

On the Minimum Character of Points in Compact Spaces

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An old problem of Arkhangel'skiĭ asks (cf. [A]) whether a compact space of countable tightness always has a point of countable character. By now we know that the answer to this question is independent of ZFC: while [F], [M] and [J1] yield models in which counterexamples exist, A. Dow [D1] shows that PFA implies the affirmative answer to Arkhangel'skiĭ's question.

However, since all known counterexamples are such that every point has character ω_1 in them, there remains the natural question whether the minimum character of points of spaces of countable tightness is at most ω_1 , in ZFC. In [D2] A. Dow has shown that this is so if one adds Cohen reals to a model of GCH, and one of our main results here says that the same conclusion holds if one adds ω_1 Cohen reals to any model of ZFC. In particular, this shows the consistency of our conjecture with arbitrary cardinal arithmetic.

Let us start by formulating a simple result that, although I have not found it in the literature, may be assumed to be folklore.

Theorem 1. *In every compact T_2 space X of countable tightness there is a point p with $\chi(p, X) \leq c = 2^\omega$.*

The proof of this follows immediately from the next lemma, which will be needed also for other purposes later.

Lemma 2. *If X is a compact T_2 space of countable tightness then there are a countable set $S \subset X$ and a non-empty closed G_δ set H in X with $H \subset \bar{S}$.*

Proof. Assume, indirectly, that no such S and H exist and define by transfinite induction on $\alpha \in \omega_1$ points $p_\alpha \in X$ and a decreasing sequence of closed G_δ sets $H_\alpha \subset X$ as follows.

Put $H_0 = X$ and $p_0 \in H_0 = X$ be arbitrary. If $\alpha \in \omega_1$ and $\{p_\beta : \beta \in \alpha\}$, $\{H_\beta : \beta \in \alpha\}$ have already been defined, let $\tilde{H}_\alpha = \bigcap \{H_\beta : \beta \in \alpha\}$, which is a non-empty closed G_δ -set, hence by our indirect assumption $\tilde{H}_\alpha \setminus \overline{\{p_\beta : \beta \in \alpha\}} \neq \emptyset$. Thus we may find a non-empty closed G_δ set $H_\alpha \subset \tilde{H}_\alpha \setminus \overline{\{p_\beta : \beta \in \alpha\}}$ and pick $p_\alpha \in H_\alpha$.

But, now, it is obvious from our construction that

$$\overline{\{p_\beta : \beta \in \alpha\}} \cap \overline{\{p_\beta : \beta \in \omega_1 \setminus \alpha\}} = \emptyset$$

for each $\alpha \in \omega_1$, i.e. $\{p_\alpha : \alpha \in \omega_1\}$ is a free sequence in X of length ω_1 , contradicting that X has countable tightness. ■

Theorem 1 now follows easily: If S and H are as in lemma 1, then $w(H) \leq w(\bar{S}) \leq c$, hence $\chi(p, H) \leq c$ for each $p \in H$, but then also $\chi(p, X) \leq c$ since $\chi(p, X) \leq \chi(p, H) \cdot \omega$ as H is a G_δ in X .

Let us now turn to the actual new results of our paper. We start with several simple definitions.

Definition 3. If $f \in C(X, I)$, i.e. f is a continuous function from X into $I = [0, 1]$, a set $A \subset X$ is called *f-small* if $\{0, 1\} \not\subset f[A]$, i.e. f does not take both values 0 and 1 on A . We call a collection \mathcal{B} of subsets of X *deep* in X if for every $f \in C(X, I)$ and $B \in \mathcal{B}$ there is some $C \in \mathcal{B}$ such that $C \subset B$ and C is *f-small*. Finally, we say that a *countable* collection of closed G_δ sets in X is *good* if it is also deep in X .

The usefulness of these definitions should be apparent from what follows. The next result gives a sufficient condition for the existence of a good family in a space.

Lemma 4. *If X is a Tychonov space such that the set $\{p \in X : \pi\chi(p, X) \leq \omega\}$ is dense in X then there is a good family in X .*

Proof. Let $p_0 \in X$ have countable π -character (i.e. $\pi\chi(p_0, X) \leq \omega$) and, using that X is Tychonov, choose \mathcal{B}_0 , a countable collection of closed G_δ sets with non-empty interior, such that \mathcal{B}_0 forms a local π -base at p_0 in X .

