\{\chi^\alpha_i\}$, $i\in\mathbb{N}$, be any Cauchy sequence in $\mathrm{Im}\,\tilde{\mathcal{D}}\subset L_2$. By the previous proposition there is a uniquely determined sequence $\{\Phi^\alpha_i\}$ in $H_{1,0}$ such that $\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_i=\chi^\alpha_i$. Then by Corollary 5.3 there is a positive constant C' such that $\Vert\Phi^\alpha_i-\Phi^\alpha_j\Vert_{1,0}\leq C'\Vert\chi^\alpha_i-\chi^\alpha_j\Vert_{0,0}$. Hence $\{\Phi^\alpha_i\}$ is Cauchy in $H_{1,0}$, converging to some $\Phi^\alpha\in H_{1,0}$. Since $\tilde{\mathcal{D}}$ is continuous, $\chi^\alpha_i=\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta_i\to\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta\in\mathrm{Im}\,\tilde{\mathcal{D}}$ if $i\to\infty$; i.e. $\mathrm{Im}\,\tilde{\mathcal{D}}\subset L_2$ is a closed subspace. □

Proposition 5.3. $\mathrm{i}\,\tilde{\mathcal{D}}:H_{1,0}\to L_2$ *is self-adjoint.*

Proof. To calculate the formal adjoint of $\tilde{\mathcal{D}}:C^\infty(\Sigma,\mathbb{D}^\alpha)\to C^\infty(\Sigma,\mathbb{D}^\alpha)$ (with respect to the L_2-scalar product), let $\Phi^\alpha=(\sigma^A,\bar\pi^{A'})\in C^\infty(\Sigma,\mathbb{D}^\alpha)\cap L_2$ and $\chi^\alpha=(\lambda^A,\bar\mu^{A'})\in C^\infty(\Sigma,\mathbb{D}^\alpha)$ be arbitrary. Then by integration by parts

$$
\begin{aligned}
\langle\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta,\chi^\alpha\rangle=&\int_\Sigma\sqrt{2}t_{AA'}\,\Bigl(\bigl(D^A{}_{B'}\bar\pi^{B'}+\frac{1}{2}\chi t^A{}_{B'}\bar\pi^{B'}+\frac{3}{2}K\sigma^A\bigr)\bar\lambda^{A'}\\
&+\bigl(D^{A'}{}_B\sigma^B+\frac{1}{2}\chi t^{A'}{}_B\sigma^B+\frac{3}{2}K\bar\pi^{A'}\bigr)\mu^A\Bigr)\mathrm{d}\Sigma\\
=&\sqrt{2}\oint_{\mathcal{S}}\bigl(\sigma^At_A{}^{A'}v_{A'B}\mu^B+\bar\pi^{A'}t_{A'}{}^Av_{AB'}\bar\lambda^{B'}\bigr)\mathrm{d}\mathcal{S}-\langle\Phi^\alpha,\tilde{\mathcal{D}}^\alpha{}_\beta\chi^\beta\rangle.
\end{aligned}
$$

Thus, the formal adjoint $\tilde{\mathcal{D}}^*:C^\infty(\Sigma,\mathbb{D}^\alpha)\to C^\infty(\Sigma,\mathbb{D}^\alpha)$ of $\tilde{\mathcal{D}}$ is just $-\tilde{\mathcal{D}}$, i.e. $\mathrm{i}\,\tilde{\mathcal{D}}$ is *formally* self-adjoint. Since Φ^α is square integrable, its Weyl spinor parts fall off as $o(\Omega^{3/2})$, and hence to ensure the vanishing of the boundary integral χ^α should be required to fall off at least as $O(\Omega^{3/2})$. In fact, since $\tilde{\mathcal{D}}$ maps $H_{1,0}$ into L_2, we require χ^α to be square integrable, too, i.e. to fall off as $o(\Omega^{3/2})$.

Thus, what we should show is only that the domain

$$
\mathrm{Dom}\bigl(\tilde{\mathcal{D}}^*\bigr):=\bigl\{\chi^\alpha\in L_2\,|\,\exists\Omega^\alpha\in L_2:\ \langle\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta,\chi^\alpha\rangle=\langle\Phi^\alpha,\Omega^\alpha\rangle\ \forall\,\Phi^\alpha\in H_{1,0}\bigr\}
$$

is just $H_{1,0}$. In fact, since the formal adjoint $(\mathrm{i}\tilde{\mathcal{D}})^*$ coincides with $\mathrm{i}\tilde{\mathcal{D}}$, their (unique) extension from the space of the smooth square integrable spinor fields to $\mathrm{Dom}(\tilde{\mathcal{D}}^*)$ and $H_{1,0}$, respectively, coincide if the domains coincide. Note that Ω^α in the definition of $\mathrm{Dom}(\tilde{\mathcal{D}}^*)$ is uniquely determined by χ^α.

Let $\chi^\alpha\in H_{1,0}$. Then $\langle\mathrm{i}\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta,\chi^\alpha\rangle=\langle\Phi^\alpha,\mathrm{i}\tilde{\mathcal{D}}^\alpha{}_\beta\chi^\beta\rangle$ for any $\Phi^\alpha\in H_{1,0}$. Thus, for any $\chi^\alpha\in H_{1,0}$, the spinor field $\Omega^\alpha:=\mathrm{i}\tilde{\mathcal{D}}^\alpha{}_\beta\chi^\beta\in L_2$ is such that $\langle\mathrm{i}\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta,\chi^\alpha\rangle=\langle\Phi^\alpha,\Omega^\alpha\rangle$ for any $\Phi^\alpha\in H_{1,0}$. Hence, $H_{1,0}\subset\mathrm{Dom}(\tilde{\mathcal{D}}^*)$.

Conversely, let $\chi^\alpha\in\mathrm{Dom}(\tilde{\mathcal{D}}^*)$. Since $H_{1,0}\subset L_2$ is dense, there exists a sequence $\{\chi^\alpha_i\}$, $i\in\mathbb{N}$, in $H_{1,0}\cap\mathrm{Dom}(\tilde{\mathcal{D}}^*)$ such that $\chi^\alpha_i\to\chi^\alpha$ in the L_2-norm. Then for any $\Phi^\alpha\in H_{1,0}$ $\langle\Phi^\alpha,\mathrm{i}\tilde{\mathcal{D}}^\alpha{}_\beta\chi^\beta_i\rangle=\langle\mathrm{i}\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta,\chi^\alpha_i\rangle\to\langle\mathrm{i}\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta,\chi^\alpha\rangle$ when $i\to\infty$. By $\chi^\alpha\in\mathrm{Dom}(\tilde{\mathcal{D}}^*)$ and the definition of $\mathrm{Dom}(\tilde{\mathcal{D}}^*)$ there exists a spinor field $\Omega^\alpha\in L_2$ such that the limit on the right is $\langle\mathrm{i}\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta,\chi^\alpha\rangle=\langle\Phi^\alpha,\Omega^\alpha\rangle$. Hence, $\langle\Phi^\alpha,\mathrm{i}\tilde{\mathcal{D}}^\alpha{}_\beta\chi^\beta_i-\Omega^\alpha\rangle\to0$ if $i\to\infty$ for any $\Phi^\alpha\in H_{1,0}$. But since $H_{1,0}$ is dense in L_2, this implies $\langle\Psi^\alpha,\mathrm{i}\tilde{\mathcal{D}}^\alpha{}_\beta\chi^\beta_i-\Omega^\alpha\rangle\to0$ if $i\to\infty$ for any $\Psi^\alpha\in L_2$. Therefore, $\mathrm{i}\tilde{\mathcal{D}}^\alpha{}_\beta\chi^\beta_i\to\Omega^\alpha\in L_2$ in the *weak* topology of L_2. Since every weakly convergent sequence is bounded, there exist positive constants C_1 and C_2 such that $\Vert\chi^\alpha_i\Vert_{0,0}\leq C_1$ and $\Vert\tilde{\mathcal{D}}^\alpha{}_\beta\chi^\beta_i\Vert_{0,0}\leq C_2$ holds. Hence $\{\chi^\alpha_i\}$ is bounded in $H_{1,0}$.

But every bounded sequence contains a *weakly* convergent subsequence, i.e. there is a subsequence $\{\chi^\alpha_{i_k}\}$, $k\in\mathbb{N}$, which converges weakly to some $\chi^\alpha_w\in H_{1,0}\subset L_2$. However, $\{\chi^\alpha_i\}$ was assumed to converge *strongly* to $\chi^\alpha\in{\rm Dom}(\tilde{\mathcal{D}}^*)\subset L_2$, the strong and the weak limits must coincide: $\chi^\alpha=\chi^\alpha_w\in H_{1,0}$; i.e. ${\rm Dom}(\tilde{\mathcal{D}}^*)\subset H_{1,0}$. □

Theorem 5.4. *Under the conditions of Lemma 5.6* $\tilde{\mathcal{D}}:H_{1,0}\to L_2$ *is a topological vector space isomorphism.*

Proof. We saw that $\tilde{\mathcal{D}}$ is continuous, and by Proposition 5.1 it is injective. Thus we should show only that it is surjective, too, because, by the open mapping theorem, any continuous bijection between two Banach spaces is a topological vector space isomorphism.

Since by Proposition 5.3 ${\rm i}\tilde{\mathcal{D}}$ is self-adjoint, ${\rm Im}\,\tilde{\mathcal{D}}={\rm Im}\,\tilde{\mathcal{D}}^*$ holds, and suppose, on the contrary, that ${\rm Im}\,\tilde{\mathcal{D}}\neq L_2$. Since by Proposition 5.2 ${\rm Im}\,\tilde{\mathcal{D}}$ is closed, this implies that $({\rm Im}\,\tilde{\mathcal{D}})^\perp$, the orthogonal complement of ${\rm Im}\,\tilde{\mathcal{D}}$ in L_2, is not empty. Thus, let $\chi^\alpha\in({\rm Im}\,\tilde{\mathcal{D}})^\perp$ be a non-zero vector. Since $H_{1,0}\subset L_2$ is dense, there is a sequence $\{\chi^\alpha_i\}$, $i\in\mathbb{N}$, in $H_{1,0}\cap({\rm Im}\,\tilde{\mathcal{D}})^\perp$ such that $\chi^\alpha_i\to\chi^\alpha$ if $i\to\infty$. Then, by the self-adjointness of ${\rm i}\tilde{\mathcal{D}}$, for any $\Phi^\alpha\in H_{1,0}$ we have that $0=\langle\chi^\alpha_i,{\rm i}\tilde{\mathcal{D}}^\alpha{}_\beta\Phi^\beta\rangle=\langle{\rm i}\tilde{\mathcal{D}}^\alpha{}_\beta\chi^\beta_i,\Phi^\alpha\rangle$. Since $H_{1,0}\subset L_2$ is dense, this implies that $\chi^\alpha_i\in\ker\tilde{\mathcal{D}}=\emptyset$. However, this yields $\chi^\alpha_i=0$ for any $i\in\mathbb{N}$, and hence that $\chi^\alpha=0$, which is a contradiction. □

This completes the proof of the existence and uniqueness of the solution of the renormalized Witten equation.

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