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On the Minimum Character of Points in Compact Spaces

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An old problem of Arkhangel'skiĭ asks (cf. [A]) whether a compact space of countable tightness always has a point of countable character. By now we know that the answer to this question is independent of ZFC: while [F], [M] and [J1] yield models in which counterexamples exist, A. Dow [D1] shows that PFA implies the affirmative answer to Arkhangel'skiĭ's question.

However, since all known counterexamples are such that every point has character ω_1 in them, there remains the natural question whether the minimum character of points of spaces of countable tightness is at most ω_1, in ZFC. In [D2] A. Dow has shown that this is so if one adds Cohen reals to a model of GCH, and one of our main results here says that the same conclusion holds if one adds ω_1 Cohen reals to any model of ZFC. In particular, this shows the consistency of our conjecture with arbitrary cardinal arithmetic.

Let us start by formulating a simple result that, although I have not found it in the literature, may be assumed to be folklore.

Theorem 1. *In every compact T_2 space X of countable tightness there is a point p with $\chi(p,X) \le c = 2^{\omega}$.*

The proof of this follows immediately from the next lemma, which will be needed also for other purposes later.

Lemma 2. *If X is a compact T_2 space of countable tightness then there are a countable set $S \subset X$ and a non-empty closed G_δ set H in X with $H \subset \overline{S}$.*

Proof. Assume, indirectly, that no such S and H exist and define by transfinite induction on $\alpha \in \omega_1$ points $p_\alpha \in X$ and a decreasing sequence of closed G_δ sets $H_\alpha \subset X$ as follows.

Put $H_0 = X$ and $p_0 \in H_0 = X$ be arbitrary. If $\alpha \in \omega_1$ and $\{p_\beta : \beta \in \alpha\}$, $\{H_\beta : \beta \in \alpha\}$ have already been defined, let $\tilde{H}_\alpha = \bigcap\{H_\beta : \beta \in \alpha\}$, which is a non-empty closed G_δ-set, hence by our indirect assumption $\tilde{H}_\alpha \setminus \overline{\{p_\beta : \beta \in \alpha\}} \neq \emptyset$. Thus we may find a non-empty closed G_δ set $H_\alpha \subset \tilde{H}_\alpha \setminus \overline{\{p_\beta : \beta \in \alpha\}}$ and pick $p_\alpha \in H_\alpha$.

But, now, it is obvious from our construction that

$$\overline{\{p_\beta : \beta \in \alpha\}} \cap \overline{\{p_\beta : \beta \in \omega_1 \setminus \alpha\}} = \emptyset$$

for each $\alpha \in \omega_1$, i.e. $\{p_\alpha : \alpha \in \omega_1\}$ is a free sequence in X of length ω_1, contradicting that X has countable tightness. ∎

Theorem 1 now follows easily: If S and H are as in lemma 1, then $w(H) \leq w(\overline{S}) \leq c$, hence $\chi(p, H) \leq c$ for each $p \in H$, but then also $\chi(p, X) \leq c$ since $\chi(p, X) \leq \chi(p, H) \cdot \omega$ as H is a G_δ in X.

Let us now turn to the actual new results of our paper. We start with several simple definitions.

Definition 3. If $f \in C(X, I)$, i.e. f is a continuous function from X into $I = [0,1]$, a set $A \subset X$ is called *f-small* if $\{0,1\} \not\subset f[A]$, i.e. f does not take both values 0 and 1 on A. We call a collection $\mathcal{B}$ of subsets of X *deep* in X if for every $f \in C(X, I)$ and $B \in \mathcal{B}$ there is some $C \in \mathcal{B}$ such that $C \subset B$ and C is f-small. Finally, we say that a *countable* collection of *closed G_δ sets* in X is *good* if it is also deep in X.

The usefulness of these definitions should be appearent from what follows. The next result gives a sufficient condition for the existence of a good family in a space.

Lemma 4. *If X is a Tykhonov space such that the set $\{p \in X : \pi\chi(p, X) \leq$ $\leq \omega\}$ is dense in X then there is a good family in X.*

Proof. Let $p_0 \in X$ have countable π-character (i.e. $\pi\chi(p_0, X) \leq \omega$) and, using that X is Tychonov, choose $\mathcal{B}_0$, a countable collection of closed G_δ sets with non-empty interior, such that $\mathcal{B}_0$ forms a local π-base at p_0 in X.

If, for some $n \in \omega$, we have already chosen $\mathcal{B}_n$ which is a countable collection of closed G_δ sets with non-empty interior, then we first choose a point $p(B) \in \operatorname{Int} B$ for each $B \in \mathcal{B}_n$ such that $\pi\chi(p(B), X) \leq \omega$ and then fix a countable collection $\mathcal{C}_B$ of closed G_δ sets with non-empty interior that forms a local π-base at $p(B)$ in X. Then we put

$$\mathcal{B}_{n+1} = \bigcup\{\mathcal{C}_B : B \in \mathcal{B}_n\}.$$

We claim that $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \omega\}$ is good, and to show this we only have to verify that B is deep in X.

Now, for each $f \in C(X, I)$ and $B \in \mathcal{B}$, if $B \in \mathcal{B}_n$ then either $f(p(B)) \neq \neq 0$ or $f(p(B)) \neq 1$. W.l.o.g. assume $f(p(B)) > 0$ and choose a neighbourhood U of $p(B)$ such that $U \subset B$ and $f(q) > 0$ for all $q \in U$. Now, there is some $C \in \mathcal{C}_B \subset \mathcal{B}_{n+1}$ with $C \subset U \subset B$, hence C is clearly f-small. This completes the proof. ■

Sapirovskiĭ (see [J2]) has shown that if a compact T_2 space X does not map continuously onto I^{ω_1} then X, and in fact every closed subspace of X, satisfies the assumption of lemma 4, in particular so does every compact T_5 space or every compact T_2 space of countable tightness.

Our next result involves MA_κ(countable), i.e. Martin's axiom restricted to countable posets and $\leq \kappa$ dense sets. As is well-known, cf. [K2], MA_κ(countable) is equivalent to the statement that the real line is not the union of $\leq \kappa$ nowhere dense sets.

Theorem 5. *Assume MA_κ(countable) and let X be a countably compact Tykhonov space of weight $\leq \kappa$ with a good family $\mathcal{B}$. Then there is a point $p \in X$ with $\chi(p, X) \leq \omega$.*

Proof. Let us consider the countable partial order $\langle \mathcal{B}, \subset \rangle$. In terms of this particular order, the fact that $\mathcal{B}$ is good, hence deep, may be formulated as follows: for each $f \in C(X, I)$ the collection $\mathcal{D}_f = \{C \in \mathcal{B} : C \text{ is } f\text{-small}\}$ is dense in $\langle \mathcal{B}, \subset \rangle$.

Now, $w(X) \leq \kappa$ implies that there is a family $\mathcal{F} \subset C(X, I)$ with $|\mathcal{F}| \leq \kappa$ that separates the points of X, i.e. for any pair of distinct points $p, q \in X$ there is an $f \in \mathcal{F}$ with $f(p) = 0$ and $f(q) = 1$.

Let us apply MA_κ(countable) to find a set $\mathcal{G} \subset \mathcal{B}$ which is generic over $\{\mathcal{D}_f : f \in \mathcal{F}\}$. We claim that $\bigcap \mathcal{G}$ is a singleton, i.e. $\bigcap \mathcal{G} = \{p\}$ for some $p \in X$.

Indeed, $\bigcap G$ is non-empty because X is countably compact, and it cannot contain two distinct points because $\mathcal{G} \cap \mathcal{D}_f \neq \emptyset$, hence $\bigcap \mathcal{G}$ is f-small, for every $f \in \mathcal{F}$. Since every $B \in \mathcal{G}$ is a G_δ set and $\mathcal{G}$ is countable so is $\bigcap \mathcal{G} = \{p\}$, i.e. $\psi(p, X) \leq \omega$. However, the countable compactness and the T_3 property of X imply that then $\chi(p, X) = \psi(p, X) \leq \omega$ as well. ■

As an immediate corollary of lemma 4 and theorem 5, and of the remark made after lemma 4, we obtain the following result.

Corollary 6. *MA_κ(countable) implies that if X is a compact T_2 space of weight $\leq \kappa$ and X has a dense set of points of countable π-character, in particular if X is T_5 or countably tight, then there is a point $p \in X$ with $\chi(p, X) \leq \omega$.*

Let us note that on one hand by adding κ^+ Cohen reals to V we obtain a model of MA_κ(countable), while by [M] or [J1] already the adjunction of a single Cohen real yields a countably tight compact T_2 space in which every point has character ω_1. This shows that the restriction on the weight of X in corollary 6 is essential.

As it was indicated in the introduction, we shall now formulate some results that hold in the extension of any ground model, V, obtained by adding ω_1 Cohen reals. Thus, the partial order we force with is $Fn(\omega_1, 2)$; if $G \subset Fn(\omega_1, 2)$ is a generic set over V then we put $G_\alpha = G \cap Fn(\omega \cdot \alpha, 2)$. We refer the reader to [K2] concerning the (simple) results on forcing that we shall use. The most important of these is the following: If $a \in V[G]$ is a countable subset of V then there is an $\alpha \in \omega_1$ such that $a \in V[G_\alpha]$.

The following is the main result of this paper.

Theorem 7. *In $V[G]$, if X is a separable compact T_2 space such that every closed G_δ set in X admits a good collection then every non-empty G_δ set F in X contains a point of character $\leq \omega_1$.*

In view of lemma 4 we immediately obtain from theorem 7 the next result.

Corollary 8. *In $V[G]$, every separable compact T_2 space X which does not map onto I^{ω_1} has a point of character ω or ω_1.*

Indeed, if H is the set of all isolated points of X then H is countable, hence $X \setminus H$ is a G_δ set. Thus if $p \in X \setminus H$ is any point of character $\leq \omega_1$ then either $\chi(p, X) = \omega$ or $\chi(p, X) = \omega_1$.

In [K1] the statement of corollary 8 was proved for 0-dimensional com-
ct T_2 spaces, using boolean algebraic methods.

Finally, applying lemma 2 we get as an immediate corollary of theorem 7
e result on the minimum character of countably tight compact T_2 spaces
entioned in the introduction.

orollary 9. *In $V[G]$, every compact T_2 space of countable tightness has
point of character $\leq \omega_1$.*

Let us now turn to the proof of theorem 7. We may assume without any
s of generality that the underlaying set of X is some cardinal κ and that
is dense in X. Also, we may assume that F is closed and has no isolated
ints in itself, since, F being a G_δ, such a point would have countable
aracter in X. Having arranged this we now formulate a lemma.

mma 10. *Let X be a space in $V[G]$ as above, $F \subset X$ be a nonempty
sed G_δ set and α be any countable ordinal. Then there is a non-empty
sed G_δ set $F^+ \subset F$ such that F^+ is f-small whenever $f \in C(X, I)$ and
$\omega \in V[G_\alpha]$.*

oof. Let $\mathcal{B}$ be a good collection in F, say $\mathcal{B} = \{B_n : n \in \omega\}$. Since F
s no isolated points we may clearly assume that for every $n \in \omega$ there are
$l \in \omega$ such that $B_k \cup B_l \subset B_n$ and $B_k \cap B_l = \emptyset$. Now, every B_n is a closed
in F, hence in X as well, consequently there is a decreasing sequence
$G^i_n : i \in \omega\}$ of regular open subsets of X such that $B_n = \bigcap\{G^i_n : i \in \omega\}$.
e may also assume that $\overline{G^{i+1}_n} \subset G^i_n$ for each $i \in \omega$. By compactness, then
e G^i_n's form a neighbourhood base for B_n in X.

Let us consider the matrix $\mathcal{A} = \{\omega \cap G^i_n : i, n \in \omega\}$ of traces; then there
a $\beta \in \omega_1$ with $\mathcal{A} \in V[G_\beta]$; of course we may assume that $\alpha \leq \beta$.

Now, if $n, m \in \omega$ then $B_n \subset B_m$ if and only if for every $i \in \omega$ there
a $j \in \omega$ with $G^j_n \subset G^i_m$, moreover, since ω is dense in X the latter is
uivalent to $\omega \cap G^j_n \subset \omega \cap G^i_m$. This implies that if we define the partial
der $\prec$ on ω by $n \prec m$ iff $B_n \subset B_m$ then $\langle \omega, \prec \rangle \in V[G_\beta]$.

Also, if $f \in C(X, I)$ and $f \restriction \omega \in V[G_\beta]$ then the set

$$\mathcal{D}^\star_f = \{n \in \omega : B_n \text{ is } f\text{-small}\}$$

an element of $V[G_\beta]$. Indeed, $n \in \mathcal{D}^\star_f$ holds iff either $m = \min\{f(x) : x \in
B_n\} > 0$ or $M = \max\{f(x) : x \in B_n\} < 1$, and by the continuity of f
d the density of ω we have $0 < m$ iff $\inf\{f(x) : x \in \omega \cap G^i_n\} > 0$ for some

$i \in \omega$ and similarly $M < 1$ iff $\sup\{f(x) : x \in \omega \cap G_n^i\} < 1$ for some $i \in \omega$. Also since $\mathcal{B}$ is good we have that $\mathcal{D}_f^*$ is dense in $\langle \omega, \prec \rangle$.

Now, $V[G]$, in fact already $V[G_{be+1}]$ contains a real that is Cohen over $V[G_\beta]$, hence, as $\langle \omega, \prec \rangle$ is forcing equivalent to the Cohen order (recall that every $n \in \omega$ has incompatible extensions under $\prec$), there is a set $H \subset \omega$ in $V[G]$ that is $\prec$-generic over $V[G_\beta]$. Let us now put

$$F^+ = \bigcap \{B_n : n \in H\}.$$

Clearly F^+ is a non-empty closed $G_\text{l}de$ set, moreover $H \cap \mathcal{D}_f^* \neq 0$ whenever $f \upharpoonright \omega \in V[G_\alpha] \subset V[G_\beta]$ implies that F^+ is f-small for every such $f \in C(X, I)$. ∎

Let us now return to the proof of theorem 7. Given X and F, we define by transfinite induction on $\alpha \in \omega_1$ a decreasing sequence of non-empty closed G_δ sets F_α as follows. Let $F_0 = F$; if α is limit we set

$$F_\alpha = \bigcap \{F_\beta : \beta \in \alpha\};$$

finally if F_α is given then we apply lemma 10 to F_α and to obtain $F_{\alpha+1} = = (F_\alpha)^+$.

We claim that then $\bigcap\{F_\alpha : \alpha \in \omega_1\}$ is a singleton. Indeed, it is non-empty by compactness, moreover it cannot contain two distinct points because for every $f \in C(X, I)$ there is some $\alpha \in \omega_1$ with $f \upharpoonright \omega \in V[G_\alpha]$, hence already $F_{\alpha+1}$ is f-small. This completes the proof of theorem 7, because the single element p of this intersection clearly has character $\leq \omega_1$.

To conclude, let us note that in theorem 7 (and also in its corollaries) we may replace the Cohen order $Fn(\omega_1, 2)$ with any ω_1-length finite support iteration of non-trivial CCC partial orders, moreover instead of the compactness of X it suffices to assume its initial ω_1-compactness, i.e. that every open cover of X of size $\leq \omega_1$ has a finite subcover.

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